On the Monotone Upper Bound Problem

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CONTENTS

The *Monotone Upper Bound Problem* asks for the maximal number $M(d, n)$ of vertices on a strictly increasing edge-path on a simple d-polytope with n facets. More specifically, it asks whether the upper bound

$$M(d, n) \ \leq \ M_{\mathrm{ubt}}(d, n)$$

provided by McMullen's [McMullen 70] *Upper Bound Theorem* is tight, where $M_{\mathrm{ubt}}(d, n)$ is the number of vertices of a dual-to-cyclic d-polytope with n facets.

It was recently shown that the upper bound $M(d, n) \leq M_{\mathrm{ubt}}(d, n)$ holds with equality for small dimensions ($d \leq 4$ [Pfeifle 04]) and for small corank ($n \leq d + 2$ [Gärtner et al. 01]). Here we prove that it is *not* tight in general: in dimension $d = 6$, a polytope with $n = 9$ facets can have $M_{\mathrm{ubt}}(6, 9) = 30$ vertices, but not more than $M(6, 9) \leq 29$ vertices can lie on a strictly increasing edge-path.

The proof involves classification results about neighborly polytopes of small corank, Kalai's [Kalai 88] concept of abstract objective functions, the Holt-Klee conditions [Holt and Klee 98], explicit enumeration, Welzl's extended Gale diagrams [Welzl 01], and randomized generation of instances, as well as nonrealizability proofs via a version of the Farkas lemma.

1. INTRODUCTION

In an attempt to understand the worst-case behaviour of the simplex algorithm for linear programming, Motzkin [Motzkin 57], in 1957, considered the maximal number $M_{\mathrm{ubt}}(d, n)$ of facets that a d-polytope with n vertices could have and claimed that the maximum is given by the cyclic d-polytopes $C_d(n)$ with n vertices; by polarity, $M_{\mathrm{ubt}}(d, n)$ is the maximal number of vertices for a simple d-polytope with n facets.

Motivated by the same problem, Klee [Klee 65], in 1965, asked for the maximal number $M(d, n)$ of vertices that could lie on a *monotone path* (that is, an edge-path that is strictly monotone with respect to a linear objective function) on a d-polytope with n facets.

Motzkin's claim was substantiated by McMullen [McMullen 70] in 1970. It seems that traditionally McMullen's result, the *Upper Bound Theorem*, was also

taken as a solution to Klee's question, the *Monotone Upper Bound Problem*. However, a priori it is only clear that for all $n > d \geq 2$ one has an inequality

$$M(d, n) \ \leq \ M_{\mathrm{ubt}}(d, n),$$

but it is not at all clear that equality always holds, that is, that for all $n > d \geq 2$ one can construct a simple dual-to-neighborly d-polytope with n facets that admits a monotone Hamilton path:

> . . . to answer the original question, we must find out whether both bounds above can be *simultaneously* tight: It is not clear that for a dual of a cyclic polytope, there can be a monotone path through all, or most, of the vertices. [Ziegler 01, page 1201]

Thus, in [Ziegler 98, Problems 3.11* and 8.41*], it was explicitly asked: *how large is $M(d, n)$? Does it coincide with $M_{\mathrm{ubt}}(d, n)$?*

The quest for "bad examples" for the simplex algorithm equipped with specified pivot rules has led to exponential lower bounds for $M(d, n)$. The most prominent one is $M(d, 2d) \geq 2^d$, as seen from the famous Klee-Minty cubes [Klee and Minty 72]. We refer to [Amenta and Ziegler 98] for a summary of such lower bounds, formulated in the framework of "deformed products." However, these lower bounds are not tight in general: for example, for $d = 4$ and $n = 8$, the Klee-Minty cubes yield $16 \leq M(4, 8) \leq M_{\mathrm{ubt}}(4, 8) = 20$, while indeed $M(4, 8) = 20$.

