Systoles of a Family of Triangle Surfaces

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CONTENTS

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We determine the systoles for a family of closed hyperbolic triangle surfaces which admit a particularly simple combinatorial description. We show that, in this family, there are exactly four surfaces which are maximal, i.e., for which the length of the systole is a local maximum in Teichmüller space. One of these surfaces gives a new example of a maximal surface.

1. INTRODUCTION

A *systole* of an oriented closed hyperbolic surface S is a simple closed geodesic on S of minimal length. The length of the systole depends on the choice of the hyperbolic metric; hence, this length defines for every $g \geq 2$, a continuous positive nonconstant function on the Teichmüller space $\mathcal{T}_g$ of hyperbolic metrics on a closed surface of genus g which is invariant under the action of the mapping class group. This function is bounded from above on $\mathcal{T}_g$ by a constant depending on g which tends to infinity as g tends to infinity [Buser and Sarnak 94]. It is not bounded from below on $\mathcal{T}_g$.

A *triangle group of type* (a, b, c) for integers $2 \leq a \leq b \leq c$ is a discrete group of isometries of the hyperbolic plane $\mathbf{H}^2$ which is generated by reflections across the sides of some triangle with angles $\pi/a, \pi/b, \pi/c$. A *triangle surface of type* (a, b, c) is a closed hyperbolic surface which is the quotient of $\mathbf{H}^2$ under a subgroup of finite index in a triangle group of type (a, b, c). Such a surface can be triangulated by hyperbolic triangles with fixed angles $\pi/a, \pi/b, \pi/c$.

The simplest triangle surfaces are closed triangle surfaces S of type (a, b, c) which admit a cyclic group Γ of orientation preserving isometries such that S/Γ is a topological 2-sphere and the projection $S \to S/\Gamma$ is a c-fold covering which is ramified at three points of S/Γ. We call such a triangle surface *elementary* and the group Γ the *basic group of isometries of* S. There may be several non-isometric elementary triangle surfaces of a given type (a, b, c).

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Define a *fundamental polygon of type* (a, b, c) to be a $2c$-gon in the hyperbolic plane $\mathbf{H}^2$ with alternating angles $2\pi/a, 2\pi/b$ and sides of equal length. Such a polygon Ω admits a cyclic group $\tilde{\Gamma}$ of order c of isometries whose elements rotate Ω about a fixed point, with a multiple of $2\pi/c$ as rotation angle. We call the fixed point of the elements of $\tilde{\Gamma}$ the *center* of Ω. If we draw $2c$ geodesic segments from the center to the vertices of the boundary $\partial\Omega$ of Ω, then these segments decompose Ω into $2c$ triangles with angles $\pi/a, \pi/b, \pi/c$. The vertices of the boundary $\partial\Omega$ of Ω will be numbered counterclockwise such that the angle at the vertices with an even number equals $2\pi/a$ and the angle at the vertices with an odd number is $2\pi/b$.

An elementary triangle surface S of type (a, b, c) is obtained from a fundamental polygon Ω of type (a, b, c) by quotienting with a group generated by suitable side pairing transformations. The basic group Γ of isometries of S lifts to the cyclic group $\tilde{\Gamma}$ of rotations of Ω; therefore, the set of these side pairing transformations which define S is invariant under conjugation with these rotations. Thus, the set $\mathcal{A}$ of vertices with an odd number is invariant under the side pairing transformations, and the same is true for the set $\mathcal{B}$ of vertices with an even number. A vertex cycle which is contained in the set $\mathcal{A}$ has exactly a elements, and a vertex cycle which is contained in the set $\mathcal{B}$ has b elements. In particular, a and b divide c.

The triangulation of Ω into $2c$ triangles with vertices at the center 0 and on the boundary $\partial\Omega$ of Ω descends to a triangulation of the quotient surface S with $m = c/a + c/b + 1$ vertices which we call the *canonical triangulation* of S. The Gauss-Bonnet formula shows that the genus g of S equals $\frac{1}{2}(c - m + 2)$.

This paper gives a precise combinatorial description of the systoles of an elementary triangle surface (Theorem 5.6 in Section 5). To prove this result, we use a computer program which computes the systoles of any given specific example. The program takes as input a choice of type (a, b, c) and a choice of side identifications for the fundamental polygon of this type and checks first whether these side identifications give rise to a smooth closed surface. If this is the case, then the program determines the systoles of the quotient surface and computes their length.

Following Schmutz [Schmutz 93], we call a point in $\mathcal{T}_g$ a *maximal surface* if the length of the systole has a local maximum at that point. Maximal surfaces always exist, and Schmutz found explicit examples [Schmutz 93]. Using our combinatorial description of systoles, we give a complete list of all elementary triangle surfaces which are maximal. There are exactly 4 examples. Three of them are of type (p, p, p) (for $p = 7, 13, 21$) and genus $g = 3, 6, 10$. The fourth is a surface of type $(6, 24, 24)$ and genus $g = 10$. The maximal surface of type $(7, 7, 7)$ is the well-known Klein surface which was analyzed earlier by Schmutz [Schmutz 93]. The two examples of type $(13, 13, 13)$ and $(21, 21, 21)$ were found by the first author [Hamenstädt 02] with purely combinatorial methods. The example of type $(6, 24, 24)$ is new. We summarize this application of our combinatorial description of systoles as follows.

Theorem 1.1. *There is exactly one maximal surface among all elementary triangle surfaces of type* $(p/\ell, p/s, p)$ *for some* $p \geq 5$ *and some divisor* $\ell \geq 2$ *of* p. *This surface is of type* $(6, 24, 24)$ *and genus* 10.

In Section 2, we deduce some combinatorial properties of elementary triangle surfaces. Moreover, we classify elementary triangle surfaces of type $(p/\ell, p, p)$ for some $\ell \geq 2$ which admit a nontrivial group Σ of orientation-preserving isometries normalizing the basic group Γ.

In Section 3, we give a combinatorial description of a class of geodesics on an elementary triangle surface which contains all systoles. In Section 4, we describe some systoles for a particular subfamily of our family of elementary triangle surfaces. With length estimates for geodesics and some explicit computations with our computer program, we find (Section 5) a combinatorial description of the systoles on every elementary triangle surface and finish the proof of our theorem. The appendices contain tables with some specific computations needed for the argument in Section 5 and a short description of our computer program.

A maximal surface is called *globally maximal* if the length of its systole is a global maximum in Teichmüller space [Schmutz 93]. Globally maximal surfaces are known for the genus $g = 2$ and for certain surfaces of finite type with cusps [Schmutz 94] (where one can make the same definitions and ask the same questions). These globally maximal surfaces can be constructed and described with number theoretical methods, in particular they are arithmetic.

On the other hand, hyperelliptic surfaces of large genus are never globally maximal [Bavard 92]. These facts, together with our analysis of a specific class of examples, seem to support the idea that (globally) maximal surfaces are arithmetic triangle surfaces with large automorphism groups. We refer to [Buser and Sarnak 94] for interesting constructions of surfaces with systoles of large length and related results.

2. FUNDAMENTAL POLYGONS FOR ELEMENTARY TRIANGLE SURFACES

This section gives a combinatorial description of all side pairings of a given fundamental polygon Ω of type $(p/\ell, p/s, p)$ $(\ell \geq s \geq 1)$ which define elementary triangle surfaces.

Lemma 2.1. *Let Ω be a fundamental polygon of type $(p/\ell, p/s, p)$ for divisors $\ell \geq s \geq 1$ of p. There is a side pairing for Ω which defines an elementary triangle surface and which identifies the edge 1 of Ω with the edge $2k$ for some $k \geq 2$ if and only if the following two conditions are satisfied.*

(i) $k = m\ell$ for some $m \in \{1, \dots, p/\ell - 1\}$ which is prime to p/ℓ.

(ii) $k - 1 = ns$ for some $n \in \{1, \dots, p/s - 1\}$ which is prime to p/s.

Proof: Let Ω be a fundamental domain of type $(p/\ell, p/s, p)$ as above, with a numbering of vertices and edges.

Choose a number $k \in \{2, \dots, p-1\}$ and define a family of side pairing transformations of $\partial\Omega$ by requiring that the (oriented) edge with odd number $2j + 1$ be mapped onto the (oriented) edge $2j + 2k$ by an orientation-reversing isometry. This isometry then maps the vertex $2m$ to the vertex $2m+2k$, the vertex $2m+1$ to the vertex $2m+2k-1$ and extends to an orientation-preserving isometry of $\mathbf{H}^2$ which maps the interior of Ω to $\mathbf{H}^2 - \Omega$. We claim that these side pairings generate a discrete torsion-free subgroup of $PSL(2, \mathbf{R})$ with fundamental domain Ω if and only if k satisfies conditions (i) and (ii).

Recall that the angles of Ω are $2\ell\pi/p$ at the even vertices and $2s\pi/p$ at the odd vertices. Thus, for every boundary identification which gives rise to a smooth hyperbolic surface, there are precisely $\ell + s$ vertex cycles. The odd vertices are divided into s pairwise-disjoint vertex cycles containing p/s elements each, and the even vertices are divided into ℓ pairwise-disjoint vertex cycles containing p/ℓ elements each. This is the case if and only if we have $k = m\ell$ for a number $m < p/\ell$ which is prime to p/ℓ and such that $m\ell - 1 = ns$ for a number $n \geq 1$ which is prime to p/s. □

Corollary 2.2. *If there exists an elementary triangle surface of type $(p/\ell, p/s, p)$ for some divisors $\ell \geq s$ of p, then ℓ and s are prime.*

From now on, we denote by $S(p/\ell, p/s, p; k)$ an elementary triangle surface which is obtained from a fundamental polygon Ω of type $(p/\ell, p/s, p)$ by a side pairing which identifies the edge 1 with the edge $2k$.

For the remainder of this section, we only consider triangle surfaces of type $(p/\ell, p, p)$ for a divisor $\ell \geq 2$ of p. By Lemma 2.1, such an elementary triangle surface is a surface $S = S(p/\ell, p, p; k)$ for a number $k \geq 2$ with the property that $k = m\ell$ for some $m \in \{1, \dots, p/\ell - 1\}$ which is prime to p/ℓ and such that $m\ell - 1$ is prime to p. The genus of S equals $\frac{1}{2}(p - \ell)$ and therefore ℓ has to be even if this is true for p.

The next lemma shows that this is the only obstruction for the existence of an elementary triangle surface of type $(p/\ell, p, p)$.

Lemma 2.3. *Let $p \geq 5$ and let $\ell \in \{2, \dots, p-1\}$ be a divisor of p. Then there is an elementary triangle surface of type $(p/\ell, p, p)$ if and only if p and ℓ have the same parity.*

Proof: Let $p \geq 5$ and let $\ell < p$ be a divisor of p. Write $q = p/\ell$. By Lemma 2.1, we have to show that we can find a number $m < q$ which is prime to q and such that $m\ell - 1$ is prime to p provided that p and ℓ have the same parity.

Assume first that $\ell \geq 2$ and that $p = \ell q$ is odd. Then ℓ and q are odd as well and $\ell + 1$ and $\ell - 1$ have 2 as their unique common divisor.

Notice that it is enough to find an arbitrary number $\tilde{m} \geq 1$ which is prime to q and such that $\tilde{m}\ell - 1$ is prime to q as well. Namely, if for such a $\tilde{m}$, we choose $j \geq 0$ in such a way that $m = \tilde{m} - jq \in \{1, \dots, q-1\}$, then m is prime to q and $m\ell - 1 = \tilde{m}\ell - 1 - jq\ell$ is prime to q as well; therefore, m is as required.

To find such a number $\tilde{m}$, let $q = q_1 \cdots q_k$ be the decomposition of q into primes. After reordering, we may assume that for some $j \in \{0, \dots, k\}$ the numbers q_i divide $\ell - 1$ for $i \leq j$, but the numbers q_i do not divide $\ell - 1$ for $i \geq j + 1$. If $j = k$, then q is prime to $\ell + 1$ since $\ell - 1$ and $\ell + 1$ have 2 as their unique common divisor and $m = q - 1$ and $m\ell - 1 = q\ell - (\ell + 1)$ are prime to q.

For the case that $j \leq k - 1$, define $u = q_{j+1} \cdots q_k$ and let $r \geq 1$ be a number which is prime to $q_1 \cdots q_j$ such that $ru \not\equiv -1 \mod q_i$ for every $i \leq j$. Then $\tilde{m} = ru + 1$ is prime to q and $ru\ell + \ell - 1$ is prime to $q\ell$ since ℓ and $\ell - 1$ are prime.

The existence of a number r as above is immediate from the following:

Let $q_1, \dots, q_j$ be pairwise distinct odd primes and let u, v be prime to $q_1, \dots, q_j$. Let $a_i \in \{0, \dots, q_i - 1\}$ and let $b_i \in \{0, \dots, q_i - 1\}$. Then there is a number $s > 0$

such that

$$sv \not\equiv a_i \bmod q_i \text{ and } svu \not\equiv b_i \bmod q_i \quad (*)$$

for every $i \le j$.

Write $Z = \mathbf{Z}/q_1 \times \cdots \times \mathbf{Z}/q_j$ and define a map Ψ of Z by $\Psi(a_1, \ldots, a_j) = (a_1 + v, \ldots, a_j + v)$. Since v is prime to $q_1, \ldots, q_j$, we have $\Psi^m a = a$ for some $m \ge 1$ and some $a \in Z$ if and only if m is a multiple of $q_1 \cdots q_j$. But this then means that the action of Ψ on Z is transitive. In particular, the orbit of v is all of Z.

