Some Experimental Results on the Frobenius Problem

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CONTENTS

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We study the *Frobenius problem*: Given relatively prime positive integers $a_1, \ldots, a_d$, find the largest value of t (the *Frobenius number*) such that $\sum_{k=1}^{d} m_k a_k = t$ has no solution in nonnegative integers $m_1, \ldots, m_d$. Based on empirical data, we conjecture that except for some special cases, the Frobenius number can be bounded from above by $\sqrt{a_1 a_2 a_3}^{5/4} - a_1 - a_2 - a_3$.

1. INTRODUCTION

Given positive integers $a_1, \ldots, a_d$ with $\gcd(a_1, \ldots, a_d) = 1$, we call an integer t *representable* if there exist nonnegative integers $m_1, \ldots, m_d$ such that

$$t = \sum_{j=1}^{d} m_j a_j \ .$$

In this paper, we discuss the *linear Diophantine problem of Frobenius*: Namely, find the largest integer which is not representable. We call this largest integer the *Frobenius number* $g(a_1, \ldots, a_d)$; its study was initiated in the 19th century. For $d = 2$, it is well known (most probably at least since Sylvester [Sylvester 84]) that

$$g(a_1, a_2) = a_1 a_2 - a_1 - a_2 \qquad (1\text{–}1)$$

For $d > 2$, all attempts for explicit formulas have proved elusive. Two excellent survey papers on the Frobenius problem are [Alfonsin 00] and [Selmer 77].

Our goal is to establish bounds for $g(a_1, \ldots, a_d)$. The literature on such bounds is vast—see, for example, [Beck et al. 02, Brauer and Shockley 62, Davison 94, Erdős and Graham 72, Selmer 77, Vitek 75]. We focus on the first nontrivial case $d = 3$; any bound for this case yields a general bound, as one can easily see that $g(a_1, \ldots, a_d) \leq g(a_1, a_2, a_3)$. All upper bounds in the literature are proportional to the product of two of the a_k. On the other hand, Davison proved the lower bound $g(a_1, a_2, a_3) \geq \sqrt{3 a_1 a_2 a_3} - a_1 - a_2 - a_3$ in [Davison 94]. Experimental data (see Figure 3) shows that this bound is sharp in the sense that it is very often very close to

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$g(a_1, a_2, a_3)$. This motivates the question whether one can establish an *upper* bound proportional to $\sqrt{a_1 a_2 a_3}^p$ where $p < 4/3$ ($p = 4/3$ would be comparable to the known bounds.). In this paper, we illustrate empirically, on the basis of more than ten thousand randomly chosen points, that $g(a_1, a_2, a_3) \le \sqrt{a_1 a_2 a_3}^{5/4} - a_1 - a_2 - a_3$.

2. SOME GEOMETRIC-COMBINATORIAL INGREDIENTS

Another motivation for the search for an upper bound proportional to $\sqrt{a_1 a_2 a_3}^p$ comes from the following formula of [Beck et al. 02], which is the basis for our study: Let a, b, c be pairwise relatively prime positive integers, and define

$$N_t(a,b,c) := \# \left\{ (m_1, m_2, m_3) \in \mathbb{Z}^3 : \; m_k \ge 0, \; a m_1 + b m_2 + c m_3 = t \right\}.$$

Then

$$N_t(a,b,c) = \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) + \sigma_{-t}(b,c;a) + \sigma_{-t}(a,c;b) + \sigma_{-t}(a,b;c), \qquad (2\text{–}1)$$

where

$$\sigma_t(a,b;c) := \frac{1}{c} \sum_{\lambda^c = 1 \ne \lambda} \frac{\lambda^t}{(\lambda^a - 1)(\lambda^b - 1)}$$

is a *Fourier-Dedekind sum*. One interpretation of $N_t(a,b,c)$ is the number of partitions of t with parts in the set $\{a,b,c\}$. Geometrically, $N_t(a,b,c)$ enumerates integer points on the triangle

$$\left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \; x_k \ge 0, \; a x_1 + b x_2 + c x_3 = 1 \right\},$$

dilated by t. The Frobenius problem hence asks for the largest integer dilate of this triangle that contains no integer point; in other words, the largest t for which $N_t(a,b,c) = 0$. It is also worth mentioning that the condition that a, b, and c are *pairwise* relatively prime is no restriction, due to Johnson's formula [Johnson 60]: if $m = \gcd(a,b)$, then

$$g(a,b,c) = m \; g\left(\frac{a}{m}, \frac{b}{m}, c\right) + (m-1)c \,. \qquad (2\text{–}2)$$