In dimension 5, we have $M(5, 7) = 12$ according to [Gärtner et al. 01]. That is, there is a realization of $C_5(7)^\Delta$ (which is combinatorially equivalent to a product of a triangle with a tetrahedron) with a monotonely increasing path through all 12 vertices. Moreover, after a projective transformation, we may assume that the path is monotone with respect to a linear function φ whose unique minimum has value 0 and whose unique maximum has value 1. Thus, a deformed prism according to [Amenta and Ziegler 98, Theorem 4.3] will have 24 vertices and a monotone path that visits all of them, proving that $24 \leq M(6, 9) \leq M_{\mathrm{ubt}}(6, 9) = 30$.

Recently the challenge has been taken up, and it has been proved that the answer to the second question in the Monotone Upper Bound Problem is "yes," that is, $M(d, n) = M_{\mathrm{ubt}}(d, n)$ does hold,

- for small dimensions, $d \leq 4$ ([Pfeifle 04]), and

- for small corank, $n - d \leq 2$ ([Gärtner et al. 01]).

In the first case, an interesting aspect is that the result cannot be achieved on dual-to-cyclic polytopes; more general dual-to-neighborly polytopes are needed. (These had been missed by Motzkin). The key to the second result is Welzl's concept of "extended Gale diagrams;" this will be crucial for our work as well.

In this paper, we give a detailed analysis of some cases of corank $n - d = 3$. The main result is that the answer to the Monotone Upper Bound Problem is "no" in general:

$$27 \ \leq \ M(6, 9) \ < \ M_{\mathrm{ubt}}(6, 9) \ = \ 30.$$

Our analysis depends on a combination of a number of different techniques and results:

- Any polytope with $M_{\mathrm{ubt}}(d, n)$ vertices is necessarily simplicial dual-to-neighborly. If $n = d + 3$ and d is even, then it must be dual-to-cyclic.

- The graphs $G = G(C_d(n)^\Delta)$ of dual-to-cyclic polytopes are given by Gale's evenness criterion. For even d, $C_d(n)^\Delta$ has a dihedral symmetry group of order $2n$.

- Any linear objective function in general position induces an acyclic orientation on G, which is an "Abstract Objective Function" (AOF) as introduced by Kalai, and satisfies the Holt-Klee (HK) conditions. Moreover, in our case, it must induce (and be given by) a directed Hamilton path in the graph.

- The symmetry classes of Hamilton paths that induce HK-AOFs are enumerated by computer.

- In terms of Welzl's "extended Gale diagrams," the realizability problem for Hamilton HK-AOFs is reformulated as a problem of three-dimensional Euclidean geometry.

- To prove that some of the Hamilton HK-AOFs of interest are indeed realizable, we use randomized generation methods.

- To prove nonrealizability of AOFs, we use a combinatorial technique that may be seen as an oriented matroid version (looking at signs only) of the Farkas lemma; to obtain short proofs, we have implemented automatic search techniques.

Our main findings may be summarized as follows.

Theorem 1.1. *Let $n = d + 3$, $d \geq 2$. Then a d-polytope with $M_{\mathrm{ubt}}(d, n)$ vertices is necessarily dual-to-neighborly; if d is even, then it is dual-to-cyclic. Hamilton HK-AOFs on such polytopes can be classified as follows:*

$d = 4$, $n = 7$: *There are 7 equivalence classes of Hamilton HK-AOFs on the graph of $C_4(7)^\Delta$; exactly 4 of them are realizable.*
In particular, $M(4,7) = M_{\mathrm{ubt}}(4,7) = 14$. Moreover, already for $d = 4$ and $n = 7$ there are nonrealizable HK-AOFs. (These are the smallest possible parameters.)

$d = 5$, $n = 8$: *There are two types of simplicial dual-to-neighborly polytopes; for the dual-to-cyclic one, realizable types of Hamilton HK-AOFs exist.*
In particular, $M(5,8) = M_{\mathrm{ubt}}(5,8) = 20$.

$d = 6$, $n = 9$: *There are 6 equivalence classes of Hamilton HK-AOFs on the graph of $C_6(9)^\Delta$; none of them are realizable.*
In particular, $M(6,9) < M_{\mathrm{ubt}}(6,9) = 30$.

