The Waring Loci of Ternary Quartics

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CONTENTS

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Let s be any integer between 1 and 5. We determine necessary and sufficient conditions that a ternary quartic be expressible as a (possibly degenerate) sum of fourth powers of s linear forms.

1. INTRODUCTION

Let F be a homogeneous polynomial (or form) of degree q in r variables. It is a classical problem to determine whether F can be expressed as a sum of powers of linear forms,

$$F = L_1^q + \cdots + L_s^q, \qquad (1–1)$$

for a specified number s. This is usually called "Waring's problem" for algebraic forms. (Normally one also allows a "degeneration" of the right-hand side in (1–1); this will be made precise later.) Now, the condition that F can be so expressed is invariant under the natural action of the group SL_r; hence, it should be equivalent to the vanishing of certain concomitants of F in the sense of the invariant theory of r-ary q-ics. It is of interest to identify these concomitants, and thus to get explicit algebraic conditions on the coefficients of F for the expression (1–1) (or its degeneration) to be possible.

In this paper, we consider the case of ternary quartics, i.e., we let $r = 3$ and $q = 4$. Since a general ternary quartic is a sum of 6 powers of linear forms, it is only necessary to consider the range $1 \leq s \leq 5$. The calculations required in this case are not prohibitively large, and it is possible to get a complete solution. This is the main result of the paper (see Theorem 4.1).

An excellent introduction to Waring's problem may be found in [Geramita 95]. A very comprehensive account of the theory is given in [Iarrobino and Kanev 99]. The problem was solved for binary forms by Gundelfinger (see [Chipalkatti 04, Grace and Young 65, Kung 86]). In Salmon's book [Salmon 60], the solution for ternary cubics is given in essence, but there it is scattered among several articles.

2. PRELIMINARIES

In this section, we establish notation and recall the representation-theoretic notions that we will need. The reader may wish to read it along with [Chipalkatti 02], where a similar set-up is used, but more detailed explanations are given. Although we work throughout with ternary quartics, on occasion I state whether a result goes through for arbitrary q and r. All the terminology from algebraic geometry follows [Hartshorne 77], in particular a variety is always irreducible as a topological space and reduced as a scheme.

Let V be a three-dimensional $\mathbf{C}$-vector space. Let V and V^* have dual bases, $\{y_0, y_1, y_2\}$ and $\{x_0, x_1, x_2\}$, respectively. We will identify $\mathbb{P}(S_4 V^*) = \mathbb{P}^{14}$ with the space of quartic forms in the x_i (up to scalars). Let $R = S_\bullet(S_4 V)$ denote the symmetric algebra on $S_4 V$, then $\mathbb{P}^{14} = \operatorname{Proj} R$. The group $SL(V)$ acts on $\mathbb{P}^{14}$ by change of variables.

For an integer s, let $W_s^\circ \subseteq \mathbb{P}^{14}$ denote the set

$$\{F \in \mathbb{P}^{14} : F = L_1^4 + \cdots + L_s^4, \text{ for some } L_i \in V^*\}, \quad (2\text{–}1)$$

and W_s its Zariski closure in $\mathbb{P}^{14}$ with its reduced scheme structure. The embedding $W_s \subseteq \mathbb{P}^{14}$ is $SL(V)$-equivariant.

Remark 2.1. In general, the W_s° are not quasiprojective. However, W_s° is the image of the rational map

$$(V^*)^s - \to \mathbb{P}^{14}, \quad (L_1, \dots, L_s) \longrightarrow \sum_{i=1}^{s} L_i^4;$$

hence it is constructible by Chevalley's theorem (see [Hartshorne 77, Chapter II]). Since W_s° is the image of an irreducible algebraic set, it is irreducible. Hence, W_s is also irreducible, i.e., it is a projective variety.

Remark 2.2. The dimensions of the varieties W_s are known by a theorem due to Alexander and Hirschowitz—see [Geramita 95, Iarrobino and Kanev 99] for references. In fact, $W_s = \mathbb{P}^{14}$ for $s \geq 6$, $\dim W_s = 3s - 1$ for $1 \leq s \leq 4$, and W_5 is a hypersurface. The degrees of $W_1, \dots, W_5$ are 16, 75, 112, 35, and 6, respectively (see [Ellingsrud and Strømme 96]). Elements of W_4 (respectively, W_5) are conventionally called Capolari (respectively, Clebsch) quartics. For $s = 6$, a general ternary quartic has ∞^3 presentations as a sum of powers, and they are parametrized by points of a prime Fano threefold of sectional genus 12 (see [Schreyer 01]).

We would like to give necessary and sufficient $SL(V)$-invariant algebraic conditions for a form F to lie in W_s. This is roughly the same as determining the structure of the ideal $I_{W_s} \subseteq R$ *qua* an $SL(V)$-representation. The parallel problem of characterizing W_s° is harder, and in Section 4.6, we only consider the case of W_2°.

2.1 Schur Functors

If λ is a partition, then $S_\lambda(-)$ denotes the corresponding Schur functor. We maintain the indexing conventions of [Fulton and Harris 91, Chapter 6].