Thus, we just have to find a point $(z_1, \ldots, z_j) \in Z$ such that $z_i \not\equiv a_i \bmod q_i$ and $z_i u \not\equiv b_i \bmod q_i$. Since $q_i \ge 3$ for all i and u is prime to every q_i, this is clearly possible and shows our claim. This finishes the proof of our lemma in the case that p is odd.

If p and ℓ are even, then the numbers $\ell + 1$ and $\ell - 1$ are prime and odd and the primes $q_1, \ldots, q_j$ defined as above are odd as well. We then can argue as before. This finishes the proof of the lemma. □

Next we look at elementary triangle surfaces with additional symmetries.

Lemma 2.4.

(i) For $p \ge 5$, a divisor $\ell \ge 2$ of p and $k \in \{\ell, \ldots, p-1\}$ the surface $S(p/\ell, p, p; k)$ is isometric to the surface $S(p/\ell, p, p; r)$ provided that $(r-1)(k-1) \equiv 1$ mod p.

(ii) For $p \ge 5$ and a divisor $\ell \ge 2$ of p, an elementary triangle surface S of type $(p/\ell, p, p)$ with basic group of isometries Γ admits a nontrivial group $\Sigma \not\subset \Gamma$ of orientation-preserving isometries which normalizes Γ if and only if $S = S(p/\ell, p, p; m\ell)$ where $p/\ell > m$ is a divisor of $m\ell - 2$ which is odd if m is even. The group Σ is then cyclic of order 2.

Proof: Let $p \ge 5$, let $\ell \in \{2, \ldots, p-1\}$ be a divisor of p and let Ω be a fundamental $2p$-gon of type $(p/\ell, p, p)$ whose angle at even vertices equals $2\pi\ell/p$ and at odd vertices equals $2\pi/p$. Write $q = p/\ell$. Write A_i for the vertex with number $2i - 1$ and B_i for the vertex with number $2i$.

Assume that the edge 1 is adjacent to the vertices $2p$ and 1. Let $m \ge 1$ be such that $k = m\ell < p$ and that m and $m\ell - 1$ are prime to q. By Lemma 2.1, we obtain an elementary triangle surface of type $(p/\ell, p, p)$ by identifying the edge 1 with the edge $2k$.

Let 0 be the center of Ω, i.e., the fixed point of the cyclic group of order p of isometries of Ω. Cut Ω open along the segments $\overline{0B_j}$. We obtain a collection of p geodesic quadrangles $Q_1, \ldots, Q_p$ where the oriented boundary of Q_j consists of the segments $\overline{B_j 0}, \overline{0B_{j-1}}$ and the edges $2j-1, 2j$.

Put $P_1 = Q_1$ and, for $1 \le j \le p-1$, define inductively a polygon P_{j+1} by glueing the quadrangle $Q_{j(k-1)+1}$ to P_j along the edge $2j(k-1)+2$ which is identified with the edge $2(j-1)(k-1)+1$ of P_j. The boundary of P_j consists of $2j+2$ edges and $2j+2$ vertices. There are $j+1$ vertices of type B (i.e., which correspond to one of the vertices B_i of Ω), j vertices of type 0 and one vertex of type A.

Since $m\ell < p-2$ and since $m\ell - 1$ is prime to q, there is a unique number $r < p$ such that $(r-1)(m\ell-1) \equiv 1$ mod p. Then $rm\ell - r - m\ell \equiv 0$ mod p and therefore $r = b\ell$ for some $b \ge 2$. The polygon P_r contains the quadrangle $Q_{(r-1)(k-1)+1}$, and this quadrangle contains the segment $\overline{0B_{(r-1)(k-1)}}$. In the chain of the $2r$ edges of the polygon P_r joining vertices of type 0 and B there are exactly $2r$ edges lying between the edges $\overline{B_1 0}$ and $\overline{0B_{(r-1)(k-1)}}$. Since the vertex $B_{(r-1)(k-1)}$ coincides with the vertex B_1, this means that on the boundary of the polygon P_r the first vertex of type B is identified with the r-th vertex of type B. In other words, the side identifications which define S from Ω where the center of Ω projects to the vertex A identifies the edge 1 with the edge $2r$. With this construction, we obtain the required isometry of $S(p/\ell, p, p; k)$ onto $S(p/\ell, p, p; r)$.

Now let $S = S(p/\ell, p, p; k)$ be an elementary triangle surface which admits a non-trivial group Σ of orientation-preserving isometries such that the basic group Γ is normal in the group G generated by Σ and Γ.

Then the action of Σ on S descends to an isometric action on the sphere S/Γ. Such an action has to preserve the singular set $\{\hat{A}, \hat{B}, \hat{0}\} \subset S/\Gamma$ of ramification points.

Since, by assumption, the elements of Σ preserve the orientation of S and hence of S/Γ, we conclude that necessarily Σ fixes the singular point $\hat{B}$ of S/Γ and permutes the two other ones. This implies that $(k-1)(k-1) \equiv 1$ mod p or equivalently that $k(k-2) \equiv 0$ mod p.

Now set $k = m\ell$. Since $m \ge 1$ is a unit in the ring $\mathbf{Z}/q$, we conclude that q divides $\ell m - 2$ and is odd if m is even. Moreover, by assumption, $m\ell < q\ell$ and therefore $q > m$.

On the other hand, for every divisor $q > m$ of $m\ell - 2$ which is odd if m is even, m is necessarily a unit in $\mathbf{Z}/q$ and $m\ell - 1$ is a unit in $\mathbf{Z}/\ell q$. This finishes the proof of part (ii) of our lemma. □

Remark 2.5. For a given type $(p/\ell, p, p)$, there are at most two elementary triangle surfaces with the properties described in the second part of Lemma 2.4. To see this, let $q \geq 2$ be arbitrary and let $\ell \geq 2$ be a number which is even if this is true for q. If there is some $m < q$ such that $m\ell \equiv 2 \bmod q$, then either ℓ and q are prime and m is unique, or the biggest common divisor of ℓ and q is 2 and there are at most two solutions.

We conclude this section with some examples.

Example 2.6. For $u \geq 1$, the surfaces $S(u+2, 2(u+2), 2(u+2); 2)$ and $S(4u, 8u, 8u; 4u+2)$ satisfy the condition in part (ii) of Lemma 2.4. These surfaces give all the examples with $\ell = 2$.

Example 2.7. For $\ell = 3$, all odd numbers which do not have 3 as a divisor are possible for q. There is a surface of type $(5, 15, 15)$ and genus $g = 6$ with $m = 4$, and a surface of type $(7, 21, 21)$ with $m = 3$.

Example 2.8. For $\ell = 4$, all numbers $q \geq 2, q \not\equiv 0 \bmod 4$ are possible for q. For $q = 2$, this yields the surface $S(2, 8, 8; 4)$ of genus $g = 2$ with $m = 1$. For $q = 3$, we obtain the surface $S(3, 12, 12; 8)$ of genus $g = 4$ with $m = 2$. For $q = 5$, there is the surface $S(5, 20, 20; 12)$ of genus $g = 8$ with $m = 3$. For $q = 6$, we obtain the surface $S(6, 24, 24; 20)$ of genus $g = 10$ which is the maximal surface from Theorem 1.1.

3. FIRST PROPERTIES OF SYSTOLES OF ELEMENTARY TRIANGLE SURFACES

In this section, we derive some first easy properties of systoles of an elementary triangle surface $S = S(p/\ell, p/s, p; k)$.

The canonical triangulation of the surface $S = S(p/\ell, p/s, p; k)$ is invariant under the basic group Γ of isometries of S. The projection of the center of Ω is a vertex 0 of the canonical triangulation which is a fixed point for the action of Γ. The remaining $s + \ell$ vertices are contained in two Γ-orbits $A_1, \dots, A_s$ and $B_1, \dots, B_\ell$. The angle of the triangles of the canonical triangulation equals π/p at the vertex 0, $s\pi/p$ at the vertices $A_1, \dots, A_s$ and $\ell\pi/p$ at the vertices $B_1, \dots, B_\ell$.

Recall that S/Γ is a topological 2-sphere which consists of two hyperbolic triangles with angles $\pi/p, s\pi/p, \ell\pi/p$ glued at their boundaries. The hyperbolic metric on S projects to a hyperbolic metric on S/Γ with 3 singular points $\hat{A}, \hat{B}, \hat{0}$. Here the point $\hat{0}$ is the projection of the vertex 0 of S, $\hat{A}$ is the projection of the vertices $A_1, \dots, A_s$, and $\hat{B}$ is the projection of the vertices $B_1, \dots, B_\ell$. The sphere S/Γ admits a natural-orientation reversing isometry $\hat{\Psi}$ of order 2 which exchanges the two triangles and leaves their common boundary pointwise fixed. Every closed geodesic on S projects to a closed geodesic on S/Γ with respect to this metric which may pass through a singular point.

Let $\triangle$ be a hyperbolic triangle with angles $\pi/p, s\pi/p, \ell\pi/p$. The triangle $\triangle$ will be viewed as a billiard table. A billiard orbit consists of smooth geodesic arcs inside $\triangle$ which are joined at points of the boundary $\partial\triangle$ according to the rule that the angle of incidence equals the angle of reflection.

A closed geodesic on S/Γ which does not pass through a vertex of the canonical triangulation corresponds to a periodic billiard orbit in $\triangle$. The *prime period* of such an orbit is the number of collisions with the boundary before returning to the original position for the first time. We divide such periodic billiard orbits into three different types.

(1) A periodic billiard orbit with an odd prime period such that none of the collisions with the boundary is perpendicular.

We call such a billiard orbit an *A-orbit*. The double of a prime A-orbit $\tilde{\gamma}$ admits a unique lift to a closed geodesic $\hat{\gamma}$ on S/Γ which is invariant under the orientation-reversing isometry $\hat{\Psi}$, and its length is twice the length of $\tilde{\gamma}$.

(2) A periodic billiard orbit whose trace consists of one piecewise geodesic arc which meets the boundary $\partial\triangle$ orthogonally at its endpoints.

We call such an orbit a *B-orbit*. A prime B-orbit $\tilde{\gamma}$ admits a unique lift to S/Γ which is invariant under the natural isometry $\hat{\Psi}$ and whose length is twice the length of the trace of $\tilde{\gamma}$.

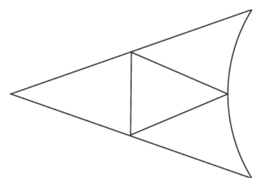
(3) A periodic billiard orbit with an even prime period such that none of the collisions with the boundary of $\triangle$ is perpendicular.

We call such an orbit a C-orbit. A prime C-orbit $\tilde{\gamma}$ admits two different lifts to S/Γ which are images of each other under the isometry $\hat{\Psi}$. The length of each of these lifts coincides with the length of $\tilde{\gamma}$.

A periodic billiard orbit $\tilde{\gamma}$ on $\triangle$ as above is *liftable to* S if there is a closed geodesic γ on S whose projection to S/Γ is a lift $\hat{\gamma}$ of $\tilde{\gamma}$ to S/Γ. Then γ is called a *lift of* $\tilde{\gamma}$ *to* S.

An example of a liftable billiard orbit is given in the next lemma.

Lemma 3.1. *There is a unique A-orbit* $\tilde{\gamma}_1$ *in* $\triangle$ *with 3 collisions with the boundary. The length of* $\tilde{\gamma}_1$ *is not larger than* $3\operatorname{arccosh}\frac{3}{2} \sim 2.8872$ *and its double is liftable.*



Proof: Let $S = S(p/\ell, p/s, p; k)$ and let Ω be a fundamental polygon of type $(p/\ell, p/s, p)$. Connect the midpoint of the edge 1 in Ω with the midpoint of the edge 3 by a simple arc, and connect the midpoint of the edge $2k$ with the midpoint of the edge $2k+2$ by a simple arc. These two arcs together project to a simple closed curve on S which is freely homotopic to a closed geodesic γ on S. The geodesic γ is necessarily a lift of the double of an A-orbit $\tilde{\gamma}_1$ in $\triangle$ of period 3. The length ℓ_1 of a lift of $\tilde{\gamma}_1$ to S is bounded from above by $6\operatorname{arccosh}\frac{3}{2}$, which is twice the minimal circumference of a triangle with vertices on the boundary of an ideal triangle in $\mathbf{H}^2$ (compare Lemma 3.2 of [Hamenstädt 02]). □

There are two natural ways to define billiard orbits which pass through a vertex of $\triangle$.

(1) The outgoing arc of an orbit through a vertex $\tilde{B}$ of $\triangle$ equals the reflection of the incoming arc along the unique geodesic line which bisects the angle at that vertex.

(2) The outgoing arc of an orbit through a vertex $\tilde{B}$ of $\triangle$ coincides with the incoming arc.

We require that a billiard orbit which passes through a vertex $\tilde{B}$ of $\triangle$ be reflected as in (1) above if the angle at $\tilde{B}$ equals π/q for an odd number $q \geq 3$, and we call such a vertex $\tilde{B}$, a *bending vertex*. If the angle at a vertex $\tilde{B}$ equals π/q for an even number $q \geq 2$, then we require that a billiard orbit which passes through $\tilde{B}$ be reflected as in (2) above, and we call $\tilde{B}$ a *reflecting vertex*. Our definition is such that each closed geodesic on our triangle surface S which possibly passes through a vertex of the canonical triangulation is the lift of a billiard orbit.

The next lemma shows that a systole on S does not pass more than once through a vertex of the canonical triangulation.