2. THE COMBINATORIAL MODEL

If a d-polytope with $d + 3$ vertices is supposed to have the maximal number $M_{\mathrm{ubt}}(d, d + 3)$ of facets, then it must be simplicial and neighborly. Thus, by polarity, we are looking at simple dual-to-neighborly d-polytopes with $d + 3$ facets.

The analysis of such polytopes P is a classical application of Gale diagrams by Perles [Grünbaum 03, Section 6.2]. It yields that if $d \geq 2$ is even, then the combinatorial type of P is uniquely that of the polar $C_d(d+3)^\Delta$ of the cyclic d-polytope with $d+3$ vertices. For odd $d \geq 3$, more combinatorial types of simple polytopes exist; for $d = 3$ as well as for $d = 5$, there is exactly one combinatorial type in addition to the dual-to-cyclic polytope (see [Altshuler and McMullen 73]).

The following yields our combinatorial model for the orientations of the graph of P that may be induced by linear objective functions (on some realization of P):

Definition 2.1. On the graph of a simple d-polytope P, let $\mathcal{O}$ be an acyclic orientation that has a unique source and sink.

(a) The orientation $\mathcal{O}$ is an *AOF orientation* of P if it has a unique sink in each non-empty face of P. In this case, $\mathcal{O}$ also has a unique source in each non-empty face ([Kalai 88], [Joswig et al. 02]). The orientation is then said to satisfy the *AOF condition*. Any linear extension of an AOF orientation is called an *abstract objective function (AOF)* on the vertices of P.

(b) The orientation $\mathcal{O}$ satisfies the *Holt-Klee conditions* (or is an *HK orientation*) if, in each k-dimensional face of P with $3 \leq k \leq d$, it admits k vertex-disjoint directed paths between the unique source and sink.

(c) The orientation $\mathcal{O}$ is an *HK-AOF orientation* if it satisfies (a) and (b) and is a *Hamilton HK-AOF orientation* if it additionally admits a directed Hamilton path from source to sink.

Any linear function in general position (that is, such that no two vertices have the same value) induces an AOF orientation on the graph of P; any such orientation is in fact an HK orientation ([Holt and Klee 98]). The negative of the linear function induces the opposite AOF orientation. Any Hamilton AOF orientation induces a *unique* abstract objective function.

If for some linear function on a d-polytope with $d + 3$ vertices there is a monotone path through $M_{\mathrm{ubt}}(d, d + 3)$ vertices, then the polytope is simple and dual-to-neighborly, and the linear function induces a Hamilton HK-AOF. So, for our problem we have to enumerate Hamilton HK-AOFs on the graphs of dual-to-neighborly d-polytopes with $d + 3$ facets, which are in fact dual-to-cyclic in the case of even dimension.

Proposition 2.2.

(a) *The graph of $C_4(7)^\Delta$ admits exactly 7 equivalence classes (with respect to symmetries of $C_4(7)^\Delta$ and global orientation reversal) of Hamilton HK-AOFs; they are displayed in Figure 1.*

(b) *The polytope $C_5(8)^\Delta$ admits exactly 1298 equivalence classes of Hamilton HK-AOFs.*

(c) *The polytope $C_6(9)^\Delta$ admits exactly 6 equivalence classes of Hamilton HK-AOFs; they are displayed in Figure 2.*

Sketch of proof: We enumerate the symmetry classes of directed Hamilton paths in the graph G of one of these polytopes, but we prune the search tree whenever the orientation induced by the partial path fails to satisfy the AOF or Holt-Klee conditions.