In [Beck et al. 02], formulas analogous to (2–1) for $d > 3$ are given. In our case ($d = 3$), a straightforward calculation shows

$$\sigma_t(a,b;c) = -\frac{1}{4c} \sum_{k=1}^{c-1} \frac{e^{\frac{\pi i k}{c}(-2t+a+b)}}{\sin\frac{\pi k a}{c} \sin\frac{\pi k b}{c}} \,.$$

In fact, $\sigma_t(a,b;c)$ is a *Dedekind-Rademacher sum* [Rademacher 64], as shown in [Beck et al. 02]. Hence we can rewrite (2–1) as

$$\begin{aligned} N_t(a,b,c) = {} & \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) \\ & + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) \\ & - \frac{1}{4a} \sum_{k=1}^{a-1} \frac{e^{\frac{\pi i k}{a}(2t+b+c)}}{\sin\frac{\pi k b}{a} \sin\frac{\pi k c}{a}} - \frac{1}{4b} \sum_{k=1}^{b-1} \frac{e^{\frac{\pi i k}{b}(2t+a+c)}}{\sin\frac{\pi k a}{b} \sin\frac{\pi k c}{b}} \\ & - \frac{1}{4c} \sum_{k=1}^{c-1} \frac{e^{\frac{\pi i k}{c}(2t+a+b)}}{\sin\frac{\pi k a}{c} \sin\frac{\pi k b}{c}} \,. \end{aligned} \qquad (2\text{–}3)$$

If we write the "periodic part" of $N_t(a,b,c)$ as

$$\begin{aligned} P_t(a,b,c) := {} & \frac{1}{4a} \sum_{k=1}^{a-1} \frac{e^{\frac{\pi i k}{a}(2t+b+c)}}{\sin\frac{\pi k b}{a} \sin\frac{\pi k c}{a}} + \frac{1}{4b} \sum_{k=1}^{b-1} \frac{e^{\frac{\pi i k}{b}(2t+a+c)}}{\sin\frac{\pi k a}{b} \sin\frac{\pi k c}{b}} \\ & + \frac{1}{4c} \sum_{k=1}^{c-1} \frac{e^{\frac{\pi i k}{c}(2t+a+b)}}{\sin\frac{\pi k a}{c} \sin\frac{\pi k b}{c}}, \end{aligned} \qquad (2\text{–}4)$$

(2–3) becomes

$$\begin{aligned} N_t(a,b,c) = {} & \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) \\ & + \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) \\ & - P_t(a,b,c). \end{aligned}$$

If we can bound $P_t(a,b,c)$ from above by, say, B, then the roots of $N_t(a,b,c)$—and hence $g(a,b,c)$—can be bounded from above:

$$\begin{aligned} g(a,b,c) \le {} & abc \left(-\tfrac{1}{2}\left(\tfrac{1}{ab} + \tfrac{1}{ac} + \tfrac{1}{bc}\right) + \sqrt{\tfrac{1}{4} A_1 - \tfrac{2}{abc}\left(\tfrac{1}{12} A_2 - B\right)} \right) \\ = {} & -\tfrac{1}{2}(a+b+c) + \sqrt{\tfrac{1}{4}(abc)^2 A_1 - \tfrac{1}{6} abc A_2 + 2B\, abc} \\ = {} & \sqrt{2B\, abc + \tfrac{1}{12}\left(a^2+b^2+c^2\right)} - \tfrac{1}{2}(a+b+c), \end{aligned}$$

where

$$A_1 = \left(\tfrac{1}{ab} + \tfrac{1}{ac} + \tfrac{1}{bc}\right)^2$$

and

$$A_2 = \left(\tfrac{3}{a} + \tfrac{3}{b} + \tfrac{3}{c} + \tfrac{a}{bc} + \tfrac{b}{ac} + \tfrac{c}{ab}\right).$$

From this computation, the question of the existence of an upper bound for $g(a,b,c)$ proportional to $\sqrt{abc}^p$ comes up naturally. Unfortunately, it is not clear how to bound the periodic part $P_t(a,b,c)$ effectively. An almost trivial bound for $P_t(a,b,c)$ yielded in [Beck et al. 02] the inequality

$$g(a,b,c) \le \frac{1}{2}\left(\sqrt{abc\,(a+b+c)} - a - b - c\right),$$

which is of comparable size to the other upper bounds for $g(a,b,c)$ in the literature. However, we believe one can obtain bounds of smaller magnitude.