Lemma 3.2. *A periodic billiard orbit on* $\triangle$ *which passes more than once through a vertex of* $\triangle$ *does not lift to a systole on* S*.*

Proof: Let ℓ_1 be the length of the prime A-orbit $\tilde{\gamma}_1$ from Lemma 3.1. By Lemma 3.1 it is enough to show that the length of a billiard orbit $\tilde{\gamma}$ in $\triangle$ which passes twice through a vertex of $\triangle$ is bigger than $2\ell_1$.

Since ℓ_1 is the smallest circumference of a triangle with vertices on the three different sides of $\triangle$, it is smaller than the circumference of every degenerate triangle with one vertex at a vertex of $\triangle$ and two identical sides which join this vertex to the opposite side. Thus, every arc in $\triangle$ which connects a vertex of $\triangle$ to the opposite side is longer than $\ell_1/2$. Since our billiard orbit $\tilde{\gamma}$ contains at least four such arcs, it is necessarily longer than $2\ell_1$. □

We can extend our definition of A, B, C-orbits to orbits passing through vertices of $\triangle$. To avoid confusion later on, we denote these extended classes by A^0, B^0, C^0. Notice that with this definition, a billiard orbit which passes through a reflecting vertex is necessarily a B^0-orbit.

Lemma 3.3.

(i) A C^0*-orbit in* $\triangle$ *does not lift to a systole on* S*.*

(ii) The double of an A^0*-orbit can only lift to a systole on* S *if this orbit is the A-orbit from Lemma 3.1.*

Proof: By Lemma 3.2, it suffices to show the lemma for A-orbits and for C^0-orbits which pass at most once through a vertex of $\triangle$. This can be done with the arguments from Section 3 of [Hamenstädt 02]. We call a curve α on S/Γ *admissible* if α is a closed curve with the additional property that every connected component of an intersection of α with one of the two triangles which make up S/Γ consists of a single geodesic segment. We require, moreover, that there is at most one pair of adjacent such segments with one endpoint at a vertex $V \in \{\hat{A}, \hat{B}, \hat{0}\}$ of S/Γ

An *admissible homotopy* of an admissible curve α is a homotopy of α through admissible curves which preserves an intersection with the set of vertices $\{\hat{0}, \hat{A}, \hat{B}\}$. We call the admissible curve α *essential* if it can not be homotoped to a *simple* admissible curve. A *simplification* of α is an admissible essential subcurve β of α such that α can be written in the form $\alpha = \beta\gamma$ where γ is nonessential and does not meet the vertices of S/Γ.

Let γ be an arbitrary closed piecewise geodesic in $\triangle$ with breakpoints only on the boundary of $\triangle$ and with an even number of segments. We assume that each of these segments meets the boundary of $\triangle$ only at its endpoints. Furthermore, we require that γ contains at most one pair of (adjacent) segments with one endpoint at a vertex of $\triangle$.

A *lift* of γ to S/Γ is a closed admissible piecewise geodesic $\hat{\gamma}$ in S/Γ which projects onto γ. By our assumption, γ has exactly two (not necessarily distinct) lifts which are mapped to one another by the orientation-reversing isometry $\hat{\Psi}$.

For these slightly extended definitions, the arguments in Section 3 of [Hamenstädt 02] are valid. They show that after finitely many simplifications of a lift $\hat{\gamma}$ of an A-orbit or of a C^0-orbit as above, we obtain a curve $\hat{\beta}$ which contains an admissible subcurve homotopic to the lift of the double of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1. But then $\hat{\gamma}$ either coincides with the lift of this orbit, or it is strictly longer than this lift; the lemma follows. □

Lemma 3.4. *A systole for $S = S(p/\ell, p/s, p)$ does not pass through a vertex of the canonical triangulation.*

Proof: The case $\ell = s = 1$ is contained in [Hamenstädt 02], so we may assume that $\ell \geq 2$.

Notice, first, that a systole can not pass through the projection x to S of the center of the fundamental polygon Ω. Every geodesic passing through x is at least as long as a geodesic obtained from a side pairing transformation for Ω, with equality only if such a side pairing identifies the edge 1 of Ω with the edge $2k = p+1$. But ℓ is a common divisor of p and k and, therefore, this is only possible if $\ell = 1$. Thus, a systole does not pass through a fixed point of the basic group Γ of isometries.

Now, let γ be a closed geodesic on S which passes through a vertex V of the canonical triangulation which is not fixed by Γ. We need to show that γ is not a systole. If γ contains an edge of the canonical triangulation, then γ has self-intersections and therefore can not be a systole [Schmutz 93]. Thus, by Lemma 3.2 and Lemma 3.3, we may assume that γ is the lift of a B^0-orbit $\tilde{\gamma}$ in $\triangle$ whose trace has one of its endpoints at a vertex $\tilde{B}$ of $\triangle$ and does not have any other intersections with the vertices of $\triangle$.

Orient the orbit $\tilde{\gamma}$ in such a way that its first segment η_1 connects the vertex $\tilde{B}$ to the opposite edge b. We consider first the case that the interior of every side of $\triangle$ contains a collision point with $\tilde{\gamma}$. The trace of $\tilde{\gamma}$ consists of at least three segments. The second segment η_2 connects the side b to a side a adjacent to $\tilde{B}$. There is also a segment with one endpoint E on the third side c of $\triangle$. By the triangle inequality, the trace of $\tilde{\gamma}$ is longer than the piecewise geodesic which contains the segments η_1, η_2, and the arc $\tilde{\eta}_3$ connecting the endpoint of η_2 to E. Notice that $\tilde{\eta}_3$ and η_1 intersect. By making $\tilde{\eta}_3$ shorter, we may assume that it meets the side c orthogonally at a (possibly) different point which we denote again by E. But then the angle at E of the triangle T inscribed in $\triangle$ which consists of the sides $\eta_2, \tilde{\eta}_3$, and a third side connecting the endpoint of $\tilde{\eta}_3$ to the beginning point of η_2 on the edge b is smaller than $\pi/2$. From strict convexity of the distance function from the beginning point of η_2 on b, we conclude that the third side of this triangle is shorter than η_1. In other words, the trace of $\tilde{\gamma}$ is longer than the circumference of the triangle T and the orbit $\tilde{\gamma}$ is longer than the double of the orbit $\tilde{\gamma}_1$ from Lemma 3.1.

Next we show that an orbit $\tilde{\gamma}$ whose trace contains a single segment passing through a bending vertex $\tilde{B}$ does not lift to a systole on S. By our above consideration, we may assume that there is a side c of $\triangle$ adjacent to $\tilde{B}$ whose interior does not intersect the trace of $\tilde{\gamma}$.

By definition of a bending vertex, the segment η_1 of the trace of $\tilde{\gamma}$ with one endpoint at $\tilde{B}$ bisects the angle at $\tilde{B}$. Thus, either $\tilde{\gamma}$ consists of the single segment η_1, or the angles at the vertices different from $\tilde{B}$ do not coincide. In both cases, our side c which is not intersected by $\tilde{\gamma}$ can be chosen to be opposite to a vertex $\tilde{0}$ with angle π/p.

As a consequence, a lift γ of $\tilde{\gamma}$ to the polygon Ω consists of a single geodesic arc connecting two preimages of $\tilde{B}$. By convexity, the shortest such arc connects two neighboring preimages of $\tilde{B}$ on the boundary of Ω. Assume that the angle at $\tilde{B}$ equals $\ell\pi/p$; then we may choose γ in such a way that it connects the vertices $2p$ and 2ℓ.

Now $\tilde{B}$ is a bending vertex, and therefore the number $q = p/\ell$ is odd. If γ is the lift of a closed geodesic on the surface $S = S(p/\ell, p/s, p; m\ell)$, then in the arrangement of the triangles of the canonical triangulation around a lift of $\tilde{B}$ there are $\frac{q-1}{2}$ quadrangles with angle $2\pi/q$ between the quadrangles with vertices $2p-1, 2p, 1$ and $2\ell-1, 2\ell, 2\ell+1$. In particular, we have $\frac{q+1}{2}m \equiv \pm 1 \bmod q$. Since $m < q$, this shows that $m = 2$ or $m = q-1$. But then our curve is homotopic to a geodesic which is induced by a side-pairing transformation; this is impossible. As a consequence, a systole on S does not pass through a bending vertex of $\triangle$.

Now let $\tilde{B}$ be a reflecting vertex. The angle at $\tilde{B}$ equals π/q for an even number q. If ξ is a generator of the basic group Γ, then the subgroup $\{\xi^{k\ell} \mid k \geq 0\}$ of Γ

fixes a lift B of $\tilde{B}$ to the surface S and acts as a group of rotations about B with rotation angle a multiple of $2\pi/q$. In particular, there is some $k > 0$ such that $\xi^{k\ell}$ is an involution which fixes B. A closed geodesic γ on S which passes through B is invariant under this involution. If γ is simple, then it passes through a second fixed point of $\xi^{k\ell}$. However, each fixed point for $\xi^{k\ell}$ is a vertex of the canonical triangulation and, therefore, γ is the lift of a B^0-orbit which contains at least two arcs through a vertex of $\triangle$. By Lemma 3.3, γ is not a systole.

Thus, a systole on an elementary triangle surface does not pass through a vertex of the canonical triangulation. □

Consider any piecewise geodesic α in $\triangle$ with the following properties:

(1) There is a pair e_1, e_2 of sides of $\triangle$ which is connected by at most one subarc of α.

(2) If e_3 is the third side of $\triangle$, then the subcurves α_1, α_2 of α which contain all arcs of α joining e_1, e_2 to e_3 are connected, and either $\alpha = \alpha_1\alpha_2$ or $\alpha_1\alpha_2$ is not connected.

We call such a curve *irreducible* (see [Hamenstädt 02] for a motivation for this notation). From Lemma 3.4 and the arguments in [Hamenstädt 02] which are equally valid in our situation, we obtain:

Corollary 3.5. *A systole in S is either a lift of the double of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1 or a lift of a B-orbit with irreducible trace.*

4. EXAMPLES OF SYSTOLES

In this section, we compute some examples of systoles on elementary triangle surfaces. We continue to use the assumptions and notation from Section 3. In particular, we denote by $\triangle$, a hyperbolic triangle with angles $\ell\pi/p, s\pi/p, \pi/p$ for an integer $p > 0$ and divisors $\ell \geq s \geq 1$ of p. Let $\tilde{0}, \tilde{A}, \tilde{B}$ be the vertex of $\triangle$ with angles $\ell\pi/p, s\pi/p, \pi/p$, respectively.

Recall the definition of irreducible curves from Section 3. A *lift* to S/Γ of an irreducible curve α is a closed piecewise geodesic curve in $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\}$ which is invariant under the natural isometry $\hat{\Psi}$ of order 2 of S/Γ exchanging the two triangles and which projects to α. Call two irreducible curves α, β in $\triangle$ *homotopic* if there are lifts of β and α to S/Γ which are freely homotopic in $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\}$.

Define an *irreducible curve of type 1* to be an irreducible curve α with the additional property that every pair of sides of $\triangle$ is connected by a segment of α. An irreducible curve which is not of type 1 will be called of *type 0*. Define a *B_1-orbit* to be a B-orbit whose trace is an irreducible curve of type 1. A B-orbit with irreducible trace of type 0 will be called a *B_0-orbit*.

The next lemma gives a useful criterion for the existence of B-orbits with irreducible trace.

Lemma 4.1. *Let α be an irreducible curve in $\triangle$ and let $\tilde{m}_{\tilde{0}}, \tilde{m}_{\tilde{A}}, \tilde{m}_{\tilde{B}}$ be the number of geodesic arcs of α connecting the edges adjacent to the vertices $\tilde{0}, \tilde{A}, \tilde{B}$. For a vertex $\tilde{C} \in \{\tilde{0}, \tilde{A}, \tilde{B}\}$, define $m_{\tilde{C}} = \tilde{m}_{\tilde{C}} + 1$ if α is of type 1 and if the edges adjacent to $\tilde{C}$ are connected by a subarc of α containing one of the endpoints of α. Define $m_{\tilde{C}} = \tilde{m}_{\tilde{C}}$, otherwise. If there is a B-orbit $\tilde{\gamma}$ whose trace is homotopic to α, then we have $m_{\tilde{0}} < p/2, m_{\tilde{A}} < p/2s$ and $m_{\tilde{B}} < p/2\ell$; moreover $p/\ell \geq 3$ unless α is of type 0 and $\min\{\tilde{m}_{\tilde{A}}, \tilde{m}_{\tilde{0}}\} = 1$.*

Proof: Let α be the irreducible trace of a B_0-orbit in $\triangle$, and let α_1 be the connected subcurve of α consisting of all geodesic segments which connect a fixed pair of edges of $\triangle$, say the edges d and e. Assume that α_1 contains one of the endpoints of α. Let m be the number of geodesic arcs of α_1.

Denote by $\tilde{C}$, the vertex of $\triangle$ adjacent to the sides d and e, and let δ be the angle of $\triangle$ at the vertex $\tilde{C}$. Choose a vertex C of the canonical triangulation of S which is a lift of $\tilde{C}$. Then C has an open contractible neighborhood U in S which is isometric to the interior of a convex $2\pi/\delta$-gon consisting of $2\pi/\delta$ isometric triangles.

There is a lift of the subarc α_1 to a connected geodesic arc in S which is entirely contained in U and neither goes through the center of U nor through any of the vertices of its boundary. Thus if k is the number of geodesic arcs in U which connect the center to one of the vertices and is intersected by α_1, then $k = 2m + 1$. Since by convexity we always have $k \leq 2\pi/\delta - 1$, we conclude that $m \leq \pi/\delta - 1$. This shows the lemma in the case that our irreducible curve is of type 0. The case of an irreducible curve of type 1 follows in the same way. □

Our next goal is to study the question of liftability for B_0-orbits. We denote by c the side of $\triangle$ which is opposite to the vertex $\tilde{0}$ with angle π/p, by a the side opposite to the vertex $\tilde{A}$ with angle π/s, and by b the side opposite to the vertex $\tilde{B}$ with angle $\ell\pi/p$.