As an additional pruning criterion, we keep a list $\mathcal{L}_F$ of *all* HK-AOF orientations for each k-face F of P for some $3 \leq k \leq \dim(P)$. Whenever we try to add a new oriented edge e to a partial Hamilton path in G, we check in all lists $\{\mathcal{L}_F : e \in F\}$ belonging to k-faces incident
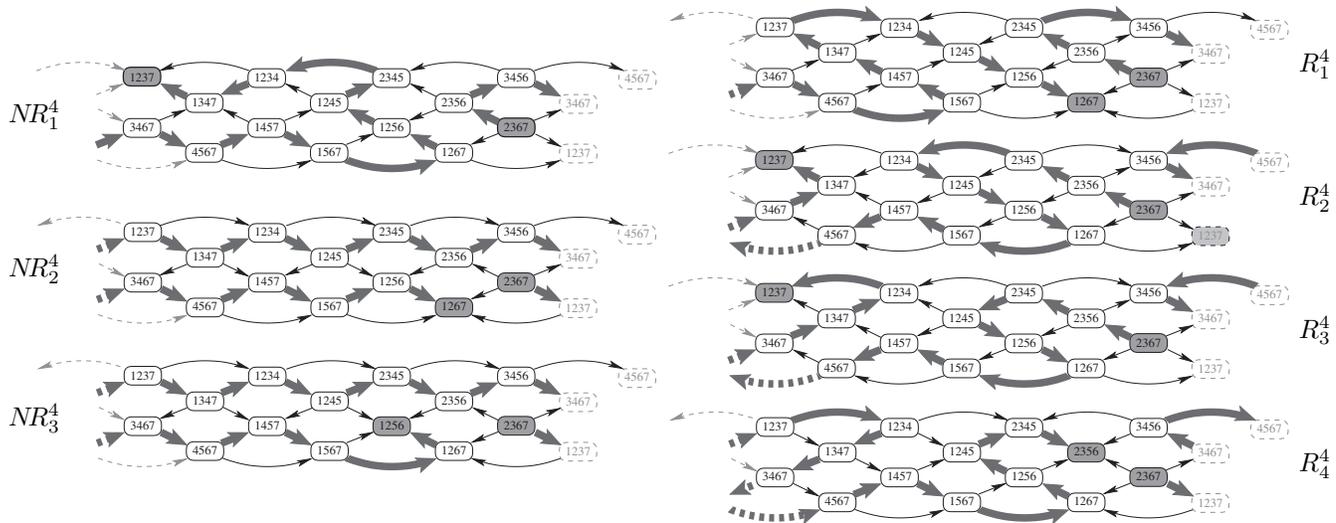
FIGURE 1. The Hamilton HK-AOFs of the graph G of $C_4(7)^{\Delta}$. (The graph G embeds into a Möbius strip [Haase and Ziegler 02].) Each vertex is labeled by its set of incident facets, which corresponds to a facet of $C_4(7)$. The bold arrows yield the monotone Hamilton paths from source to sink. An arrow $v \rightarrow w$ means that w is higher than v; so, for example, NR_1^4 corresponds to

$$2367 < 2356 < 3456 < 3467 < 4567 < 1457 < 1245 < 2345 < 1234 < 1347 < 1237.$$

NR_1^6: $\quad 458 < 258 < 238 < 278 < 478 < 078 < 058 < 038 < 018 < 014 < 012 < 016 < 036 < 034 < 345 <$
$\quad\quad\quad < 234 < 347 < 147 < 127 < 167 < 678 < 367 < 567 < 056 < 456 < 256 < 236 < 123 < 125 < 145$

NR_2^6: $\quad 038 < 238 < 123 < 236 < 234 < 034 < 345 < 347 < 478 < 147 < 014 < 018 < 012 < 016 < 036 <$
$\quad\quad\quad < 367 < 167 < 678 < 567 < 056 < 256 < 456 < 145 < 458 < 058 < 258 < 125 < 127 < 278 < 078$

NR_3^6: $\quad 038 < 238 < 236 < 036 < 016 < 056 < 256 < 567 < 367 < 167 < 678 < 078 < 278 < 478 < 147 <$
$\quad\quad\quad < 127 < 123 < 012 < 125 < 258 < 058 < 458 < 456 < 145 < 345 < 347 < 234 < 034 < 014 < 018$