3. SPECIAL CASES

On the path to such "better" bounds, we first have to exclude some cases which definitely yield Frobenius numbers of size a_k^2. One of these cases is triples (a,b,c), such that c is representable by a and b: by (1–1), we obtain in this case $g(a,b,c) = ab - a - b$.

A second case of triples (a,b,c) that we need to exclude are those for which $a|(b+c)$. Brauer and Shockley [Brauer and Shockley 62] proved that, in this case,

$$g(a,b,c) = \max\left(b\left\lfloor\frac{ac}{b+c}\right\rfloor, c\left\lfloor\frac{ab}{b+c}\right\rfloor\right) - a.$$

Here $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

An even less trivial example of special cases was given by Lewin [Lewin 75], who studied the Frobenius number of *almost arithmetic sequences*: If $m,n>0$, $\gcd(a,n)=1$, and $d \le a$, then

$$g\left(a, ma+n, ma+2n, \ldots, ma+(d-1)\,n\right) = \left(m\left\lfloor\frac{a-2}{d-1}\right\rfloor + m - 1\right)a + (a-1)\,n.$$

For arithmetic sequences ($m=1$), this formula goes back to Roberts [Roberts 56]; for consecutive numbers ($m=n=1$), it is due to Brauer [Brauer 42]. For the special case $d=3$, we obtain

$$g(a, ma+n, ma+2n) = \left(m\left\lfloor\frac{a}{2}\right\rfloor - 1\right)a + (a-1)\,n.$$

As a function in a, $b := ma+n$, $c := ma+2n$, this Frobenius number grows proportionally to ab, which means an upper bound proportional to $\sqrt{abc}^{\,p}$ with $p < 4/3$ cannot be achieved. Hence in our computations and conjectures about upper bounds for $g(a,b,c)$, we will exclude the cases of one of the numbers being representable by the other two, one number dividing the sum of the other two, and almost arithmetic sequences. Finally, as noted above, thanks to (2–2) we may assume without loss of generality that a, b, and c are pairwise coprime. The triples (a,b,c) that are not excluded will be called *admissible*.

4. COMPUTATIONS

In the present section, we discuss the computation of the Frobenius number. For convenience we computed the number $f(a,b,c) = g(a,b,c) + a + b + c$. It is not hard to see that $f(a,b,c)$ is the largest integer that cannot be represented by a linear combination of a, b, and c with *positive* integer coefficients. The respective counting function,

$$\overline{N}_t(a,b,c) := \#\left\{(m_1,m_2,m_3)\in\mathbb{Z}^3 : m_k > 0,\ am_1 + bm_2 + cm_3 = t\right\},$$

can also be found in [Beck et al. 02] and is closely related to $N_t(a,b,c)$:

$$\begin{aligned}\overline{N}_t(a,b,c) &= \frac{t^2}{2abc} - \frac{t}{2}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)\\ &\quad + \frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+\frac{3}{c}+\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)\\ &\quad - \frac{1}{4a}\sum_{k=1}^{a-1}\frac{e^{\frac{\pi i k}{a}(-2t+b+c)}}{\sin\frac{\pi k b}{a}\sin\frac{\pi k c}{a}} - \frac{1}{4b}\sum_{k=1}^{b-1}\frac{e^{\frac{\pi i k}{b}(-2t+a+c)}}{\sin\frac{\pi k a}{b}\sin\frac{\pi k c}{b}}\\ &\quad - \frac{1}{4c}\sum_{k=1}^{c-1}\frac{e^{\frac{\pi i k}{c}(-2t+a+b)}}{\sin\frac{\pi k a}{c}\sin\frac{\pi k b}{c}}\,.\end{aligned}$$