Lemma 4.2. *Let $\tilde\eta$ be the trace of a B_0-orbit in $\triangle$ which consists of at least 3 segments. We divide $\tilde\eta$ into two connected subarcs $\tilde\eta_1, \tilde\eta_2$. The subarc $\tilde\eta_1$ consists of $r_1 \geq 1$ segments connecting the edge e to the edge f of $\triangle$, and the subarc $\tilde\eta_2$ consists of $r_2 \geq 1$ segments connecting the edge e to the edge g of $\triangle$. Then $\tilde\eta$ admits a lift to a closed geodesic on $S = S(p/\ell, p/s, p; k)$ if and only if one of the following possibilities is satisfied.*

	e	f	g	
(i)	a	b	c	$r_1 + r_2 k \equiv 0 \bmod p$
(ii)	a	c	b	$r_2 + r_1 k \equiv 0 \bmod p$
(iii)	b	a	c	$r_1 - r_2 k + r_2 \equiv 0 \bmod p$
(iv)	b	c	a	$r_2 - r_1 k + r_1 \equiv 0 \bmod p$
(v)	c	a	b	$k(r_1 + r_2) - r_2 \equiv 0 \bmod p$
(vi)	c	b	a	$k(r_1 + r_2) - r_1 \equiv 0 \bmod p$

Proof: We show only case (ii) of the lemma; the other cases follow by the same arguments.

Let Ω be a fundamental $2p$-gon of type $(p/\ell, p/s, p)$. Let $k \geq 2$ be such that the side pairings for Ω which induce our surface S in such a way that the center of Ω corresponds to the vertex $\tilde 0$, identify the edge 1 with the edge $2k$. By Lemma 2.1, $k = m\ell$ for some $m \geq 1$, and $k - 1 = ns$ for some $n \geq 1$.

Let $\tilde\eta$ be as in case (ii) of the lemma. Denote by $\tilde\alpha_1$ the connected subarc of $\tilde\eta$ which consists of $r_1 - 1$ segments connecting the edge c to the edge a of $\triangle$ and which contains one of the endpoints of $\tilde\eta$. Denote by $\tilde\alpha_2$ the connected subarc $\tilde\eta - \tilde\alpha_1$ of $\tilde\eta$; it consists of $r_2 + 1 \geq 2$ segments and has one endpoint on the edge c of $\triangle$.

Assume that the billiard orbit with trace $\tilde\eta$ lifts to a closed geodesic on S. This geodesic then lifts to a piecewise geodesic α in Ω which we can choose in such a way that a lift of the endpoint of $\tilde\alpha_2$ on the edge c is contained in the edge 2 of Ω. Then α consists of r_1 connected arcs $\alpha_{1,1}, \dots, \alpha_{1,r_1-1}, \alpha_2$ in Ω. The arc α_2 is a lift of $\tilde\alpha_2$. Its first intersection with the geodesics in Ω which connect 0 to the vertices of Ω is a lift of the edge a which connects $\tilde 0$ to $\tilde B$.

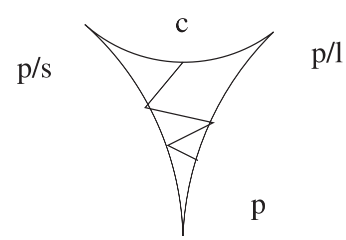
This means that α_2 connects the edge 2 to the edge $2r_2 + 3$. Similarly, since α is the lift of a closed curve on S, the arc $\alpha_{1,i}$ connects the edge $2r_2 + 2ki + 2$ to the edge $2r_2 + 2ki + 3$ $(i = 1, \dots, r_1 - 1)$. The endpoints of α_{1,r_1-1} and α_2 are identified by a side pairing of Ω and consequently, we have $2r_2 + 2r_1 k + 2 \equiv 2 \bmod 2p$ and hence p divides $r_2 + r_1 k$, proving the lemma. □

Let Ω be a fundamental polygon of type $(p/\ell, p/s, p)$. The side pairings for Ω which induce the surface S define a particular B_0-orbit $\tilde\gamma_0$ which can be described as follows. Assume that the side pairings identify the edge 1 with the edge $2k$ for $k \in [2, p/2]$. Let $\tilde 0$ be the vertex of $\triangle$ corresponding to the center of Ω. Then $\tilde\gamma_0$ has the following properties:

(a) The trace of $\tilde\gamma_0$ is a piecewise geodesic which consists of exactly k geodesic segments.

(b) One of the two endpoints of the trace of γ_0 lies on the edge c opposite to $\tilde 0$ and is the only intersection with this edge.

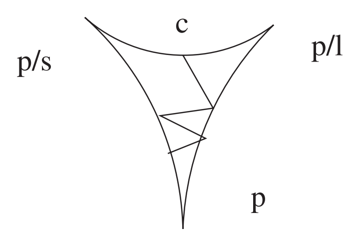
(c) The segment with one endpoint on c has its second endpoint on the edge joining $\tilde 0$ to $\tilde A$.



Similarly, if the side pairings identify the edge 1 with the edge $2k$ for some $k \in [p/2, p-1]$ then the side pairings for S define a B_0-orbit whose trace satisfies (b) above, and moreover

(a′) The trace of $\tilde\gamma_0$ is a piecewise geodesic which consists of exactly $p - k$ geodesic segments.

(c′) The segment with one endpoint on c has its second endpoint on the edge joining $\tilde 0$ to $\tilde B$.



We call such an orbit a *side pairing orbit*. In other words, a side pairing orbit is a liftable orbit whose trace has properties (a), (b), (c) or (a′), (b), (c′).

For $\ell = 1$, there are three different side pairing orbits. They are of the same length if and only if S admits a nontrivial group Σ of orientation-preserving isometries normalizing the basic group Γ (see [Hamenstädt 02]). For $\ell \geq 2$ and a surface S of type $(p/\ell, p, p)$, there are exactly two different side pairing orbits. They are of the same length if and only if S is one of the surfaces from Lemma 2.4. If $s \geq 2$, then a side pairing orbit is unique.

The following two corollaries are immediate consequences of Lemma 4.2.

Corollary 4.3. *Let $\tilde{\eta}$ be a liftable B_0-orbit in $\triangle$ with the additional property that one of the edges of $\triangle$ contains one endpoint of the trace of $\tilde{\eta}$ and no other collision point. Then $\tilde{\eta}$ is a side pairing orbit.*

Proof: We need to show that if $\tilde{\eta}$ is a liftable B_0-orbit in $\triangle$ as in the corollary, then the edge of $\triangle$ which contains one endpoint of the trace of $\tilde{\eta}$ and no other collision point is opposite a vertex with angle π/p.

Our orbit $\tilde{\eta}$ corresponds to the case $r_1 = 1$ in Lemma 4.2, and we need to show that either $f = c$, or $f = a$ and $s = 1$, or $f = b$ and $\ell = 1$. However, since ℓ is a common divisor of k and p and s is a common divisor of $k-1$ and p, this is immediate from the table in Lemma 4.2. □

Corollary 4.4. *Let $\tilde{\eta}$ be a B_0-orbit in $\triangle$ whose trace has 5 intersections with the boundary and such that both endpoints lie on the same edge of $\triangle$. Then $\tilde{\eta}$ admits a lift to a closed geodesic on S if and only if one of the following two possibilities holds.*

(i) $\ell = 2, s = 1$, the endpoints of the trace of $\tilde{\eta}$ lie on an edge adjacent to the vertex with angle $\ell\pi/p$ and $S = S(p/\ell, p/s, p; \frac{1}{2}(p-r))$ or $S = S(p/\ell, p/s, p; \frac{1}{2}(2p-r))$.

(ii) $\ell = 2, s = 1$, the endpoints of the trace of $\tilde{\eta}$ lie on the edge opposite to the vertex with angle $\ell\pi/p$ and S is one of the surfaces described in Lemma 2.4.

Proof: Let $\tilde{\eta}$ be as in Lemma 4.2 with $r_1 = r_2 = 2$. Assume that $\tilde{\eta}$ lifts to a closed geodesic on S. Since $k = m\ell$ for some $m \geq 1$ and ℓ divides p, we conclude from the list in Lemma 4.2 that ℓ divides 2 and necessarily $\ell = 2, s = 1$. On the other hand, we have $m\ell < p$. Thus, in the first part of Lemma 4.2, only the cases $2m\ell = p-2$ or $2m\ell = 2p-2$ are possible. The corollary now follows from Lemma 4.2. □

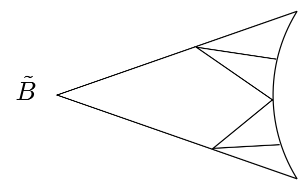
In the next lemma we compare the length of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1 with the length of a specific B-orbit in a triangle $\triangle$ with angles $\ell\pi/p, \pi/p, \pi/p$ for some $\ell \geq 2$.

Lemma 4.5. *Let $\ell \geq 2, s = 1$ and let $\tilde{\gamma}_2$ be a B-orbit in $\triangle$ with the following properties.*

(i) The trace of $\tilde{\gamma}_2$ has 5 intersections with the boundary.

(ii) Both endpoints of the trace of $\tilde{\gamma}_2$ as well as a third intersection point lie on the edge of $\triangle$ opposite to the vertex $\tilde{B}$ with angle $\ell\pi/p$.

If $\ell \geq 3$, then the length of $\tilde{\gamma}_2$ is larger than the length of the double of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1. For $\ell = 2$, equality holds.



Proof: Let $\tilde{\gamma}_2$ be a B-orbit in $\triangle$ as in the statement of the lemma; it is necessarily unique. Denote by ℓ_2 the length of the trace ζ of $\tilde{\gamma}_2$.

Draw the perpendicular g from the vertex $\tilde{B}$ of $\triangle$ with angle $\ell\pi/p$ to the opposite side b; it meets b at the midpoint E of b. The orbit $\tilde{\gamma}_2$ is invariant under the reflection Λ along g and hence the point E is the unique intersection point of ζ with b which is not an endpoint of ζ.

Consider the subtriangle $\tilde{\triangle}$ of $\triangle$ with vertices $\tilde{A}, \tilde{B}, E$. The angle of $\tilde{\triangle}$ at the vertex $\tilde{B}$ is $\ell\pi/2p$, and the angle at $\tilde{A}$ equals π/p. Denote by ℓ_0 the length of the side b of $\triangle$ opposite to $\tilde{B}$, by h the length of the perpendicular g from $\tilde{B}$ to b. Hyperbolic trigonometry for the triangle $\tilde{\triangle}$ gives

$$\sinh\frac{\ell_0}{2} = \sinh h\frac{\sin \ell\pi/2p}{\sin \pi/p},$$

and hence $\ell_0/2 < h$ for $\ell = 1$, $\ell_0/2 = h$ for $\ell = 2$ and $\ell_0/2 > h$ for $\ell \geq 3$.

Let Ψ be the reflection in the hyperbolic plane across the hyperbolic geodesic which contains the side c of $\triangle$ connecting $\tilde{A}$ to $\tilde{B}$. Then $\tilde{\triangle}$ and $\Psi\tilde{\triangle}$ form together a geodesic quadrangle Q with a right angle at the vertex E and the angle $2\pi/p$ at the vertex $\tilde{A}$. The length $\ell_2/2$ of $\zeta \cap \tilde{\triangle}$ equals the distance between E and the side Ψb of $\Psi\tilde{\triangle} \subset Q$.

Hyperbolic trigonometry shows that

$$\sinh\frac{\ell_2}{2} = \sinh\frac{\ell_0}{2}\sin 2\pi/p.$$

Again let $\tilde{\gamma}_1$ be the A-orbit as in Lemma 3.1 of length ℓ_1; it is invariant under the reflection Λ along the perpendicular g from $\tilde{B}$ to b. Thus $\tilde{\gamma}_1$ meets the fix point set g of Λ perpendicularly and has E as a collision point. This means that the length $\ell_1/2$ of $\tilde{\gamma}_1 \cap \tilde{\triangle}$ equals the distance between the point E and the side Ψg of $\Psi\tilde{\triangle} \subset Q$.

Hyperbolic trigonometry shows that

$$\sinh \frac{\ell_1}{2} = \sinh h \sin \ell\pi/p.$$

Thus $\ell_2 \geq \ell_1$ if and only if $\frac{\sin \ell\pi/2p}{\sin \pi/p} \sin 2\pi/p \geq \sin \ell\pi/p$ which implies our lemma. □

The next lemma gives a description of systoles for a specific family of elementary triangle surfaces.

Lemma 4.6. *Let $\tilde{\gamma}_3$ be a B_0-orbit in $\triangle$ with the following properties.*

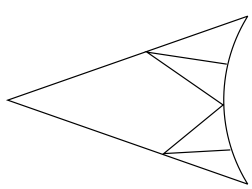
(i) The trace of $\tilde{\gamma}_3$ has 5 intersections with the boundary.

(ii) Both endpoints of the trace of $\tilde{\gamma}_3$ lie on the same edge of $\triangle$.

Then $\tilde{\gamma}_3$ admits a lift to a systole γ_3 of S if and only if one of the following two possibilities is satisfied.

(1) $\ell = 2, s = 1$, S is one of the surfaces described in Lemma 2.4, and the lifts to S of the double of the A-orbit $\tilde{\gamma}_1$ are systoles of S. Moreover the edge of $\triangle$ containing the endpoints of the trace of $\tilde{\gamma}_3$ is opposite to the vertex $\tilde{B}$ with angle $\ell\pi/p$.