Let $J \subseteq R$ be a homogeneous $SL(V)$-stable ideal (for instance, I_{W_s}), and J_d its degree d part. We have a direct sum decomposition

$$J_d = \bigoplus_\lambda (S_\lambda V)^{N_\lambda} \subseteq S_d(S_4 V). \quad (2\text{–}2)$$

Since V is three-dimensional, each λ is of the form $(m + n, n)$ for some integers m, n. If we can locate the degrees d which generate J, and then specify the inclusions in (2–2), then J is completely specified. These inclusions are encoded by the concomitants of ternary quartics.

2.2 Concomitants

Write a_I for the monomial $y_0^{i_0} y_1^{i_1} y_2^{i_2}$, where $I = (i_0, i_1, i_2)$ is of total degree 4. Then the $\{a_I\}$ form a basis of $S_4 V$, and R is the polynomial algebra $\mathbf{C}[\{a_I\}]$. Now $S_4 V \otimes S_4 V^*$ contains the trace element, which, when written out in full, appears as

$$\mathbb{F} = \sum_{|I|=4} a_I \otimes x_0^{i_0} x_1^{i_1} x_2^{i_2}.$$

We may treat the $\{a_I\}$ as independent indeterminates, so $\mathbb{F}$ is the precise formulation of the concept of a "generic" ternary quartic.

Now write $u_0 = x_1 \wedge x_2$, $u_1 = x_2 \wedge x_0$, $u_2 = x_0 \wedge x_1$, which form a basis of $\wedge^2 V^*$. Consider an inclusion

$$S_{(m+n,n)} V \xrightarrow{\varphi} S_d(S_4 V)$$

of $SL(V)$-representations, which will correspond to an equivariant inclusion

$$\mathbf{C} \longrightarrow S_d(S_4 V) \otimes S_{(m+n,n)} V^*.$$

Let Φ denote the image of 1 under this map. Written out in full, it is a form of degree d, m, n, respectively, in the three sets of variables a_I, x_i, u_i. This follows because $S_{(m+n,n)} V^*$ has a basis derived from standard tableaux (see [Fulton and Harris 91, Chapter 6]). We will write this image as $\Phi(d,m,n)$; classically it is called a concomitant of degree d, order m, and class n of ternary quartics. For instance, $\mathbb{F}$ itself is a $\Phi(1,4,0)$ and the Hessian of $\mathbb{F}$ is a $\Phi(3,6,0)$. For fixed d,m,n, the number of linearly independent concomitants equals the multiplicity of $S_{(m+n,n)}\,V$ in $S_d(S_4\,V)$.

Remark 2.3. The name "concomitant" was introduced by Sylvester (see [Sylvester 04]). The terminology is sometimes further refined—if either m or n is zero, then Φ is accordingly called a contravariant or a covariant. If m,n are both zero, then it is an invariant.

We may regard Φ as a form in u_i, x_i with coefficients in R_d. Then the subspace of R_d generated by these coefficients coincides with the image of the inclusion φ above. We can evaluate Φ at a specific form F by substituting its actual coefficients for the letters a_I. Then we say that Φ vanishes at F if this evaluated form is identically zero.

3. APOLARITY

We briefly explain the connection between the expression of F as a sum of powers, and the existence of schemes "apolar" to F (also see [Dolgachev and Kanev 93, Ehrenborg and Rota 93, Iarrobino and Kanev 99]).

Let $A = S_\bullet V$ denote the symmetric algebra on V. A linear form $L \in V^*$ can be considered as a point in $\mathbb{P}V^* = \operatorname{Proj} A$. If $Z = \{L_1,\dots,L_s\}$ is a collection of points in $\mathbb{P}V^*$, then $I_Z \subseteq A$ denotes their ideal. For every $k \geq 0$, there is a coproduct map

$$S_4\,V^* \longrightarrow S_k\,V^* \otimes S_{4-k}\,V^*.$$

(It is zero unless $0 \leq k \leq 4$.) This gives rise to a map

$$S_4\,V^* \longrightarrow \operatorname{Hom}(S_k\,V, S_{4-k}\,V^*), \quad F \longrightarrow \alpha_{k,F}. \qquad (3\text{–}1)$$

Taking a direct sum over all k, we have a map

$$S_4\,V^* \longrightarrow \operatorname{Hom}(A, \bigoplus_{i=0}^{4} S_i\,V^*), \quad F \longrightarrow \alpha_F.$$

For a fixed F, $\ker \alpha_F$ is a homogeneous ideal in A. Classically, a form in $\ker \alpha_F$ is said to be apolar to F. The passage between apolarity and Waring's problem is forged by the following beautiful theorem of Reye, to whom the concept of apolarity should be credited.

Theorem 3.1. *With notation as above,*

$$F \in \mathit{span}\{L_1^4,\dots,L_s^4\} \iff I_Z \subseteq \ker \alpha_F \iff (I_Z)_4 \subseteq \ker \alpha_{4,F}.$$

Of course, Reye's own formulation in [Reye 74] is rather different. A modern proof can be found in [Iarrobino and Kanev 99, Theorem 5.3 B]. An analogous result holds for any q,r.