NR_4^6: $\quad 038 < 238 < 236 < 036 < 016 < 056 < 256 < 567 < 367 < 167 < 678 < 278 < 078 < 478 < 147 <$
$\quad\quad\quad < 127 < 123 < 012 < 125 < 258 < 058 < 458 < 456 < 145 < 345 < 347 < 234 < 034 < 014 < 018$

NR_5^6: $\quad 038 < 058 < 258 < 125 < 256 < 056 < 456 < 458 < 145 < 345 < 034 < 234 < 347 < 147 < 014 <$
$\quad\quad\quad < 018 < 012 < 016 < 036 < 236 < 367 < 567 < 167 < 678 < 478 < 078 < 278 < 127 < 123 < 238$

NR_6^6: $\quad 018 < 058 < 458 < 258 < 125 < 012 < 127 < 278 < 078 < 038 < 238 < 123 < 234 < 034 < 345 <$
$\quad\quad\quad < 347 < 478 < 147 < 014 < 145 < 456 < 256 < 236 < 036 < 367 < 678 < 567 < 167 < 016 < 056$

FIGURE 2. Representatives for the six equivalence classes of Hamilton HK-AOFs on $C_6(9)^{\Delta}$. Each vertex $\boldsymbol{p}$ is given by the three-element set $N_{\boldsymbol{p}}$ of the indices of facets *not* incident to it.

to e whether there still exists an HK-AOF orientation containing e and discard all other orientations of that k-face.

This strategy was implemented in C++ within the `polymake` programming environment by Gawrilow and Joswig [Gawrilow and Joswig 98–03, Gawrilow and Joswig 00]; this produced the results listed above. □

3. EXTENDED GALE DIAGRAMS

Welzl's *extended Gale diagram* [Welzl 01, Gärtner et al. 01] encodes the values of a linear objective function on a d-dimensional polytope with n facets into an $(n-d)$-dimensional diagram. For this, we start from a sequence $(\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n, \boldsymbol{g})$ of points in $\mathbb{R}^d$: the $\boldsymbol{w}_i$'s represent the n facet-defining hyperplanes $\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{w}_i^T \boldsymbol{x} = 1\}$,
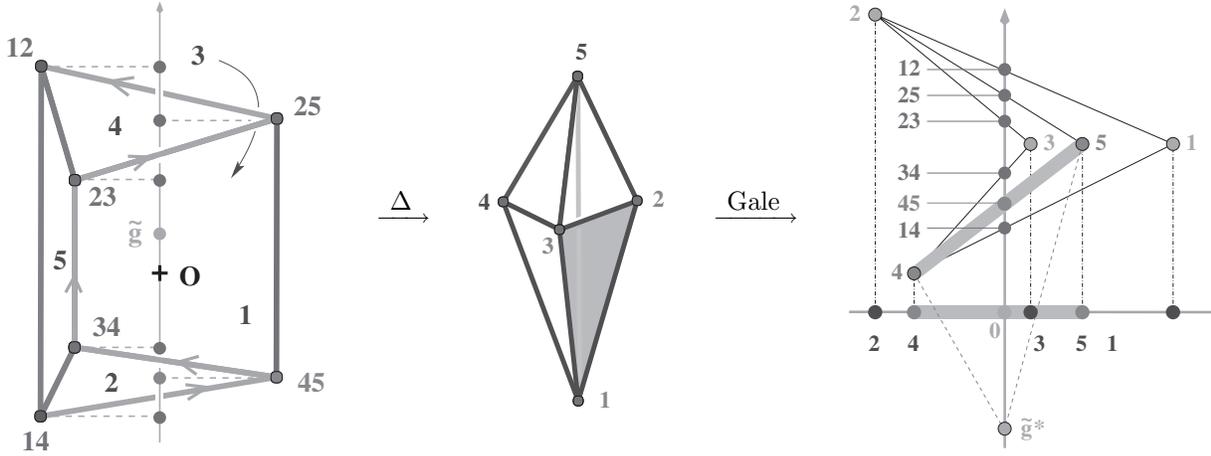
FIGURE 3. An instance of the extended Gale diagram. Left: A simple polytope P whose vertices are labeled with the facets they are *not* incident to and the ordering $14 < 45 < 34 < 23 < 25 < 12$ of the vertices induced by the linear objective function $\tilde{\boldsymbol{g}}^T$. Middle: The simplicial polar polytope P^Δ, whose vertices are labeled like the corresponding facets of P. Right: On the base line, a Gale transform of the vertices of P^Δ: complements of facets of P^Δ correspond to positive circuits (minimal linear dependencies) of $(\text{vert } P^\Delta)^*$. Adding $\tilde{\boldsymbol{g}}$ results in a lifting of the Gale transform such that the intersection heights for facet complements of P^Δ encode the ordering of the vertices of P by $\tilde{\boldsymbol{g}}^T$.