(2) $\ell = 2$, $p \equiv 2 \mod 4$, $S = S(p/2, p, p; \frac{1}{2}(p-2))$ and the edge of $\triangle$ containing the endpoints of the trace of $\tilde{\gamma}_3$ is adjacent to the vertex $\tilde{B}$ with angle $\ell\pi/p$. In addition there is no liftable irreducible B-orbit for S which is strictly shorter than $\tilde{\gamma}_3$.



Proof: As before, denote by $\tilde{B}$ the vertex of $\triangle$ with angle $\ell\pi/p$, let $\tilde{A}$ be the vertex with angle $s\pi/p$ and let $\tilde{0}$ be the remaining vertex. Denote by b, a, c the edges of $\triangle$ which are opposite to $\tilde{B}, \tilde{A}, \tilde{0}$. The proof of the lemma is divided into 2 steps.

Step 1: Let $\tilde{\gamma}_2$ be a B_0-orbit in $\triangle$ as in the statement of the lemma with the additional property that the endpoints of the trace of $\tilde{\gamma}_2$ lie on the edge of $\triangle$ opposite to the vertex with angle $\ell\pi/p$. Such an orbit is necessarily unique.

By Corollary 4.4, $\tilde{\gamma}_2$ is liftable if and only if $\ell = 2$, $s = 1$, and S is one of the surfaces from Lemma 2.4. On the other hand, by Lemma 4.5, the length ℓ_2 of the trace of $\tilde{\gamma}_2$ is equal to the length of the unique A-orbit $\tilde{\gamma}_1$ of period 3.

Step 2: Let $\tilde{\gamma}_3$ be as in the lemma with the additional property that the edge of $\triangle$ containing the endpoints of the trace of $\tilde{\gamma}_3$ is adjacent to the vertex $\tilde{B}$. By Corollary 4.4, $\tilde{\gamma}_3$ admits a lift to a closed geodesic γ on the surface S if and only if $S = S(p/2, p, p; \frac{1}{2}(p-2))$. This implies, in particular, that $p \equiv 2 \mod 4$ and hence S is not a surface from the list in Lemma 2.4.

Since $\ell = 2$ the discussion in the proof of Lemma 4.5 shows that the trace of the orbit $\tilde{\gamma}_3$ is shorter than $\tilde{\gamma}_1$ and the length of γ is smaller than the length of a lift of the A-orbit $\tilde{\gamma}_1$. From Corollary 3.5, we infer that every liftable billiard orbit which is shorter than $\tilde{\gamma}_3$ is necessarily an irreducible B-orbit. This shows case (2) in the statement of our lemma. □

Example. The surface $S(9, 18, 18; 14)$ of type $(9, 18, 18)$ and genus $g = 8$ admits systoles of the type described under (ii) above. It has, moreover, the interesting property that there is a second set of 18 systoles whose free homotopy classes are given by the side identifications of the fundamental polygon Ω which identify the sides 1 and 28.

5. LENGTH ESTIMATES FOR SYSTOLES

In this section, we give a precise combinatorial description of the billiard orbits which lift to systoles on an elementary triangle surface S. From this, we deduce Theorem 1.1

We showed in Section 3 that a systole on S is either the lift of an irreducible B-orbit or the lift of the double of the A-orbit from Lemma 3.1. Our strategy is to give explicit length estimates for irreducible B-orbits and use a comparison argument.

Recall from Section 4 the definition of homotopy for irreducible arcs in a hyperbolic triangle $\triangle$. Two homotopic arcs correspond to freely homotopic curves on the thrice punctured sphere $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\}$, and therefore we can also speak of homotopic irreducible arcs if these arcs are contained in two nonisometric triangles. More precisely, given two hyperbolic triangles $\triangle, \triangle'$ with (positive) angles α, β, γ and α', β', γ', there is a natural homeomorphism of $\triangle$ onto $\triangle'$ which maps the vertex with angle α, β, γ to the vertex with angle α', β', γ'. We call two billiard orbits $\tilde{\gamma}, \tilde{\gamma}'$ in $\triangle$ and $\triangle'$ *homotopic* if this homeomorphism maps the trace of $\tilde{\gamma}$ to an arc which is homotopic to the trace of $\tilde{\gamma}'$.

Lemma 5.1. *Let $\triangle, \triangle'$ be triangles with angles α, β, γ and $\alpha' \leq \alpha, \beta' \leq \beta, \gamma' \leq \gamma$. Let $\tilde{\gamma}, \tilde{\gamma}'$ be two homotopic B-orbits in $\triangle, \triangle'$. Then the length of γ is not bigger than the length of γ'.*

Proof: The trace of a B-orbit in $\triangle$ is the shortest curve in $\triangle$ in its homotopy class. Let γ_0 be the trace of such a B-orbit in a hyperbolic triangle T_0 with arbitrary angles $\alpha_0, \beta_0, \gamma_0$. If T is a hyperbolic triangle with angles $\alpha \leq \alpha_0, \beta \leq \beta_0, \gamma \leq \gamma_0$ then there is a B-orbit in T whose trace is homotopic to γ_0. We claim that the length of γ is longer than the length of γ_0, with equality only if $T = T_0$.

It is sufficient to consider a triangle T with angle $\alpha < \alpha_0$ close to α and angles $\beta = \beta_0, \gamma = \gamma_0$. There is an isometric embedding of T_0 into T which maps the two sides of T_0 adjacent to β_0 to the two sides of T adjacent to β. If α is sufficiently close to α_0, then every geodesic segment of the B-orbit γ in T intersects T_0. This means that the components of $\gamma - T_0$ are either single arcs containing an endpoint of γ or they consist of two geodesic segments with endpoints on the same side of T_0 which meet on a side of T. Remove the components containing an endpoint of γ and replace every remaining component by a single geodesic arc contained in the boundary of T_0. The resulting curve $\tilde{\gamma}$ is shorter than γ and homotopic to γ_0 as a curve in T_0. Since γ_0 is the shortest curve in its homotopy class, we conclude that γ is longer than γ_0. □

Recall from Section 3 that we divided the family of irreducible B-orbits into two subfamilies, the B_0-orbits and B_1-orbits. We use Lemma 5.1 to describe all B_0-orbits in $\triangle$ which admit a lift to a systole on some elementary triangle surface S.

Lemma 5.2. *Let $\tilde{\eta}$ be a B_0-orbit in $\triangle$ which lifts to a systole on S. Then either $\tilde{\eta}$ is a side pairing orbit, or $\tilde{\eta}$ and S are as in Lemma 4.6.*

Proof: We show in 3 steps that every liftable B_0-orbit whose trace consists of at least 5 segments and is not a side pairing orbit is longer than the double of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1 and therefore does not lift to a systole. With the notation from Lemma 4.2, such an orbit satisfies $\min\{r_1, r_2\} \geq 2$ and $r_1 + r_2 \geq 5$. The case $\ell = 1$ is contained in [Hamenstädt 02], so we may assume that $\ell \geq 2$.

Step 1: Let $\tilde{\gamma}_5$ be an orbit with $r_1 + r_2 = 5$. Without loss of generality, we may assume that $r_1 = 2$ and $r_2 = 3$. Note that by Lemma 4.1, the angle of $\triangle$ at the vertex G adjacent to e and f is at most $\pi/5$, and the angle at the vertex F adjacent to g and e is at most $\pi/7$.

Tables 2 and 3 (Appendix B) contain a list of lengths of the trace of $\tilde{\gamma}_5$ in triangles with angles of the form $\ell\pi/p, s\pi/p, \pi/p$ which we computed with our computer program. From Lemma 5.1, Lemma 3.1, and this list, we see that for $p/\ell \geq 9$ the length of the trace of $\tilde{\gamma}_5$ is longer than the length of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1.

Now let $s = 1, \ell \geq 2$. For $e = b$, the trace of the orbit $\tilde{\gamma}_5$ contains a subcurve homotopic to the trace of the orbit $\tilde{\gamma}_2$ from Lemma 4.5 and therefore, $\tilde{\gamma}_5$ is longer than $\tilde{\gamma}_1$.

By Lemma 4.2, for $s = 1$ and $e \neq b$ the orbit $\tilde{\gamma}_5$ is liftable to S only in the following cases.

(1) $g = b$, $p/\ell \geq 5$ and $\ell = 3$.

(2) $f = b$, $p/\ell \geq 7$ and $\ell = 2$.

However, as we can see from Tables 2 and 3, in these cases the trace of the orbit is longer than the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1.

For $s \geq 2$, the orbit $\tilde{\gamma}_5$ is liftable only if $\ell \leq 5$. Table 1 (Appendix A) contains a complete list of all surfaces $S(p/\ell, p/s, p; k)$ with $\ell \leq 5, p/\ell \leq 8$ and $s \geq 2$. The orbit $\tilde{\gamma}_5$ exists and is liftable only for the surfaces $S(3, 5, 15; 10)$ and $S(4, 10, 20; 15)$ in this list. From the table in Appendix B, for these two cases the trace of $\tilde{\gamma}_5$ is longer than the A-orbit $\tilde{\gamma}_1$.

Step 2: Let $\tilde{\gamma}_6$ be the orbit whose trace consists of 6 segments and corresponds to the case $r_1 = 3 = r_2$. Such an orbit can only exist if the angles at the vertices G and F are at most $\pi/7$. In the case $s = 1$, we obtain as in Step 1 above from Lemma 4.5 and the list in Appendix B, that the trace of $\tilde{\gamma}_6$ is longer than the A-orbit $\tilde{\gamma}_1$. If $s \geq 2$, then Lemma 5.1 and the list in Appendix B imply that this orbit is longer than a systole provided that $p/s \geq 8$. Table 1 (Appendix A) contains a complete list of all surfaces with $s \geq 2$, $p/s \leq 7$. The orbit is only liftable for the surface $S(4, 10, 20; 15)$, and in this case it is longer than the double of the A-orbit $\tilde{\gamma}_1$.

Step 3: Let $\tilde{\gamma}_7$ be a B_0-orbit with $\min\{r_1, r_2\} = 2$ and $r_1 + r_2 = 6$. Assume without loss of generality that $r_1 = 2$ and $r_2 = 4$. The angle at the vertex G adjacent to the edges f and e is at most $\pi/5$, the angle at the vertex

F adjacent to e and g is at most $\pi/9$, and the angle at the vertex E adjacent to f and g is at most $\pi/3$. Since the trace of $\tilde{\gamma}_7$ contains a subarc which is homotopic to the trace of $\tilde{\gamma}_5$, we conclude that the trace of $\tilde{\gamma}_7$ is longer than the A-orbit $\tilde{\gamma}_1$ provided that either $p/\ell \geq 9$ or $s = 1$ and $g \neq b$.

In the case $s = 1, g = b$, we again use numerical computations given in Appendix B to show that the trace of $\tilde{\gamma}_7$ is longer than $\tilde{\gamma}_1$.

Now let $s \geq 2$. From the list in Appendix A and Lemma 5.1 the orbit $\tilde{\gamma}_7$ is longer than a systole whenever $p/s \geq 11$. Appendix A contains a list of all surfaces with $p/s \leq 10$ and $s \geq 2$. Comparison with the list in Appendix B shows that for each of these surfaces, the trace of the orbit is longer than the A-orbit $\tilde{\gamma}_1$.

As a consequence, a liftable B_0-orbit which is not a side pairing orbit and whose trace either consists of 5 or 6 segments or contains a subarc homotopic to the trace of $\tilde{\gamma}_7$ is longer than a systole. The lemma now follows from the fact that the trace of every liftable B_0-orbit which is not a side pairing orbit and which has at least 7 segments contains a subarc homotopic to the trace of $\tilde{\gamma}_7$. □

With a similar argument we can show that B_1-orbits never lift to systoles on elementary triangle surfaces. Again we begin our analysis with the question of liftability.

Lemma 5.3. *Let $\tilde{\eta}$ be a B_1-orbit in $\triangle$. Denote by $\tilde{\eta}_1\tilde{\beta}\tilde{\eta}_2$ the trace of $\tilde{\eta}$ where the subarc $\tilde{\eta}_1$ consists of $r_1 \geq 1$ segments connecting the edge e to the edge f of $\triangle$, the subarc $\tilde{\beta}$ contains a unique segment joining f to the third edge g, and $\tilde{\eta}_2$ consists of $r_2 \geq 1$ segments joining e to g. Then $\tilde{\eta}$ admits a lift to a closed geodesic on S if and only if $S = S(p/\ell, p/s, p; k)$ and if one of the following possibilities is satisfied.*

	e	f	g	
(i)	a	b	c	$k(r_2+1) - r_1 - 1 \equiv 0 \ mod\ p$
(ii)	a	c	b	$k(r_1+1) - r_2 - 1 \equiv 0 \ mod\ p$
(iii)	b	a	c	$k(r_2+1) + r_2 - r_1 \equiv 0 \ mod\ p$
(iv)	b	c	a	$k(r_1+1) + r_2 - r_1 \equiv 0 \ mod\ p$
(v)	c	a	b	$k(r_2 - r_1) + r_1 + 1 \equiv 0 \ mod\ p$
(vi)	c	b	a	$k(r_1 - r_2) + r_2 + 1 \equiv 0 \ mod\ p$

Proof: As before, we show the lemma only for the case (ii) above. Let $\tilde{\alpha}_2$ be the subarc of the trace of $\tilde{\eta}$ which contains $\tilde{\beta}$ and $\tilde{\eta}_2$. Then there is a lift of $\tilde{\eta}$ to a fundamental polygon Ω of type $(p/\ell, p/s, p)$ which consists of $r_1 + 1$ arcs $\alpha_2, \alpha_{1,1}, \dots, \alpha_{1,r_1}$ as follows: The arc α_2 connects the edge 1 to the edges $2r_2 + 2$, and the arc α_{1j} connects the edge $2r_2 - 2jk + 1$ to the edge $2r_2 + 2 - 2jk$. Then $\tilde{\eta}$ lifts to a closed geodesic on S if and only if $2r_2 + 2 - 2r_1k - 2k + 1 \equiv 1 \bmod p$, or equivalently, if $k(r_1 + 1) - r_2 - 1 \equiv 0 \bmod p$. □

Corollary 5.4. *A B_1-orbit whose trace has subarcs $\tilde{\eta}_1, \tilde{\eta}_2$ as in Lemma 5.3 with the same number of segments (i.e., for which $r_1 = r_2$) is not liftable.*

Proof: Let $\tilde{\eta}$ be a B_1-orbit whose trace consists of $2r + 1$ segments and which corresponds to the case $r_1 = r_2 = r$ in Lemma 5.3. If $S = S(p/\ell, p/s, p; k)$, then k does not divide p and therefore only the case $rk \equiv r \bmod p$ is possible by Lemma 5.3. But then $r(k-1) \equiv 0 \bmod p$ which is impossible since $k - 1$ can not divide p. □

Lemma 5.5. *A B_1-orbit does not lift to a systole on S.*

Proof: We proceed exactly as in the proof of Lemma 5.3. As before, we only have to treat the case $\ell \geq 2$ [Hamenstädt 02].