It is natural to relax the requirement that Z be a reduced scheme, which motivates the following definition.

Definition 3.2. Let $F \in \mathbb{P}^{14}$, and $Z \subseteq \mathbb{P}V^*$ a closed subscheme with (saturated) ideal I_Z. We say that Z is apolar to F, if $I_Z \subseteq \ker \alpha_F$.

To restate Reye's theorem, F lies in W_s° iff F admits a reduced zero-dimensional apolar scheme of length s. We now tentatively introduce the locus

$$X_s^\circ = \{F \in \mathbb{P}^{14} : \ker \alpha_F \supseteq I_Z \text{ for some } Z \in \operatorname{Hilb}^s(\mathbb{P}V^*)\}.$$

Let X_s denote the closure of X_s° with its reduced scheme structure. A priori, X_s is a closed algebraic subset of $\mathbb{P}^{14}$ which only contains W_s. However, they turn out to be equal.

Lemma 3.3. *We have $W_s = X_s$ for all s, in particular X_s is irreducible.*

Proof: This is essentially proven in [Iarrobino and Kanev 99, Proposition 6.7]; here we will sketch the argument. Let $F \in X_s^\circ$, with apolar scheme $Z \in \operatorname{Hilb}^s(\mathbb{P}^2)$. It is known that $\operatorname{Hilb}^s(\mathbb{P}^2)$ is irreducible, which implies that Z is smoothable (i.e., it admits a flat deformation to a smooth scheme). Then [Iarrobino and Kanev 99, Proposition 6.7 A] implies that $F \in W_s$. It follows that $X_s^\circ \subseteq W_s$, and so $X_s \subseteq W_s$. □

Remark 3.4. The proof shows that the analogous result is true for $r = 2,3$ and all q. There are examples for $r = 6$ (see [Iarrobino and Kanev 99, Corollary 6.28]) where X_s is reducible and hence strictly contains W_s.

We will now use apolarity to relate X_s with degeneracy loci of certain morphisms of vector bundles on $\mathbb{P}^{14}$. Globally, the map in (3–1) gives a morphism of vector bundles

$$\alpha_k : S_k\,V \otimes \mathcal{O}_{\mathbb{P}^{14}}(-1) \longrightarrow S_{4-k}\,V^*. \qquad (3\text{–}2)$$

Up to a twist, α_k is dual to α_{4-k}.

Lemma 3.5. *If F belongs to X_s, then rank $\alpha_{k,F} \leq s$ for any k.*

Proof: Since the rank is lower-semicontinuous as a function of F, we may assume $F \in X_s^\circ$. Then $(I_Z)_k \subseteq \ker \alpha_{k,F}$, hence

$$\operatorname{rank} \alpha_{k,F} = \operatorname{codim}(\ker \alpha_{k,F}, A_k) \leq \operatorname{codim}((I_Z)_k, A_k) \leq s.$$

□

If $\psi : \mathcal{F} \longrightarrow \mathcal{E}$ is a morphism of vector bundles on $\mathbb{P}^{14}$, let $Y(s,\psi)$ denote the scheme $\{\operatorname{rank} \psi \leq s\}$, whose ideal sheaf is locally generated by the $(s+1)\times(s+1)$-minors of a matrix representing ψ. Let $Y_{\mathrm{red}}(s,\psi)$ denote the underlying reduced scheme. We will shorten this to Y or Y_{red} if no confusion is likely.

Remark 3.6. By the lemma above, $X_s \subseteq Y_{\mathrm{red}}(s,\alpha_k)$ for any k. Hence, we have a containment

$$X_s \subseteq \bigcap_k Y_{\mathrm{red}}(s,\alpha_k). \tag{3–3}$$

For binary forms, this is an equality. By a result of Schreyer [Schreyer 01, Theorem 2.3], we already have an equality of sets $W_s = Y(s,\alpha_2)$ for ternary quartics.

Example 3.7. This is an example where the containment (3–3) is strict; I owe it to the referee. Let $r = 3, q = 8$, and $s = 14$. For $k \neq 4$, either the source or the target of α_k has rank < 14, so $Y(14,\alpha_k) = \mathbb{P}^{44}$. Now $Y(14,\alpha_4)$ is a hypersurface in $\mathbb{P}^{44}$, hence the right-hand side of (3–3) is 43-dimensional. In contrast, X_{14} is only 41-dimensional. In general, it is not known for which q, r, s equality holds in (3–3).

3.1 Symmetric Bundle Maps

In the sequel, we will exploit the fact that α_2 is a twisted symmetric morphism.