$i \in \{1, 2, \ldots, n\}$, of a full-dimensional polytope $P \subset \mathbb{R}^d$ with $\boldsymbol{0} \in \text{int } P$, and $\boldsymbol{g} \in \mathbb{R}^d$ encodes a linear objective function $\boldsymbol{g}^T \in (\mathbb{R}^d)^*$.

With this interpretation of the input, the extended Gale diagram produces a sequence $(\boldsymbol{w}_1^*, \boldsymbol{w}_2^*, \ldots, \boldsymbol{w}_n^*, \tilde{\boldsymbol{g}}^*)$ of $n + 1$ labeled vectors in $\mathbb{R}^{n-d}$ that encodes both the face lattice of P and the orientation $\mathcal{O}_{\boldsymbol{g}}$ of the graph of P induced by $\boldsymbol{g}^T$. It is calculated as follows:

(1) Replace $\boldsymbol{g}$ by some positive scalar multiple $\tilde{\boldsymbol{g}} = c\boldsymbol{g}$ such that $\tilde{\boldsymbol{g}}^T\boldsymbol{x} < 1$ for all $\boldsymbol{x} \in P$; equivalently, $\tilde{\boldsymbol{g}} \in \text{int } P^\Delta$.

(This step is optional, and will be modified later. In Welzl's original version of extended Gale diagrams, it ensures that the "lifting heights" defined below can be made positive.)

(2) Calculate the standard Gale transform $(\boldsymbol{w}_1^*, \boldsymbol{w}_2^*, \ldots, \boldsymbol{w}_n^*, \tilde{\boldsymbol{g}}^*)$ of the point sequence $(\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n, \tilde{\boldsymbol{g}})$.

Definition 3.1. Let $P = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{w}_i^T\boldsymbol{x} \leq 1, \ 1 \leq i \leq n\}$ be a polytope, let $\mathcal{A}_{\boldsymbol{g}} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n, \boldsymbol{g}) \subset \mathbb{R}^d$ be the sequence of its facet normal vectors augmented by $\boldsymbol{g}$, and let $\mathcal{A}_{\boldsymbol{g}}^* = (\boldsymbol{w}_1^*, \boldsymbol{w}_2^*, \ldots, \boldsymbol{w}_n^*, \tilde{\boldsymbol{g}}^*) \subset \mathbb{R}^{n-d}$ be the extended Gale diagram of this sequence, whose rows form a basis for the space of affine dependencies among the columns of $\mathcal{A}_{\boldsymbol{g}}$.

For every vertex $\boldsymbol{p}$ of P let $N_{\boldsymbol{p}} \subset \{1, 2, \ldots, n\}$ index the $\boldsymbol{w}_i$ that correspond to the facets of P that are not incident to $\boldsymbol{p}$. The *intersection height* $z_{\boldsymbol{p}}$ of $\boldsymbol{p}$ is $z_{\boldsymbol{p}} = -(\tilde{\boldsymbol{g}}^*)^T\boldsymbol{z}_{\boldsymbol{p}}$, where $\boldsymbol{z}_{\boldsymbol{p}} = \mathbb{R}\tilde{\boldsymbol{g}}^* \cap \text{conv}\{\boldsymbol{w}_i^* : i \in N_{\boldsymbol{p}}\}$ is the intersection point of the line $\mathbb{R}\tilde{\boldsymbol{g}}^*$ with the convex hull of the $\boldsymbol{w}_i^*$'s indexed by $N_{\boldsymbol{p}}$. (See Figures 3 and 6.)