We found that every liftable B_0-orbit which is different from a side pairing orbit is longer than the double of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1. This is not true for B_1-orbits. We find surfaces $S(p/\ell, p/s, p; k)$ for $s \geq 2$ admitting a liftable B_1-orbit which is shorter than $\tilde{\gamma}_1$. However, a systole for such a surface is a lift of the unique side pairing orbit. We describe these surfaces:

Step 1: Let $\tilde{\eta}_4$ be a B_1-orbit whose trace consists of 4 segments and which is liftable to $S = S(p/\ell, p/s, p; k)$. Without loss of generality, and using the notations from Lemma 5.3, we may assume that $r_1 = 1$ and $r_2 = 2$. Since both k and p are multiples of $\ell \geq 2$, by Lemma 5.3 one of the following cases is satisfied.

(1) $f = b$, $3k \equiv 2 \bmod p$ and $s = 1$, $\ell = 2$.

(2) $g = b$, $2(k-1) \equiv 1 \bmod p$ and $s = 1$, $\ell = 3$.

(3) $g = b$, $k \equiv -2 \bmod p$ and $s = 1$, $\ell = 2$.

(4) $f = b$, $k \equiv 3 \bmod p$ and $\ell = 3$, $s = 2$ or $s = 1$.

Assume first that $s \geq 2$. Then necessarily case (4) holds, and therefore we have $f = b$, $c = e$, $g = a$ and, in particular, $p/\ell \geq 6$. The surface of this form with the largest angles is the surface $S(6, 9, 18; 3)$. The trace of the orbit $\tilde{\eta}_4$ is shorter than the A-orbit $\tilde{\gamma}_1$. However, explicit computation with our computer program shows that the systoles of $S(6, 9, 18; 3)$ are the lifts of the unique side pairing orbit.

On the other hand, by Lemma 5.1 and Appendix B the trace of the orbit $\tilde{\eta}_4$ is longer than the upper bound

3arccosh $\frac{3}{2}$ for the length of the A-orbit $\tilde{\gamma}_1$ whenever $p/\ell \geq 11$. All surfaces with $s \geq 2$ and $p/\ell \leq 10$ are listed in Appendix A. In addition to the surface $S(6,9,18;3)$, there are only two more examples in this list for which the orbit $\tilde{\eta}_4$ is liftable, namely the surfaces $S(8,12,24;3)$ and $S(10,20,30;3)$. Appendix B shows that for these two surfaces the trace of the orbit $\tilde{\eta}_4$ is longer than the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1.

Now let $s = 1$ and let $S = S(p/\ell, p, p; k)$ be such that $\tilde{\eta}_4$ is liftable to S. Then one of the possibilities (1)–(4) above is satisfied.

In case (1), we have $3(k-1) \equiv -1 \mod p$, and hence by Lemma 2.4 the surface has a side pairing orbit whose trace consists of 2 segments. This orbit is shorter than $\tilde{\eta}_4$. Similarly, in case (3), there is a side pairing orbit whose trace consists of 2 segments and which is shorter than $\tilde{\eta}_4$.

In case (2), we have $g = b$, $\ell = 3$ and $2(k-1) \equiv 1 \mod p$, and thus by Lemma 2.4 the surface is isometric to a surface with a side pairing orbit whose trace consists of 3 segments. The trace of this side pairing orbit is homotopic to the trace of the orbit which we obtain from $\tilde{\eta}$ by replacing the first and the second segment by a single arc. In particular, the side pairing orbit is shorter than $\tilde{\eta}_4$.

In case (4), the orbit exists for $p/\ell \geq 6$ and $p \geq 18$. Since it is liftable only if $k \equiv 3 \mod p$, we conclude that p is odd; in particular we may assume that $p/\ell \geq 7$ and $p \geq 21$. From Lemma 5.1 and the list in Appendix B, we conclude that the orbit is longer than the double of the A-orbit $\tilde{\gamma}_1$.

Step 2: Let $\tilde{\eta}_5$ be a B_1-orbit with $r_1 = 1, r_2 = 3$. Such an orbit exist only if the angle at the vertex F is at most $\pi/8$. By Lemma 5.3, if $\tilde{\eta}_5$ is liftable then one of the following cases is satisfied.

(1) $f = b$ and $4k \equiv 2 \mod p$ or $2k \equiv 4 \mod p$.

(2) $f = a$ and $4k \equiv -2 \mod p$ or $2k \equiv -2 \mod p$.

(3) $f = c$ and $2k \equiv 4 \mod p$ or $2k \equiv -2 \mod p$.

Appendix B shows that for $p/\ell \geq 9$ the orbit is longer than an upper bound for the length of the double of the A-orbit $\tilde{\gamma}_1$.

We again treat first the case $s \geq 2$ which is only possible if $\ell = 4, s = 3$ and $2k \equiv 4 \mod p$. The list in the appendix shows that there is no such surface with $p/\ell \leq 8$ for which $\tilde{\eta}_5$ is liftable.

Now for $s = 1$, we observe that the orbit $\tilde{\eta}_5$ is longer than the B_0-orbit whose trace consists of 4 segments and is obtained from $\tilde{\eta}_5$ by replacing the arc $\tilde{\eta}_1\tilde{\beta}$ by a single segment. Thus by Lemma 4.5, for $e = b$ the orbit is longer than the double of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1.

If $e \neq b$, then necessarily $p/\ell \geq 4$ and $\ell = 4$ or $\ell = 2$. We first consider the case $\ell = 4$ which is only possible if $2k \equiv 4 \mod p$, and therefore if p/ℓ is odd. Appendix B shows that the trace of the orbit is longer than $\tilde{\gamma}_1$. For $\ell = 2$, the triangle with the biggest possible angle corresponds to a surface of type $(4,8,8)$. If $\tilde{\eta}$ is liftable, then there is a side pairing orbit whose trace consists of 2 segments and is shorter than $\tilde{\eta}_5$. The remaining surfaces are treated in the tables in Appendix B.

Step 3: Consider now the B_1-orbit $\tilde{\eta}_6$ whose trace has 6 segments and which corresponds to the case $r_1 = 1, r_2 = 4$. This orbit can only exist if the angle at the vertex F is at most $\pi/10$. The tables in Appendix B show that the trace of $\tilde{\eta}_6$ is longer than the upper bound 3arccosh $\frac{3}{2}$ for the length of the A-orbit $\tilde{\gamma}_1$ provided that $p/\ell \geq 5$ and $p/s \geq 10$ or $p/s \geq 13$.

As before, if $s = 1$ and $e = b$, then necessarily the trace of $\tilde{\eta}_6$ is longer than $\tilde{\gamma}_1$. The remaining cases are listed in Appendix B. On the other hand, Appendix A contains a complete list of all surfaces of type $(p/\ell, p/s, p)$ with $s \geq 2$, $p/s \leq 10$ or $p/\ell \leq 4$ and $p/s \leq 12$. Appendix B shows that for each of these, the trace of the orbit $\tilde{\eta}_6$ is longer than the orbit $\tilde{\gamma}_1$. In particular, the trace of no B_1-orbit which lifts to a systole can have a subarc which is homotopic to the trace of $\tilde{\eta}_6$.

Step 4: Consider the B_1-orbit $\tilde{\eta}_7$ whose trace has 6 segments given by $r_1 = 2, r_2 = 3$. This orbit can only exist if the angle at the vertex G is at most $\pi/6$ and the angle at the vertex F is at most $\pi/10$. The trace of the orbit contains a subarc homotopic to the trace of $\tilde{\eta}_5$. From Lemma 4.5, the argument in Step 2 above, and Appendix B, we conclude that for $s = 1$, $\tilde{\eta}_7$ is longer than the double of the A-orbit $\tilde{\gamma}_1$.

The case $s \geq 2$ follows as before from the list in Appendix A and a length comparison, using Appendix B. As a conclusion, the orbit $\tilde{\eta}_7$ is always longer than the double of the A-orbit $\tilde{\gamma}_1$. In particular, the trace of no B_1-orbit which lifts to a systole can have a subarc homotopic to the trace of $\tilde{\eta}_7$. The lemma follows as in the proof of Lemma 5.2. □

We can now summarize our considerations as follows:

Theorem 5.6. *Let S be an elementary triangle surface of type $(p/\ell, p/s, p)$ for some $\ell \geq 2, 1 \leq s < \ell$. Then* *every systole γ of S satisfies one of the following three possibilities.*

(i) γ is the lift of the double of the A-orbit $\tilde{\gamma}_1$ in $\triangle$ of period 3.

(ii) γ is the lift of a side pairing orbit.

(iii) $\ell = 2$ and γ and S are as in Lemma 4.6.

In Section 1, we called a closed hyperbolic surface S *maximal* if the length of its systole is a local maximum in Teichmüller space. A surface S can only be maximal if the lengths of its systoles locally parameterize Teichmüller space near S. Since the dimension of the Teichmüller space of closed surfaces of genus $g \geq 2$ equals $6g - 6$, this implies that a maximal surface has at least $6g - 5$ systoles (see [Schmutz 93]). This observation together with Theorem 5.6 implies that among our elementary triangle surfaces of type $(p/\ell, p, p)$ for some $\ell \geq 2$, there is at most one maximal example.

Proposition 5.7. *An elementary triangle surface of type $(p/\ell, p, p)$ for some $\ell \geq 2$ which is different from $S(6, 24, 24; 20)$ is not maximal.*

Proof: Let S be an elementary triangle surface of type $(p/\ell, p, p)$ for some $p \geq 6$ and divisors $\ell \geq 2$, $s \geq 1$ of p. Denote by ℓ_1 the length of the A-orbit $\tilde{\gamma}_1$, and let ℓ_0 be the minimal length of the trace of a side pairing orbit for S. We first analyse the surfaces listed in part (ii) of Lemma 2.4.

Case 1: Surfaces with additional symmetries. In Lemma 2.4 we gave a combinatorial description of all surfaces which admit a cyclic group Σ of order 2 of isometries normalizing the basic group Γ. Among them, we find the surfaces $S = S(h, 2h, 2h; 2)$ for some $h \geq 3$ as in the example in Section 2. They correspond to the case $m = 1$ in Lemma 2.4 and have a side pairing orbit $\tilde{\gamma}_0$ whose trace consists of exactly two segments. The trace of this side pairing orbit is shorter than the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1. The orbit $\tilde{\gamma}_0$ admits $2h$ different lifts to a closed geodesic on S, and this set of geodesics is invariant under the action of Σ on S. By Theorem 5.6, these lifts are exactly the systoles of our surface $S(h, 2h, 2h; 2)$. In other words, the surface $S(h, 2h, 2h; 2)$ has $2h$ systoles and is not maximal.

Recall from Lemma 2.4 that every surface S with a group Σ of symmetries as in Lemma 2.4 which is different from one of the surfaces $S(h, 2h, 2h; 2)$ is determined by a number $\ell \geq 2$, a number $m \geq 1$, and a divisor $q > m$ of $m\ell - 2$. The side pairings for S identify the edge 1 with the edge $2m\ell$. By Theorem 5.6, the length of the systole of S equals $\min\{2\ell_0, 2\ell_1\}$.

We claim that there are only finitely many among those surfaces S with additional symmetries for which $\ell_0 \leq \ell_1$ and which do not belong to the family $\{S(h, 2h, 2h; 2) \mid h \geq 3\}$. We write $p = \ell q$, and we estimate the length ℓ_0 for $S = S(q, p, p; m\ell)$ as follows: Let Ω be a fundamental polygon of type (q, p, p) with center 0, and let γ be a geodesic arc of minimal length in Ω which connects the edges 1 and $2m\ell$ of Ω. Let $T \subset \Omega$ be the hyperbolic triangle with one vertex at 0, one vertex C_1 at the intersection of γ with the edge 1, and one vertex C_2 at the orthogonal projection of the center 0 onto γ. Then T has a right angle at C_2 and the length of the side of T connecting C_1 to C_2 equals $\ell_0/2$. The angle α of T at the vertex 0 is contained in $((m\ell - 1)\pi/p, m\ell\pi/p)$.