Generally, let T be a smooth complex projective variety. Let $\mathcal{E}$ be a rank e vector bundle and $\mathcal{L}$ a line bundle on T. Assume that $\psi : \mathcal{E} \longrightarrow \mathcal{E}^* \otimes \mathcal{L}$ is a twisted symmetric bundle map (i.e., $\psi^* \otimes \mathcal{L} = \psi$). Define $Y = Y(s,\psi)$ as above. Then assuming Y is nonempty,

$$\operatorname{codim}(Y', T) \leq \frac{(e-s)(e-s+1)}{2}, \tag{3–4}$$

for every component Y' of Y. Moreover, if equality holds for every component, then Y is Cohen-Macaulay. In that case, the class of Y in the Chow ring of T is given by a determinantal formula (see [Harris and Tu 84]). Let $z_k = c_k(\mathcal{E}^* \otimes \sqrt{\mathcal{L}})$, then $[Y]$ equals 2^{e-s} times the $(e-s)\times(e-s)$ determinant whose (i,j)-th entry is $z_{(e-s-2i+j+1)}$.

The minimal free resolution of Y (assuming equality in (3–4)) is deduced in [Józefiak et al. 80]. All that we need is the beginning portion

$$S_{\lambda_s}\mathcal{E} \otimes \mathcal{L}^{\otimes(-s-1)} \to \mathcal{O}_T \to \mathcal{O}_Y \to 0, \tag{3–5}$$

where λ_s denotes the partition $(2,\ldots,2) = (2^{s+1})$ with $s+1$ parts.

We will apply this set-up to α_2, with $\mathcal{E} = S_2 V \otimes \mathcal{O}_{\mathbb{P}^{14}}(-1)$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{14}}(-1)$.

4. THE IDEAL OF W_S

We proceed to state the main theorem, and then explain the calculations entering into it. The concomitants will be written in the symbolic notation—see [Chipalkatti 02, Grace and Young 65] for an explanation of this formalism. Define the following concomitants of ternary quartics:

$$\begin{aligned}
\Phi(2,4,2) &= \alpha_x^2\,\beta_x^2\,(\alpha\,\beta\,u)^2\\
\Phi(2,0,4) &= (\alpha\,\beta\,u)^4\\
\Phi(3,6,0) &= \alpha_x^2\,\beta_x^2\,\gamma_x^2\,(\alpha\,\beta\,\gamma)^2\\
\Phi(3,3,3) &= \alpha_x\beta_x\gamma_x(\alpha\,\beta\,\gamma)\,(\alpha\,\beta\,u)\,(\alpha\,\gamma\,u)\,(\beta\,\gamma\,u)\\
\Phi(3,2,2) &= \gamma_x^2\,(\alpha\,\beta\,\gamma)^2\,(\alpha\,\beta\,u)^2\\
\Phi(3,0,0) &= (\alpha\,\beta\,\gamma)^4\\
\Phi(3,0,6) &= (\alpha\,\beta\,u)^2(\alpha\,\gamma\,u)^2(\beta\,\gamma\,u)^2\\
\Phi(4,0,2) &= (\alpha\,\gamma\,\delta)^2(\beta\,\gamma\,\delta)^2(\alpha\,\beta\,u)^2\\
\Phi(4,1,3) &= \alpha_x(\alpha\,\gamma\,\delta)^2(\beta\,\gamma\,u)^2(\alpha\,\beta\,\delta)(\beta\,\delta\,u)\\
\Phi(4,4,0) &= \alpha_x\beta_x\gamma_x\delta_x\,(\alpha\,\beta\,\gamma)(\alpha\,\beta\,\delta)(\beta\,\gamma\,\delta)(\alpha\,\gamma\,\delta)\\
\Phi(4,2,4) &= \alpha_x\beta_x\,(\alpha\,\gamma\,\delta)(\beta\,\gamma\,\delta)(\alpha\,\beta\,u)^2(\gamma\,\delta\,u)^2\\
\Phi_I(5,0,4) &= (\alpha\,\beta\,\gamma)^4(\delta\,\epsilon\,u)^4\\
\Phi_{II}(5,0,4) &= (\alpha\,\beta\,\gamma)^2(\delta\,\epsilon\,u)^2(\alpha\,\delta\,\epsilon)^2(\beta\,\gamma\,u)^2\\
\Phi(5,2,0) &= \alpha_x\beta_x(\alpha\,\beta\,\gamma)^2(\alpha\,\delta\,\epsilon)(\beta\,\delta\,\epsilon)(\gamma\,\delta\,\epsilon)^2\\
\Phi(6,0,0) &= (\alpha\,\beta\,\gamma)^2(\delta\,\epsilon\,\zeta)^2(\alpha\,\epsilon\,\zeta)^2(\beta\,\gamma\,\delta)^2.
\end{aligned}$$