Observation 3.2. After a linear transformation we may assume that $\tilde{\boldsymbol{g}}^* = (0, 0, \ldots, 0, -1)$. The intersection height $z_{\boldsymbol{p}}$ of a vertex $\boldsymbol{p}$ is then given as the last coordinate of the point where the $(n - d)$-axis meets the affine plane $H_{\boldsymbol{p}}$ through the points $\{\boldsymbol{w}_i^* : i \in N_{\boldsymbol{p}}\}$.

Proposition 3.3. *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be vertices of P. Then $\boldsymbol{q}$ is higher than $\boldsymbol{p}$ with respect to the linear objective function given by $\boldsymbol{g}$,*

$$\boldsymbol{g}^T\boldsymbol{p} \ < \ \boldsymbol{g}^T\boldsymbol{q},$$

if and only if the intersection height of $\boldsymbol{q}$ is larger than that of $\boldsymbol{p}$,

$$z_{\boldsymbol{p}} \ < \ z_{\boldsymbol{q}}.$$

Example 3.4. Let P be a triangular prism in $\mathbb{R}^3$ (see Figure 3, left) with $n = 5$ facets. The polar P^Δ is the polytope of Figure 3 (middle) with 5 vertices, and the Gale transform of P^Δ consists of 5 points in $\mathbb{R}^{5-3-1} = \mathbb{R}^1$ (Figure 3, right, base line). We obtain the extended Gale diagram in $\mathbb{R}^2$ by additionally encoding a linear objective function via a level hyperplane that does not

intersect P, which corresponds to a point in the relative interior of P^Δ. Proposition 3.3 says that, in the extended Gale diagram, the value $\boldsymbol{g}^T\boldsymbol{p}$ of the objective function is encoded by the height of the intersection of $\mathbb{R}\tilde{\boldsymbol{g}}^*$ with the triangle spanned by the points $\boldsymbol{w}_i^*$ that correspond to facets of P that do not contain $\boldsymbol{p}$.

4. FINDING REALIZATIONS

Proposition 4.1.

(a) *The equivalence classes R_1^4–R_4^4 of Hamilton HK AOF orientations of the graph of $C_4(7)^\Delta$ (as given by Figure 1) are realizable. In particular, $M(4,7) = M_{\mathrm{ubt}}(4,7) = 14$.*

(b) *There exist realizable Hamilton HK AOF orientations of the graph of $C_5(8)^\Delta$.*
In particular, $M(5,8) = M_{\mathrm{ubt}}(5,8) = 20$.

(c) *There exist realizations of $C_6(9)^\Delta$ with 27 vertices on a monotone path.*

Sketch of proof: The realizations in (a) and (b) were found by the following procedure. For each polytope $P = C_d(d+3)^\Delta$, randomly generate a Gale transform $\mathcal{G}(P^\Delta) = (\boldsymbol{v}_1^*, \boldsymbol{v}_2^*, \ldots, \boldsymbol{v}_{d+3}^*)$ of P^Δ, and, for each vertex $\boldsymbol{p}$ of P, express the intersection height $z_{\boldsymbol{p}}$ as a linear function of the *lifting heights* h_i, where $\big(\boldsymbol{w}_i^* = (\boldsymbol{v}_i^*, h_i) : 1 \le i \le d+3\big)$ is an extended Gale transform of P. Now check whether the linear program

$$z_{\boldsymbol{p}} - z_{\boldsymbol{q}} \le -1, \qquad \text{for all oriented edges } e = (\boldsymbol{p}, \boldsymbol{q}) \text{ in } \mathcal{O},$$

in the variables $h_1, h_2, \ldots, h_{d+3}$ is feasible, for $\mathcal{O}$, one of the Hamilton HK AOF-orientations of P. If so, the polar dual of the Gale transform of $\mathcal{G}(P^\Delta)$ yields a realization of the combinatorial type of P, and the lifting heights solving the linear program yield a linear objective function that induces the orientation $\mathcal{O}$ on this realization. If not, repeat.