Let $\triangle$ be the hyperbolic triangle with angles $\ell\pi/p, \pi/p, \pi/p$. The length ℓ_2 of the side of T connecting 0 to C_1 is not smaller than the distance h of a vertex $\tilde{A}$ of $\triangle$ with angle π/p to the opposite edge. From hyperbolic trigonometry, we obtain: $\sinh \frac{\ell_0}{2} \geq \sinh h \cdot \sin \alpha$.

To compute h, let ℓ_3 be the length of the side of $\triangle$ adjacent to the vertex $\tilde{B}$ with angle $\ell\pi/p$. Then $\sinh h = \sinh \ell_3 \cdot \sin \ell\pi/p$ and

$$\cosh \ell_3 = \frac{\cos \ell\pi/2p \cdot \cos \pi/p}{\sin \ell\pi/2p \cdot \sin \pi/p}.$$

For $t > 0$, we have $\sinh t \geq \cosh t - 1$ and therefore

$$\sinh \ell_0/2 \geq \Big(\frac{\cos \ell\pi/p \cdot \cos \pi/p}{\sin \ell\pi/p \cdot \sin \pi/p} - 1\Big) \sin \ell\pi/p \cdot \sin \alpha.$$

Recall that $\alpha \in ((m\ell - 1)\pi/p, m\ell\pi/p)$ where $m \leq q - 1$ and q divides $m\ell - 2$. In particular, we have $\pi/q \leq \alpha \leq \pi - \pi/q$ and $\sin \alpha$ is bounded from below by $\sin \pi/q$. Together with the above inequality, this implies that there is a number $r_0 > 0$ such that $\ell_0 > \ell_1$ for every surface $S = S(p/\ell, p, p; m\ell)$ as above with $\ell \geq r_0$. On the other hand, for every fixed $\ell < r_0$ and every fixed $r_1 > 0$ we obtain from the fact that q divides $m\ell - 2$ that there are only finitely many such surfaces with $m \leq r_1$. Once again using the above inequality, this shows that indeed among the surfaces $S = S(p/\ell, p, p; m\ell)$ with additional symmetries, there are only finitely many for which $\ell_0 \leq \ell_1$.

We computed the systoles for these surfaces with our computer program and found that the surface $S(6, 24, 24; 20)$ is the only one among the surfaces in Lemma 2.4 for which equality $\ell_0 = \ell_1$ holds.

As a consequence of this observation, Lemma 4.6, and Theorem 5.6, the surfaces $S(4h, 8h, 8h; 4h + 2)$ $(h \geq 1)$

have exactly $2p = 16h = 4g + 4$ systoles of length $\min\{2\ell_0, 2\ell_1\}$. In other words, among these surfaces only the surface $S(4, 8, 8; 6)$ has more than $6g - 5$ systoles. However, if a surface S is maximal, then the lengths of its systoles locally parameterize Teichmüller space near S. This is not the case for the surface $S(4, 8, 8; 6)$ (this was stated by Schmutz in [Schmutz 93], but it also follows from the arguments in [Hamenstädt 02]). Thus none of the surfaces $S(4h, 8h, 8h; 4h+2)$ is maximal. Thus, there is no maximal surface with additional symmetries of type $(p/2, p, p)$ for some $p \geq 6$.

For $\ell \geq 3$ and $(\ell, p/\ell) \neq (4, 6)$, the surfaces $S(p/\ell, p, p; m\ell)$ with additional symmetries either have $p = 2g + \ell - 2$ systoles (if $\ell_1 < \ell_0$) or $2p = 4g + 2\ell - 2$ systoles (if $\ell_0 < \ell_1$) and are not maximal.

Case 2: Surfaces without additional symmetries. Let S be a surface of type $(p/2, p, p)$ which is different from one of the surfaces $\tilde{S}(h, 2h, 2h; 2)$. If the B_0-orbit $\tilde{\gamma}_3$ whose trace has 5 intersections with the boundary of $\triangle$ as in (2) of Lemma 4.6 is liftable, then the lift of $\tilde{\gamma}_3$ to S is a systole provided that there is no side pairing orbit for S which is shorter than $\tilde{\gamma}_3$. In particular, every systole is either a lift of a side pairing orbit of minimal length or a lift of $\tilde{\gamma}_3$, and there are at most $4g + 4$ systoles (where again g is the genus of S). As before, we conclude that S is not maximal.

In the case that $\tilde{\gamma}_3$ is not liftable or for surfaces S of type $(p/\ell, p, p)$ for some $\ell \geq 3$, the length of the systoles of S equals $\min\{2\ell_0, 2\ell_1\}$ and we conclude, as before, that there are at most $4g + 4$ systoles and that S is not maximal. Namely, the set of $2p$ curves which are lifts of the A-orbit γ_1 and of one side pairing orbit for $\triangle$ is not parameterizing for Teichmüller space. (We can also argue that we have $4g+4 \geq 6g-5$ only if $g \leq 4$, and the surfaces of small genus can be checked explicitly.) □

By Proposition 5.7, among the elementary triangle surfaces of type $(p/\ell, p, p)$ for some $\ell \geq 2$, only the surface $S = S(6, 24, 24; 20)$ of genus $g = 10$ with 72 systoles can be maximal. To finish the proof of the Theorem 1.1, we have to show that the lengths of the systoles of S are locally parameterizing for the Teichmüller space $\mathcal{T}_{10}$ near S [Schmutz 93].

Let Ω be a fundamental polygon of type $(6, 24, 24)$ for S which consists of 48 triangles with angles $\frac{\pi}{6}, \frac{\pi}{24}, \frac{\pi}{24}$. The side pairings for Ω which define S identify the edges $2i + 1$ and $2i + 40$. The triangulation of Ω into these 48 triangles induces a triangulation of S with 6 vertices. The cyclic group of order 24 of rotations of Ω about the center with rotation angle $\pi/12$ descends to a group Γ of automorphisms of S which leaves two of the six vertices of the canonical triangulation fixed and permutes the remaining 4 vertices. The surface S admits, in addition, a group Σ of order 2 of isometries which normalizes the group Γ and exchanges the two fixed points of Γ.

The surface S has 72 systoles $\gamma_1, \ldots, \gamma_{72}$ which can be divided into 3 orbits $\mathcal{O}_i$ $(i = 0, 1, 2)$ under the action of Γ. The orbit $\mathcal{O}_0$ consists of the 24 lifts of the double of the A-orbit from Lemma 3.1 and is Σ-invariant. The orbits $\mathcal{O}_1, \mathcal{O}_2$ consist of lifts of side pairing orbits and are exchanged by the generator $\mathcal{J}$ of the group Σ.

Let ℓ_i be the function on the Teichmüller space $\mathcal{T}_{10}$ which assigns to a surface M the length of the geodesic γ_i on M. The differential of ℓ_i is dual with respect to the *Weil Petersen Kähler form* to the tangent X_i of the *earthquake path* along the geodesic γ_i [Wolpert 82]. Thus, the lengths of the geodesics γ_i locally parameterize $\mathcal{T}_{10}$ near S if and only if the tangent space T_S of $\mathcal{T}_{10}$ at S is spanned by the vectors $X_1, \ldots, X_{72}$, and this is the case if and only if the rank of the $(72, 72)$-matrix $A = (a_{i,j})$ with entries $a_{i,j} = d\ell_i(X_j)$ equals 54.

Following Wolpert [Wolpert 82], the entries $a_{i,j}$ of the matrix A can be computed as follows: For $i \neq j$, the geodesics γ_i and γ_j intersect transversely in at most one point, with oriented intersection angle $\theta_{i,j}$ measured from γ_j to γ_i. The differential $X_j(\ell_i)$ of the length ℓ_i of γ_i in the direction of the earthquake path along γ_j equals $a_{i,j} = \cos\theta_{i,j}$ [Wolpert 82].

Using the symmetries of our surface S, the matrix A can easily be determined explicitly with our computer program. The computation shows that the rank of A equals indeed 54.

We can simplify our computation by using the symmetries of the surface S in a more essential way. Namely, let ξ be a generator of the cyclic group Γ. It acts as an isometry (with respect to the Weil Petersen metric) of order 24 on T_S, and therefore its eigenvalues are the 24th roots of unity. For $0 \leq j \leq 12$, we denote by Z_j the generalized eigenspace which corresponds to the complex conjugate pair of eigenvalues $e^{2\pi ij/24}$, $e^{-2\pi ij/24}$.

The isometry $\xi^{12} = \zeta$ is an involution of S. Its fixed points are precisely the 6 vertices of the canonical triangulation. By the Riemann Hurwitz formula, the quotient S/ζ is a surface of genus 4.

For each $g \geq 2$, the tangent space of the Teichmüller space $\mathcal{T}_g$ of Riemann surfaces of genus g at a surface M can be identified naturally with the space of holomorphic quadratic differentials on M. The isometry ζ is a biholomorphic automorphism of the Riemann surface S, and hence it acts on the space of holomorphic quadratic differentials. Each such differential which is preserved under the action of ζ descends to a meromorphic quadratic differential on the surface S/ζ with at most simple poles at the projections of the 6 fixed points of ζ.

On the other hand, every meromorphic quadratic differential on S with at most simple poles at these six points lifts to a ζ-invariant holomorphic quadratic differential on S. Thus, by the Riemann Roch theorem, the (real) dimension of the subspace $\oplus_{j=0}^{6} Z_{2j}$ of T_S of ζ-invariant quadratic differentials on S is 30 and consequently, the dimension of $\oplus_{j=0}^{5} Z_{2j+1}$ equals 24.

With the same argument, we conclude that the eigenspace of $\xi^8, \xi^6, \xi^4, \xi^3, \xi^2, \xi$ with respect to the eigenvalue 1 is of dimension $18, 12, 6, 6, 2, 0$. Thus, we can determine the dimensions of some of the spaces Z_i.

On the other hand, our argument is valid for every choice of a generator for the cyclic group Γ, and therefore the dimension of a generalized eigenspace with respect to a pair of complex conjugate eigenvalues $e^{2\pi i j/24}, e^{-2\pi i j/24}$ only depends on the order of the eigenvalue, i.e., on the smallest number $p > 0$ such that $pj \equiv 0 \bmod 24$. We can thus obtain a complete list of the dimensions of all the generalized eigenspaces.

Order of eigenvalue	1	2	3	4	6	8	12	24
Dimension	0	2	6	4	6	6	4	4

Note that there are two pairs of complex conjugate eigenvalues of order 8 and of order 12, and there are four pairs of order 24. Thus, the table gives a complete description of the representation of Γ on T_S. Moreover, the generalized eigenspaces are orthogonal with respect to the Weil Petersen Kähler form.

The generator $\mathcal{J}$ of the cyclic group Σ acts as an involution on S and normalizes Γ. The proof of Lemma 2.4 shows that for a suitable choice of the generator ξ of Γ, we have $\mathcal{J} \circ \xi \circ \mathcal{J} = \xi^{19}$. In particular, $\mathcal{J}$ permutes the generalized eigenspaces of ξ with respect to eigenvalues of order $24, 12, 8$ and preserves the remaining ones.

On the other hand, there are 6 fixed points for the action of $\mathcal{J}$ on S and hence, the eigenspace of $\mathcal{J}$ with respect to the eigenvalue 1 is of dimension 30. This determines the representation of the semidirect product G of Γ and Σ on T_S. Namely, the eigenspace of $\mathcal{J}$ with respect to the eigenvalue -1 intersects the sum of the generalized eigenspaces for Γ of order $24, 12, 8, 6, 4, 3, 2$ in a subspace of dimension $8, 4, 6, 2, 2, 2, 0$. We then can use this knowledge on the representation of G on T_S to determine the rank of our matrix A. Namely, let H_i $(i = 0, 1, 2)$ be the subspace of T_S which is spanned by the tangents of the earthquake paths along the geodesics γ_{24i+j} $(1 \leq j \leq 24)$. The spaces H_i are invariant under Γ, and $H = \sum_i H_i$ is the span of the tangents of the earthquake paths along all the geodesics γ_i. We claim that H contains the generalized eigenspaces for the eigenvalues of order 12 and 24.

Consider our matrix A. For a suitable numbering of the geodesics γ_i, the geodesic γ_1 intersects the geodesics $\gamma_{25}, \gamma_{29}, \gamma_{30}, \gamma_{34}$ transversely in a single point, and it intersects no other of the geodesics γ_{24+j} $(1 \leq j \leq 24)$. By invariance of the intersection angles under the symmetries of our polygon Ω, there are numbers $1 > c > b > 0$ such that $a_{j,24+j} = -b, a_{j,28+j} = -c$, $a_{j,29+j} = c$ and $a_{j,33+j} = b$ for some $c > b > 0$ (indices are taken modulo 24). Let X be a complex eigenvector for the action of Γ on H_0 with respect to a 24th root of unity α. There is a complex number $\kappa \neq 0$ such that $X = \kappa \sum_{i=1}^{24} \alpha^{i-1} X_i$. Thus, if $X = 0$ in H_0, then $d\ell_{25}(X) = 0$ and hence by the above we have $\frac{b}{c}(\alpha^9 - 1) = \alpha^4 - \alpha^5$. Since $\frac{b}{c} \in (0, 1)$ there is a 24th root of unity $\alpha \neq 1$ solving this equation only if either the order of α is 6 or 8. In particular, the intersection of H_i with a generalized eigenspace Z for an eigenvalue of order 12 or 24 is 2. By our above calculation, the dimension of Z is 4, and therefore from invariance under the action of Γ we conclude that the space Z is contained in H if and only if the intersections $Z \cap H_i$ do not all coincide. By invariance under $\mathcal{J}$, it is therefore enough to show that the image under $\mathcal{J}$ of the intersection $Z \cap H_1$ is *not* contained in H_1.