The following illustrates our algorithm.

```
STEP 0: Initiate the intervals I1, I2, I3 for
        the selection of the arguments a,b,c;

STEP 1: Draw at random integers a,b,c from
        I1, I2, I3, respectively;

STEP 2: Test a,b,c for coprimality and for
        almost arithmetic sequences;

STEP 3: IF (a,b,c are not pairwise coprime) or
        IF (a,b,c are almost arithmetic)
        {discard a,b,c and GOTO STEP 1}
        ELSE {SET delta <- min(a,b,c);
        GOTO STEP 4};

STEP 4: Compute z=sqrt(a*b*c),
        SET mb <- INT(sqrt(3)*z)+delta;
        SET t <- mb;

STEP 5: Compute NB(t,a,b,c);

STEP 6: IF (NB(t,a,b,c)>0)
```

```
        {SET t <- t-1, and GOTO STEP 5}
        ELSE {GOTO STEP 7};

STEP 7: SET f <- t;

STEP 8: IF(mb-f < delta) {SET
        mb <- mb+delta
        t <- mb
        GOTO STEP 5}
        ELSE {GOTO STEP 9};

STEP 9: PRINT f(a,b,c) <- f;
        STOP.
```

For example, for $a = 7, b = 13, c = 30$ the program yields the Frobenius number $f(7, 13, 30) = 95$, or $g(7, 13, 30) = 45$. This program was tested against arguments which yield known results, and found to be correct.

Our program is to choose at random arguments a, b, c in a certain range (in our case $[1, 750]$), and test the triplets for admissibility. For admissible triplets a, b, c, we compute the Frobenius number $f(a, b, c)$ based on the straightforward observation that, once we have $a = \min(a, b, c)$ consecutive integers which are representable, we know that every integer beyond that interval is representable as well. We start searching for roots of $\overline{N}_t(a, b, c)$ at the lower bound $\sqrt{3abc}$. If a root is found at an integer f, we repeat this search until we find an interval of a integers t with $\overline{N}_t(a, b, c) > 0$, that is, an interval of a representable integers. At this stopping point, the integer f is the sought-after Frobenius number $f(a, b, c)$.

We have created a `PARI-GP` program[1], following the above algorithm. The program proved to be quite efficient, since most of the values of $f(a, b, c)$ were found to be close to the lower bound $\sqrt{3abc}$, as shown in the analysis below. The Dedekind-Rademacher sums appearing in (2–3) can be computed very efficiently because they satisfy a reciprocity law ([Rademacher 64], for computational complexity see also [Knuth 77]), which allows us to calculate their values similar in spirit to the Euclidean algorithm. This implies that for a given t, $\overline{N}_t(a, b, c)$ can be computed with our rather simple algorithm in $O(\log(c))$ time, assuming that $c = \max(a, b, c)$. Hence if $f(a, b, c)$ is close to the lower bound $\sqrt{3abc}$—which, again, happens in the vast majority of cases—we obtain $f(a, b, c)$ in $O(a \log(c))$ time. On the other hand, we can of course not assume that $f(a, b, c)$ is close to $\sqrt{3abc}$; still we get, at worst, a computation time of $O(ab \log(c))$. What makes this analysis even more appealing is that it applies to the general case of the Frobenius problem. As mentioned above, there is an analog for (2–1) and (2–3) for $d > 3$ [Beck et al. 02], which again is a lattice-point count in a polytope and as such is known (for fixed d) to be computable in $O\left(p\left(\log a_1, \ldots, \log a_d\right)\right)$ time for some polynomial p [Barvinok 94]. With an analogous algorithm for the general case, we would hence be able to compute $f(a_1, \ldots, a_d)$ in $O\left(a_1 a_2\, p\left(\log a_1, \ldots, \log a_d\right)\right)$ time, where $a_1 < a_2 < \cdots < a_d$. As in the three-variable case—in fact, even more so—most Frobenius numbers will be situated very close to the lower bound $\sqrt{3a_1a_2a_3}$, which means that in the vast majority of cases, we can expect a computation time of $O\left(a_1\, p\left(\log a_1, \ldots, \log a_d\right)\right)$.