Now form the lists

$$\begin{aligned}
\mathcal{U}_1 &= \{\Phi(2,4,2), \Phi(2,0,4)\}\\
\mathcal{U}_2 &= \{\Phi(3,6,0), \Phi(3,0,6), \Phi(3,3,3), \Phi(3,2,2), \Phi(3,0,0)\}\\
\mathcal{U}_3 &= \{\Phi(4,4,0), \Phi(4,2,4), \Phi(4,1,3), \Phi(4,0,2)\}\\
\mathcal{U}_4 &= \{\Phi_I(5,0,4) - 3\,\Phi_{II}(5,0,4), \Phi(5,2,0)\}\\
\mathcal{U}_5 &= \{3\,\Phi(6,0,0) - \Phi(3,0,0)^2\}.
\end{aligned}$$

$$\begin{array}{ccccc} 0 & \to H^0(\mathbb{P}^{14}, \mathcal{I}_{W_1}(2)) & \to H^0(\mathbb{P}^{14}, \mathcal{O}_{\mathbb{P}^{14}}(2)) & \to H^0(W_1, \mathcal{O}_{W_1}(2)) \\ & & \| & \| \\ & & S_2(S_4\, V) & S_8\, V \end{array}$$

FIGURE 1.

If $\mathcal{U}$ is such a list, then $\mathcal{U}|_F = 0$ (respectively, $\mathcal{U}|_F \neq 0$) means that all elements of $\mathcal{U}$ vanish at F (respectively, at least one element is nonzero at F). With notation as above, our main theorem is the following:

Theorem 4.1. *Let F be a ternary quartic, and $1 \leq s \leq 5$. Then*

$$F \in W_s \iff \mathcal{U}_s|_F = 0.$$

This statement is closer to the classical roots of the subject, but in fact something stronger is true. Let $\mathfrak{A}_s \subseteq R$ denote the ideal generated by all the coefficients of all the elements in $\mathcal{U}_s$. Then the saturation of $\mathfrak{A}_s$ is I_{W_s}; in other words, the elements of $\mathcal{U}_s$ define W_s scheme-theoretically. In fact, no saturation is needed for $s = 1, 2, 5$; I do not know if it is needed for $s = 3, 4$. All of this will follow from the analysis below.

In the sequel, it is frequently necessary to calculate plethysms and tensor products of $SL(V)$-representations; this was done using John Stembridge's SF package for Maple. All commutative algebra computations were done in Macaulay-2.

4.1 Case $s = 1$

The locus W_1 is the quartic Veronese embedding of $\mathbb{P}V^*$. It is well known that its ideal is generated in degree 2, and we have an exact sequence as shown in Figure 1 above. Decomposing $S_2(S_4\, V)$, we see that $H^0(\mathbb{P}^{14}, \mathcal{I}_{W_1}(2))$ must be isomorphic to $S_{(6,2)}V \oplus S_{(4,4)}V$. Sometimes we will abbreviate the latter as $(6,2) \oplus (4,4)$.

To specify the inclusion $S_{(6,2)} \hookrightarrow S_2(S_4\, V)$ is to specify a concomitant $\Phi(2,4,2)$. There is only one copy of $S_{(6,2)}$ inside $S_2(S_4)$, hence there is a unique such Φ up to a constant. Now observe that $\alpha_x^2\, \beta_x^2\, (\alpha\, \beta\, u)^2$ is a (legal) symbolic expression of the right degree; moreover it is not identically zero. This is tantamount to checking that it is a nonzero element in the "bracket algebra" (see [Sturmfels 93, Section 3]), which was done in Macaulay-2. Thus we have found $\Phi(2,4,2)$. The other concomitant $\Phi(2,0,4)$ is found in the same way, and this finishes the calculation for $s = 1$.

Remark 4.2. In general, given d, m, n, it is possible to get all possible symbolic expressions which would be candidates for concomitants, by solving a system of Diophantine equations. However, in practice it is much easier to concoct such expressions by hand, especially if the multiplicity of $S_{(m+n,n)}$ in $S_d(S_4)$ is small.

4.2 Case $s = 2$

First, we calculate the ideal I_{W_2} by explicit elimination. Let

$$F = \sum_{|I|=4} a_I\, x^I, \quad L_i = p_{i0}\, x_0 + p_{i1}\, x_1 + p_{i2}\, x_2 \ \text{ for } i = 1, 2,$$

where a_I, p_{ij} are indeterminates. Write $F = L_1^4 + L_2^4$, and equate coefficients. We obtain polynomial expressions $a_I = f_I(p_{10}, \ldots, p_{22})$, defining a morphism

$$f : \mathbf{C}\,[\{a_I\}] \longrightarrow \mathbf{C}\,[\{p_{ij}\}].$$

Then I_{W_2} equals $\ker f$. The actual Macaulay-2 computation shows that all the ideal generators are in degree 3, and $\dim\,(I_{W_2})_3 = 148$.