Since the Weil Petersen Kähler form ω is invariant under the action of $\mathcal{J}$, this can easily be checked as follows: Let α be a root of unity of order 24 and let $X = \sum_{i=1}^{24} \alpha^{i-1} X_{i+24} \in H_1$ be an eigenvector for ξ with respect to the eigenvalue α. If $\mathcal{J}X \in H_1$, then $\mathcal{J}X = \kappa \sum_{i=1}^{24} \alpha^{19(i-1)} X_{i+24}$ for some $\kappa \in \mathbf{C}$ and therefore for each $Z \in H_0$, we have

$$\sum_{i=1}^{24} \alpha^{i-1} \omega(Z, X_{i+24}) = \kappa \sum_{i=1}^{24} \alpha^{19(i-1)} \omega(JZ, X_{i+24}).$$

This however is not the case.

As a summary, the orthogonal complement of H in T_S with respect to the Weil Petersen metric is contained in the sum of the generalized eigenspace for eigenvalues of order at most 8. Thus, if we denote once again by ξ a generator of Γ, then it is enough to show that the dimension of the space $\{\sum_{i=0}^{2} \xi^{8i} X \mid X \in H\}$ equals 18, and the dimension of $\{\sum_{i=0}^{3} \xi^{6i} X \mid X \in H\}$ is 14. In other words, we can verify that the lengths of the systoles of S locally parametrize $\mathcal{T}_{10}$ near S by computing the rank of an explicitly given $(24, 24)$-matrix and an explicitly given $(18, 18)$-matrix. This computation can easily be done numerically. It can also be carried out

type	triangle surfaces
$p/\ell = 3 = s$	$S(3,4,12;4), S(3,5,15;10), S(3,7,21;7), S(3,8,24;16)$, $S(3,10,30;10), S(3,11,33;22)$
$p/\ell = 4 = s$	$S(4,5,20;5), S(4,7,28;21), S(4,9,36;9), S(4,11,44;33)$
$p/\ell = 4, s = 2$	$S(4,6,12;3), S(4,10,20;15)$
$p/\ell = 5 = s$	$S(5,6,30;6), S(5,7,35;21), S(5,8,40;16), S(5,9,45;36)$, $S(5,11,55;11)$
$p/\ell = 6 = s$	$S(6,7,42;7), S(6,11,66;55)$
$p/\ell = 6, s = 3$	$S(6,8,24;4), S(6,10,30;10)$
$p/\ell = 6, s = 2$	$S(6,9,18;3), S(6,9,18;15), S(6,15,30;5)$
$p/\ell = 7 = s$	$S(7,8,56;8)$
$p/\ell = 8 = s$	$S(8,9,72;9)$
$p/\ell = 8, s = 4$	$S(8,10,40;5)$
$p/\ell = 8, s = 2$	$S(8,12,24;3), S(8,12,24;15), S(8,20,40;15), S(8,20,40;35)$

TABLE 1. List of triangle surfaces.

by a further reduction as above using the matrix which determines the restriction of the Weil Petersen Kähler form to the space H_0.

APPENDIX A

See Table 1 for a list of triangle surfaces of type $(p/\ell, p/s, p)$ for $\ell > s > 1$, such that either

1) $p/\ell \leq 8$ and $p/s \leq 10$ or
2) $p/\ell \leq 8$ and $\ell \leq 5$ or
3) $p/\ell \leq 4$ and $p/s \leq 12$.

APPENDIX B

See Table 2 for a classification of lengths of B_0-orbits and Table 3 for a classification of Lengths of B_1-orbits.

APPENDIX C: THE COMPUTER PROGRAM FUNDA

In this appendix, we give a short description of our computer program *funda* which computes the systoles of an elementary triangle surface and their lengths.

Our program uses the combinatorial description of an elementary triangle surface from Section 2 and computes for a given type $(p/\ell, p/s, p)$, the coordinates in the hyperboloid model of the hyperbolic plane of a fundamental polygon Ω of this type.

Side identifications can be defined by simply prescribing the number of the edge with which the edge 1 is identified. If the result of these identifications is a smooth closed surface, then the program computes the lengths of all closed geodesics which have the particular combinatorial type described in Corollary 3.5.

Let Ω be a fundamental polygon of type $(p/\ell, p/s, p)$ for some $p \geq 5$ and some divisors $\ell \geq s$ of p, and let $S = S(p/\ell, p/s, p; k)$ be an elementary triangle surface of type $(p/\ell, p/s, p)$. A lift $\bar{\gamma}$ to Ω of a closed geodesic γ on S consists of $n \geq 1$ geodesic arcs with endpoints on the boundary $\partial\Omega$ of Ω. Since we are only interested in systoles, by the results in Section 3 we may assume that every endpoint of such a segment is contained in the interior of a boundary edge.

Since a closed geodesic γ on S is unique in its free homotopy class, its lift $\bar{\gamma}$ to Ω is uniquely determined by the sequence of edges containing the endpoints of its consecutive segments. For a choice of an orientation and a basepoint for γ denote by k_i, the number of the side of Ω from which the ith segment emanates and by K_i be the side where the ith segment ends. Then, $k_{i+1} = K_i + 2k - 1 \mod 2p$ if K_i is odd, and $k_{i+1} = K_i - 2k + 1 \mod 2p$ if K_i is even, and γ is determined by the string

$$k_1 - K_1, k_2 - K_2, \ldots, k_n - K_n.$$

In Section 3, we obtained some information on the combinatorial type of a systole. To use this for our program, we extend the notion of the canonical triangulation for S to Ω. An edge of this triangulation will be called *inner* if the edge connects the center of Ω to one of the vertices on the boundary.

Recall that the boundary of Ω only contains vertices of two different types, namely odd edges which we call *edges of type 1* and even edges which we call *edges of type 2*. We call an inner edge of the triangulation an *edge of*

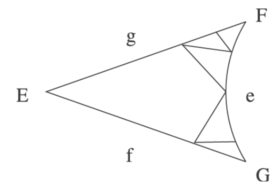
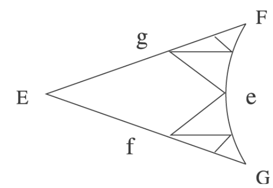
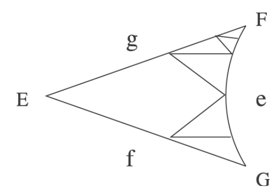
(G, F, E)	length of B_0-Orbit	length of A-Orbit	illustration
$r_1 = 2, r_2 = 3$			
(9,9,9)	2.894		
(5,15,15)	2.811	2.667	
(6,18,18)	2.952		
(14,7,14)	2.808	2.750	
(16,8,16)	2.919		
(5,15,3)	2.504	2.139	
(20,10,4)	2.895	2.546	
$r_1 = r_2 = 3$			
(7,7,12)	2.890		
(8,7,7)	2.940		
$r_1 = 2, r_2 = 4$			
(5,10,10)	2.849	2.613	
(5,11,11)	2.909		
(11,11,3)	2.904		
(7,9,9)	2.929		
(5,15,3)	2.727	2.139	
(7,14,3)	2.900		
(20,10,3)	2.892		
(5,20,4)	2.932		
(36,9,4)	2.952		
(6,12,4)	2.909		
(5,30,6)	3.071		
(5,9,45)	2.799	2.648	
(45,9,5)	3,014		
(6,9,15)	2.890		

TABLE 2. Lengths of B_0-orbits.

type i $(i = 1, 2)$ if it connects the center to a vertex of type i. We now divide the geodesic segments in Ω with endpoints in interior points of the boundary edges into mutually disjoint subsets as follows.

(1) A *segment of type* i $(i = 1, 2)$ connects two adjacent edges of $\partial\Omega$ and intersects an inner edge of type i of the canonical triangulation.

(2) A segment which does not pass through the center and which intersects more than one inner edge is a *segment of type* ii $(i = 1, 2)$ if the first and the last inner edge intersected is of type i $(i = 1, 2)$.

(3) A segment which is not of one of the types above will be called a *segment of type* 12.

Using this terminology, we obtain from Corollary 3.5 that the lift $\bar{\gamma}$ to Ω of a systole on S satisfies one of the following possibilities.

(I) $\bar{\gamma}$ consists of an arbitrary segment of type ii $(i = 1, 2)$ and possibly some additional segments of type j for some $j \in \{1, 2\}$.

(II) $\bar{\gamma}$ consists of a string of edges of type 1 followed by a string of edges of type 2.

(III) $\bar{\gamma}$ consists of two edges of type 12 which separate (possibly empty) strings of edges of type 1 from (possibly empty) strings of edges of type 2.

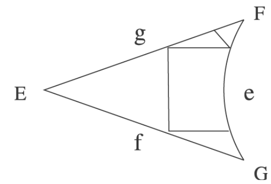
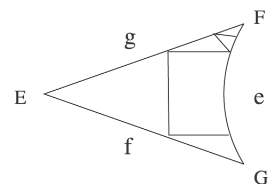
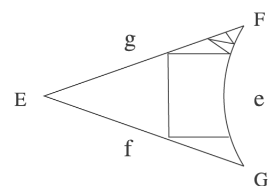
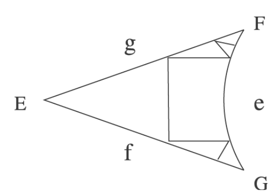
(G, F, E)	length of B_1-Orbit	length of A-Orbit	illustration
$r_1 = 1, r_2 = 2$			
(11,11,11)	2.901		
(9,6,18)	2.599	2.692	
(12,8,24)	2.824	2.778	
(20,10,30)	2.936		
(21,7,21)	2.786	2.776	
(27,9,27)	2.910		
$r_1 = 1, r_2 = 3$			
(7,9,9)	2.905		
(5,10,10)	2.826	2.613	
(6,12,12)	3.013		
(5,20,20)	3.053		
(16,8,19)	2.963		
$r_1 = 1, r_2 = 4$			
(11,11,3)	2.937		
(10,10,4)	2.985		
(5,10,10)	2.8882		
(4,13,13)	2.905		
(4,12,3)	2.518	1.992	
(5,15,3)	2.835	2.139	
(7,21,3)	3.087		
(4,20,5)	2.956		
(5,20,4)	3.070		
(4,12,6)	2.791	2.432	
(6,12,4)	2.978		
(4,10,20)	2.717	2.546	
(5,11,6)	2.919		
(6,10,5)	2.904		
$r_1 = 2, r_2 = 3$			
(7,21,3)	3.145		
(6,12,4)	3.026		

TABLE 3. Lengths of B_1-orbits.

We say that a geodesic is of type j ($j = I, II, III$) if it admits a lift to Ω with the properties as for j above. Notice that a geodesic may be of more than one type.

We determine the systoles of a given surface numerically by computing the lengths of all closed geodesics of type I, II, III. By Lemma 4.1 there are only finitely many such geodesics for each of these types. The geodesics of the shortest length from this family are the systoles of our surface. For our computations, we use the hyperboloid model of the hyperbolic plane. Unparameterized geodesics in this model correspond precisely to 2-dimensional linear subspaces in $\mathbf{R}^3$ which have nontrivial intersection with the hyperboloid. Every such subspace can be represented by a choice for its normal vector. Using the hyperbolic cross product, we can determine whether or not two distinct geodesics intersect. If they do, we can compute the coordinates of their intersection point again using the hyperbolic cross product.

Side identifications for a fundamental polygon Ω are isometries of the hyperbolic plane. The length of the geodesic on S which corresponds to a side pairing Φ equals the translation length of Φ along its unique invariant geodesic and is determined by the trace of Φ as a linear map. Namely, this translation length simply equals

$$\operatorname{arccosh}\left|\frac{\operatorname{trace}(\Phi)}{2}\right|.$$

To compute the length of a geodesic whose lift $\bar{\gamma}$ to Ω consists of more than one segment we proceed as follows. Assume for simplicity that $\bar{\gamma}$ consists of exactly two segments. This lift then can be represented as x-y, y'-x', where $x, y \in \{1, \dots, 2p\}$ and y' is the side which is identified with y, x' is the side which is identified with x. To compute the length of the geodesic x-y, y'-x', we first determine the isometry Φ which maps the side y' onto the side y. Next we compute the isometry Ψ which maps the side x onto the side $x'' = \Phi(x')$. The length of the geodesic x-y, y'-x' can then be calculated as above from the trace of Ψ. With this method (combining the n segments of the lift of the geodesic to one big segment), we compute the length of any closed geodesic on S which is represented as above.

In the same way, we can also compute the length of billiard orbits in an arbitrary hyperbolic triangle. The triangle is determined by its ordered triple of angles. The orbit is given by a word in the edges of the triangle which does not contain a subword made out of two identical edges.

To determine the systoles of our surface, we have to look in a systematic way for irreducible B-orbits which lift to closed geodesics on S. For example, to find all the geodesics of type I we check for every $m \in \{1, 2, 3, \dots, p\}$ whether or not there exists a geodesic of type $x \in \{11, 22\}$ connecting the sides 1 and $2m$. We use that fact that for a given m the segment $1 - (2m)$ is of the following type:

$$\text{segment type} = \begin{cases} 1 & \text{if } m = 1 \\ 11 & \text{if } 2 \le m \le p/2 \\ 22 & \text{if } p/2 < k_1' \le p-1 \\ 2 & \text{if } m = p \end{cases}$$

We then attach to our initial segment at most $p-1$ additional segments of type 1 or type 2 in a combinatorial pattern prescribed by our side pairings until we obtain a closed curve and compute its length with the above procedure.

The computer program is available at www.math.uni-bonn.de/˜ursula

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