The computational complexity of the Frobenius problem is very interesting and still gives rise to ongoing studies. Davison [Davison 94] provided an algorithm for the three-variable case ($a < b < c$) which runs in $O(\log b)$ time. The general case is still open. While Kannan [Kannan 92] proved that there is a polynomial-time algorithm (polynomial in $\log a_1, \ldots, \log a_d$) to find $g(a_1, \ldots, a_d)$ for fixed d, no such algorithm is known for $d > 3$. The fastest general algorithm of which we are aware is due to Nijenhuis [Nijenhuis 79] and runs in $O(d\, a \log a)$ time, where $a = \min(a_1, \ldots, a_d)$. Hence, while our primitive algorithm is not competitive for the three-variable case of the Frobenius problem, it might be worthwhile to develop it further in the general case.

We initially implemented our program as an MS-DOS QUICK BASIC program and experienced some interesting problems due to floating-point errors: Computing generalized Dedekind sums can get challenging for large arguments. These problems were only discovered when we reimplemented the algorithm in `PARI-GP`, which has an extended precision arithmetic and also keeps track of roundoff errors effectively. It is worth mentioning that both Knuth's algorithm [Knuth 77] for the computation of Rademacher-Dedekind sums and Davison's algorithm [Davison 94] for computing $g(a, b, c)$ are integer algorithms and therefore are very stable.

With our program, we generated at random 10000 admissible triplets. Our main question is the relation of the Frobenius number $f = f(a, b, c)$ to $z := \sqrt{abc}$. The following is a statistical description of the ratios $R := f/z$.

4.1 Descriptive Statistics

Q_1 and Q_3 are the first and third quartiles, respectively. We see in Table 2 that 50% of the cases have a ratio smaller than 2.01, and 75% have ratio smaller than 2.30.

[1]Our program can be downloaded at www.math.binghamton.edu/matthias/frobcomp.html.

a	b	c	$f(a,b,c)$	$z = \sqrt{abc}$	$\sqrt{3}\,z$	$z^{5/4}$	$R = f/z$
487	733	738	121755	16231.0	28112.9	183202	7.50140
229	483	662	64901	8557.0	14821.1	82300	7.58457
223	307	698	52657	6912.7	11973.2	63032	7.61740
244	357	619	56067	7343.0	12718.5	67974	7.63542
509	541	557	95788	12384.7	21450.9	130649	7.73439
262	349	699	61861	7994.7	13847.2	75597	7.73776
475	611	679	109183	14037.9	24314.4	152802	7.77773
248	305	439	45274	5762.5	9980.9	50207	7.85671
265	488	509	65434	8113.2	14052.5	77000	8.06514
274	401	695	70596	8738.6	15135.6	84489	8.07868
368	415	599	77374	9564.5	16566.2	94586	8.08972
281	341	502	57790	6935.6	12012.8	63293	8.33241
315	488	559	77734	9269.8	16055.8	90958	8.38571
305	319	652	67142	7964.7	13795.3	75242	8.42995
393	452	619	89830	10486.0	18162.3	106112	8.56664
313	532	579	84150	9819.0	17007.0	97743	8.57012
301	479	725	87903	10224.0	17708.5	102808	8.59773
655	671	679	150043	17274.9	29921.1	198048	8.68558
296	731	749	110834	12730.5	22049.9	135225	8.70618
359	520	619	94318	10749.6	18618.9	109457	8.77406
337	346	701	79559	9040.9	15659.3	88159	8.79989
320	469	491	77556	8584.2	14868.4	82628	9.03469
335	668	669	112894	12235.6	21192.6	128685	9.22672
379	389	748	97998	10501.4	18188.9	106306	9.33194

TABLE 1.

In the following figures, we present a box-plot and a histogram of the variable R.

Variable	N	Mean	Median	StDev	Min	Max	Q_1	Q_3
R	10000	2.283	2.012	0.737	1.736	9.332	1.940	2.299

TABLE 2.

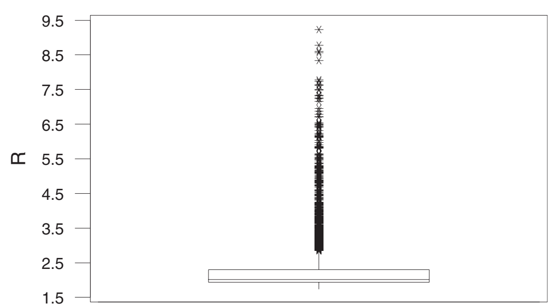

FIGURE 1. Box-plot.