The inclusion $W_2 \subseteq Y(2, \alpha_3) = Y$ implies $I_Y \subseteq I_{W_2}$. Now Y is a rank variety of dimension 6 in the sense of Porras [Porras 96]; in particular it is reduced. (It is the locus of those F which can be written as forms in only two variables by a change of coordinates.) By [loc. cit.], its ideal I_Y has a resolution given by the Eagon-Northcott complex of the map

$$S_3 V \otimes R(-1) \longrightarrow V^* \otimes R.$$

The beginning portion of this resolution is

$$\ldots \to \wedge^3(S_3 V) \otimes R(-3) \to R \to R/I_Y \to 0.$$

Hence, I_Y is generated by the 120-dimensional piece

$$\wedge^3(S_3) = (6,3) \oplus (6,0) \oplus (4,2) \oplus (0,0),$$

which is a subrepresentation of

$$R_3 = S_3(S_4 V) = (12,0) \oplus (10,2) \oplus (9,3) \oplus (8,4) \oplus (6,6) \oplus (6,3) \oplus (6,0) \oplus (4,2) \oplus (0,0).$$

The quotient of the inclusion $(I_Y)_3 \subseteq (I_{W_2})_3$ is a 28-dimensional representation, so it can only be $S_{(6,6)}$. Hence

$$(I_{W_2})_3 = (6,6) \oplus (6,3) \oplus (6,0) \oplus (4,2) \oplus (0,0).$$

The concomitants are then calculated as in the previous case.

Remark 4.3. The Gordan-Noether theorem (see [Olver 99, page 234]) implies that $F \in Y(2, \alpha_3) = Y$ iff the Hessian of F (which is $\Phi(3,6,0)$) identically vanishes. However, the Hessian does not define Y as a scheme in the following sense. Let $\mathfrak{H} \subseteq R$ denote the ideal generated by the coefficients of $\Phi(3,6,0)$; then the saturation of $\mathfrak{H}$ is strictly smaller than I_Y. (This was verified in Macaulay-2.) Hence, Proj $(R/\mathfrak{H})$ is a nonreduced scheme with the same support as Y.

4.3 Case $s = 3$

Matters are greatly simplified due to the following lemma.

Lemma 4.4. *As schemes, $W_3 = Y(3, \alpha_2)$; in particular the latter is a reduced scheme.*

Proof: As a first step, we show that $W_3 = Y_{\text{red}}(3, \alpha_2)$. Let $F \in Y_{\text{red}}$. If $\text{rank}\, \alpha_{1,F} \leq 2$, then F is a binary quartic in disguise, and then it has infinitely many apolar schemes of length 3 (see [Iarrobino and Kanev 99, Section 1.3]). If $\text{rank}\, \alpha_{1,F} = 3$, then the existence of a length 3 apolar scheme follows from [Iarrobino and Kanev 99, Theorem 5.31]. (To summarize the situation, the Buchsbaum-Eisenbud structure theorem implies that $\ker \alpha_F$ is generated as an ideal by 3 conics and 2 quartics. The subideal generated by the 3 conics defines the apolar scheme.) In either case, $F \in W_3$. This shows that $W_3 = Y_{\text{red}}(3, \alpha_2)$.

Now $Y = Y(3, \alpha_2)$ is irreducible of dimension 8, so equality holds in the codimension estimate (3–4). Hence, Y is Cohen-Macaulay, and has no embedded components. By the determinantal formula, $\deg Y = 112$ which is the same as $\deg W_3$. Hence, Y must be reduced. □

The resolution (3–5) in Section 3.1 implies that I_{W_3} is generated up to saturation by the following submodule of R_4:

$$S_{(2,2,2,2)}(S_2\, V) = (6,4) \oplus (4,3) \oplus (4,0) \oplus (2,2).$$

The concomitants are calculated as before.

As in the previous case, I tried to calculate I_{W_3} by direct elimination in Macaulay-2, but the program failed to terminate successfully.

4.4 Case $s = 4$

This is similar to the previous case.

Lemma 4.5. *As schemes, $W_4 = Y(4, \alpha_2)$; in particular the latter is a reduced scheme.*

Proof: Assume $F \in Y_{\text{red}}(4, \alpha_2)$. If either $\text{rank}\, \alpha_{1,F} \leq 2$ or $\text{rank}\, \alpha_{2,F} \leq 3$, then $F \in W_3$ by the previous argument. Hence, we may assume $\text{rank}\, \alpha_{1,F} = 3, \text{rank}\, \alpha_{2,F} = 4$. We would like to show that F admits an apolar scheme of length 4. Let $U = \ker \alpha_{2,F}$, which is a two-dimensional subspace of $S_2 V$. There are now three subcases.

- If the generators of U do not have a common linear factor, then they define a complete intersection scheme of length 4 which is apolar to F. If the generators do have a common linear factor, then up to a change of variables, there are only two possibilities.
- $U = (y_0 y_1, y_0 y_2)$. Then necessarily $F = x_0^4 + q(x_1, x_2)$, for some quartic (binary) form q. It is now immediate that $\ker \alpha_{3,F}$ contains a cubic form $u(y_1, y_2)$. The ideal $(y_0 y_1, y_0 y_2, u)$ defines an apolar length 4 scheme.
- $U = (y_0^2, y_0 y_1)$, which forces $F = x_0 x_2^3 + q(x_1, x_2)$ for some quartic form q. Then $\ker \alpha_{3,F}$ contains a cubic form $u(y_1, y_2)$, which is a linear combination of $y_1^3, y_1^2 y_2, y_1 y_2^2$. Now the ideal $(y_0^2, y_0 y_1, u)$ defines the required length 4 scheme.

We have shown that $W_4 = Y_{\text{red}}(4, \alpha_2)$. The rest of the proof is similar to the previous lemma. □

It follows that I_{W_4} is generated up to saturation by the following submodule of R_5:

$$S_{(2,2,2,2,2)}(S_2\, V) = (4,4) \oplus (2,0).$$

There are two copies $S_{(4,4)}$ inside $S_5(S_4)$, hence a two-dimensional space of concomitants of degree $(5,0,4)$. A basis for this space is given by $\Phi_I(5,0,4)$ and $\Phi_{II}(5,0,4)$. Choose a typical form in W_4, say $F = x_0^4 + x_1^4 + x_2^4 + (x_0 + x_1 + x_2)^4$ and evaluate both concomitants at F. It is found that $\Phi_I - 3\Phi_{II}$ identically vanishes on F.

Similarly, there are two copies of $S_{(2,0)}$ in $S_5(S_4)$. However, it turns out that $\Phi(5,2,0)$ itself vanishes on F, so no linear combination is needed.

4.5 Case $s = 5$

Clebsch showed in [Clebsch 61] that W_5 is a hypersurface in $\mathbb{P}^{14}$; here we calculate its invariant equation.

Let $Y = Y(5, \alpha_2)$, then $W_5 \subseteq Y_{\text{red}}$. Let $\mathcal{C}$ denote the equation of the scheme Y. Since $\mathcal{C}$ is given by the determinant of

$$\alpha_2 : S_2\, V \otimes \mathcal{O}_{\mathbb{P}^{14}}(-1) \longrightarrow S_2\, V^*,$$

it has degree 6. Decomposing $S_6(S_4\, V)$, we see that it contains a two-dimensional subspace of trivial representations. Now $\Phi(6,0,0)$ and $\Phi(3,0,0)^2$ generate this subspace, hence $\mathcal{C}$ must be their linear combination. To determine this combination, specialize both of them at

$$F = x_0^4 + x_1^4 + x_2^4 + (x_0 + x_1 + x_2)^4 + (x_0 - x_1 + x_2)^4,$$

which is an element of W_5. It turns out that $\mathcal{C} = 3\,\Phi(6,0,0) - \Phi(3,0,0)^2$. Now if $\mathcal{C}$ were not a prime element of the ring R, then it would have an *invariant* factor of degree ≤ 3. The only candidate for such a factor is $\Phi(3,0,0)$ (because ternary quartics have no invariant of degree 2), but we have seen that it does not divide $\mathcal{C}$. Hence, $\mathcal{C}$ is irreducible, and it defines W_5. Usually $\mathcal{C}$ is called the catalecticant of ternary quartics. This completes the discussion of Theorem 4.1.

4.6 A Description of W_2°

In general, W_s° is only expressible as a complicated boolean expression in closed sets, and it is not easy to characterize it algebraically. Here we attempt such a characterization for $s = 2$.

Let $F \in W_2 \setminus W_2^\circ$, then F is apolar to a nonreduced length two subscheme Z of $\mathbb{P}V^*$. Up to a change of coordinates, $I_Z = (y_0, y_1^2)$. This forces $F = c_1\, x_2^4 + c_2\, x_1 x_2^3$, for some constants c_i. Since F has no apolar scheme of length one, $c_2 \neq 0$; so $F = x_2^3(\frac{c_1}{c_2} x_2 + x_1)$. Hence,

$$W_2 \setminus W_2^\circ = \{L_1^3 L_2 : L_i \text{ are linearly independent}\}.$$

Now let

$$B = (W_2 \setminus W_2^\circ) \cup W_1 = \{L_1^3 L_2 : L_i \in V^*\},$$

which is an irreducible projective variety of dimension 4. Geometrically, B is the union of tangent planes to W_1. The inclusions $W_1 \subseteq B \subseteq W_2$ imply $I_{W_2} \subseteq I_B \subseteq I_{W_1}$.

As in Section 4.2, we calculate the generators of I_B by explicit elimination. Its minimal resolution begins as

$$R(-3) \otimes M_8 \,\oplus R(-4) \otimes M_{570} \,\oplus R(-5) \otimes M_{66} \rightarrow$$
$$R(-2) \otimes M_{15} \,\oplus R(-3) \otimes M_{56} \rightarrow R \rightarrow R/I_B \rightarrow 0,$$

where M_i is an i-dimensional $SL(V)$-representation. We need to identify M_{15} and M_{56}. Since $(I_B)_2 \subseteq (I_{W_1})_2$, on dimensional grounds $M_{15} = S_{(4,4)}$. Consider the chain $(I_{W_2})_3 \subseteq (I_B)_3 \subseteq (I_{W_1})_3$. The irreducible decompositions of the end terms are already known, hence the middle term is forced:

$$(I_B)_3 = (8,4) \oplus (6,6) \oplus (6,3) \oplus (6,0) \oplus (4,2) \oplus (0,0).$$