In the box-plot, the bottom line of the box corresponds to the first quartile Q_1. The top line of the box corresponds to the third quartile Q_3. There are 980 points above the value of $R = 3$. Only 24 points, which are listed in Table 1, have a value of R greater than 7.5.

In Figure 3, we present all the points (z, f). Notice that all the points are above the straight line $\sqrt{3}z$, which illustrates Davison's lower bound. The upper bound is, however, convex. It is included in the figure as the graph $z^{5/4}$.

5. CONJECTURES AND CLOSING REMARKS

Randomly chosen admissible arguments tend to yield a Frobenius number f smaller than the expected number (mean) which is estimated to be $2.28z$. The distribution of $R = f/z$ is very skewed (positive asymmetry) as seen in Figure 1. Since 10000 random points yielded $f < z^{5/4}$, or $g(a,b,c) < \sqrt{abc}^{5/4} - a - b - c$, the probability that a future randomly chosen admissible triplet with $z = \sqrt{abc} < 20000$ will yield $f \geq z^{5/4}$ is smaller than 1/10000.

In general, our data suggests that one can obtain an upper bound of smaller magnitude than what the above cited results state. Again, the upper bounds in the literature are comparable to an upper bound proportional to $\sqrt{abc}^{4/3}$. We believe the following is true.

Conjecture 5.1. *There exists an upper bound for (a,b,c) proportional to $\sqrt{abc}^{p}$ where $p < \frac{4}{3}$, valid for all admissible triplets (a,b,c).*

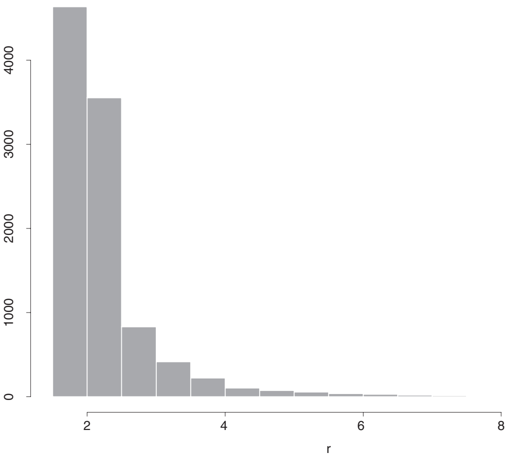

FIGURE 2. Histogram

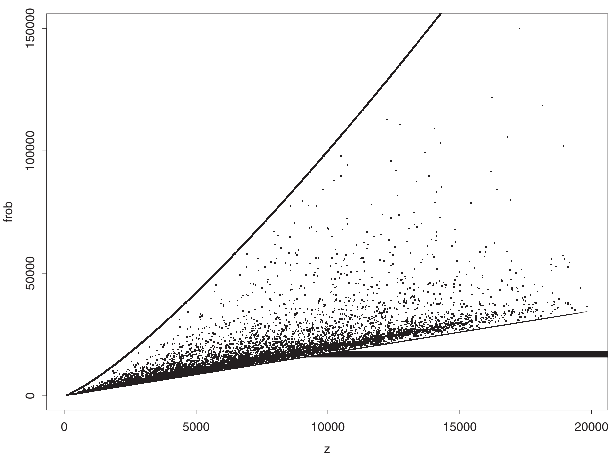

FIGURE 3. $f = f(a, b, c)$ as a function of $z = \sqrt{abc}$.

In fact, our data suggests, more precisely, that for all admissible triplets (a, b, c),

$$g(a, b, c) \leq \sqrt{abc}^{5/4} - a - b - c.$$

It is very improbable that a randomly chosen admissible triple (a, b, c), such that $\sqrt{abc} < 20000$, will yield $g(a, b, c) > \sqrt{abc}^{5/4} - a - b - c$. However, we remark that there might be specific structures of triples (a, b, c), close to almost arithmetic, for which $g(a, b, c) > \sqrt{abc}^{5/4} - a - b - c$. This is generally not the case.

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