Now M_8 is a submodule of

$$M_{15} \otimes R_1 = (8,4) \oplus (6,3) \oplus (4,2) \oplus (2,1) \oplus (0,0),$$

hence $M_8 = S_{(2,1)}$. This implies that the submodule

$$(8,4) \oplus (6,3) \oplus (4,2) \oplus (0,0) \subseteq (I_B)_3$$

is generated by M_{15}. Hence, M_{56} (the module of new generators in degree 3) must be $(6,6) \oplus (6,0)$. Define

$$\mathcal{V} = \{\Phi(2,0,4), \Phi(3,0,6), \Phi(3,6,0)\},$$

following the generators of I_B. Since $W_2^\circ = (W_2 \backslash B) \cup W_1$, we deduce the following:

Proposition 4.6. *For a ternary quartic F,*

$$F \in W_2^\circ \iff (\mathcal{U}_2|_F = 0 \,\wedge\, \mathcal{V}|_F \neq 0) \vee (\mathcal{U}_1|_F = 0).$$

The cases $s > 2$ do not seem so accessible, partly because there are a great many possibilities for the structure of a nonreduced length s scheme.

5. A FOLIATION OF $Y(2, \alpha_3)$

This section is something of a digression, since it does not concern Waring's problem. However, it is consonant with a dominant theme in classical invariant theory: those properties of a form which are independent of coordinates should be detectable by the vanishing of concomitants.

Let us write Y for $Y(2, \alpha_3)$. A point F in Y is really a binary form up to a change of variables. Hence, for a general such F, the curve $\{F = 0\} \subseteq \mathbb{P}^2$ is a set of four concurrent lines, which can be assigned a cross-ratio. This motivates the following definition: for $t \in \mathbf{C}$, let $\Omega^{(t)}$ denote the Zariski closure of the set

$$\{L_1 L_2 (L_1 + L_2)(L_1 + t\, L_2) : L_i \in V^*\}$$

in Y (with the reduced scheme structure). This is a hypersurface in Y for a fixed t; and the family $\{\Omega^{(t)}\}$ defines a foliation over a dense open set of Y. Following a venerated tradition (see [Hartshorne 77, Chapter IV, Section 4]), we define

$$j(t) = \frac{4\,(t^2 - t + 1)^3}{27\,t^2(t-1)^2}, \quad \text{for } t \neq 0, 1;$$

and $j(0) = j(1) = \infty$. Then $\Omega^{(t)} = \Omega^{(t')}$ iff $j(t) = j(t')$.

Now we can calculate the ideal of $\Omega^{(t)}$ by elimination as in Section 4.2, and decompose it as a representation. This goes through without complications, hence I will omit the details and merely state the result.

Let $\mathfrak{J}^{(t)} \subseteq R$ denote the ideal of $\Omega^{(t)}$, evidently $I_Y \subseteq \mathfrak{J}^{(t)}$. Since the generators of I_Y are already known from Section 4.2, it is enough to describe the generators of the quotient $Q^{(t)} = \mathfrak{J}^{(t)}/I_Y$. The computation shows that $Q^{(t)}$ is generated as a graded R/I_Y module by an *irreducible* representation $M^{(t)}$. The degree in which $M^{(t)}$ appears and its structure depend on $j(t)$, in fact

$$M^{(t)} = \begin{cases} S_{(4,4)} & \text{in degree 2 if } j(t) = 0, \\ S_{(6,6)} & \text{in degree 3 if } j(t) = 1, \\ S_{(12,12)} & \text{in degree 6 if } j(t) \neq 0, 1. \end{cases}$$

Now it is a routine matter to identify the concomitant corresponding to $M^{(t)}$. Define

$$E_j = \begin{cases} (1-j)\,\Phi(2,0,4)^3 + 6j\,\Phi(3,0,6)^2, & \text{for } j \text{ finite;} \\ -\Phi(2,0,4)^3 + 6\,\Phi(3,0,6)^2, & \text{for } j = \infty. \end{cases}$$

(The definitions of Φ are those in the beginning of Section 4.) Then we have the following result.

Theorem 5.1. *For a ternary quartic F,*

$$F \in \Omega^{(t)} \quad \textit{if and only if}$$
$$\{\Phi(3,6,0), \Phi(3,3,3), \Phi(3,2,2), \Phi(3,0,0), E_{j(t)}\}|_F = 0.$$

Notice that $\Omega^{(2)} = W_2$ and $j(2) = 1$. In this case, the result agrees with Theorem 4.1 (as it should).

Remarks 5.2.

1. The roles played by $\Phi(2,0,4)$ and $\Phi(3,0,6)$ are very similar to those of the Eisenstein series g_2, g_3 in the classical theory of elliptic functions. I do not know if one can demonstrate a precise connection between the two theories.

2. It is tempting to conjecture that there is a similar story to be told for quartic forms in any number of variables. For instance, (conjecturally) there should be a continuously moving concomitant of quaternary quartics which detects the cross-ratio of four coaxial planes.

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