The realizations in (c) were found by lifting randomly generated instances $\{v_1^*, v_2^*, \ldots, v_9^*\}$ of Gale diagrams of $C_6(9)$ to the rational surface $h = \prod_{i=1}^9 1/\ell_i(\boldsymbol{x})$, where $\ell_i(\boldsymbol{x})$ is the equation of the line $\mathbb{R}(v_i^* - v_{i+1}^*)$ for $1 \le i \le 9$, where $v_{10}^* := v_1^*$. $\qquad \square$

Remark 4.2. Following a suggestion of Walter Morris, we also searched for long ascending paths on the product $P = \Delta^2 \times \Delta^2 \times \Delta^2$ of three triangles. This is a simple six-dimensional polytope with 9 facets, 27 vertices, and 81 edges. Its graph admits many Hamilton paths that induce HK-AOF orientations ($800{,}928$ in total, 4944 of them beginning on a given directed edge), but we have not been able to find realizations of P with more than 25 vertices on an ascending path.

5. PROVING NONREALIZABILITY

Our strategy for proving nonrealizability of orientations may be summarized as follows: for each candidate orientation $\mathcal{O}$ of the graph of a polytope P (of even dimension d, with $d+3$ facets), we assume that there is a realization of P and a linear objective function $\boldsymbol{g}^T$ that induces $\mathcal{O}$ on the graph of P. Each oriented edge of $\mathcal{O}$ then imposes a linear inequality on the lifting heights of the extended Gale diagram of $(P, \boldsymbol{g}^T)$. For some orientations $\mathcal{O}$, a combinatorial version of the Farkas Lemma implies that these inequalities are inconsistent, thereby proving the nonrealizability of $\mathcal{O}$.

5.1 Inequalities Induced by Edges

We start with some notation for vector configurations in $\mathbb{R}^2$ and $\mathbb{R}^3$. The shorthand $[d+3]$ will denote $\{1, 2, \ldots, d+3\}$.

Convention 5.1. For $i \in [d+3]$, we write $\boldsymbol{i}$ for a vector $(x_i, y_i)^T \in \mathbb{R}^2$ and $\boldsymbol{i}^\perp$ for the vector $(y_i, -x_i)^T$ orthogonal to $\boldsymbol{i}$ that is obtained by rotating $\boldsymbol{i}$ in the clockwise direction. With this convention, the following relations hold for scalar products:

$$\begin{aligned} \boldsymbol{i}\boldsymbol{j}^\perp &= x_i y_j - x_j y_i \\ &= \det(\boldsymbol{i}, \boldsymbol{j}) \\ &= -\det(\boldsymbol{j}, \boldsymbol{i}) \\ &= -\boldsymbol{j}\boldsymbol{i}^\perp \\ &= -\boldsymbol{i}^\perp\boldsymbol{j}. \end{aligned}$$

We further abbreviate

$$\begin{aligned} ij^\perp &:= \operatorname{sign}(\boldsymbol{i}\boldsymbol{j}^\perp), \\ [\boldsymbol{i}\boldsymbol{j}\boldsymbol{k}] &:= \det\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \end{pmatrix}, \text{ and} \\ [ijk] &:= \operatorname{sign}([\boldsymbol{i}\boldsymbol{j}\boldsymbol{k}]). \end{aligned}$$

Lemma 5.2.

(a) *If $\boldsymbol{i}, \boldsymbol{i} + \boldsymbol{j}, \boldsymbol{j} \in \mathbb{R}^2$ come in counterclockwise order around 0, then $ij^\perp = +$.*

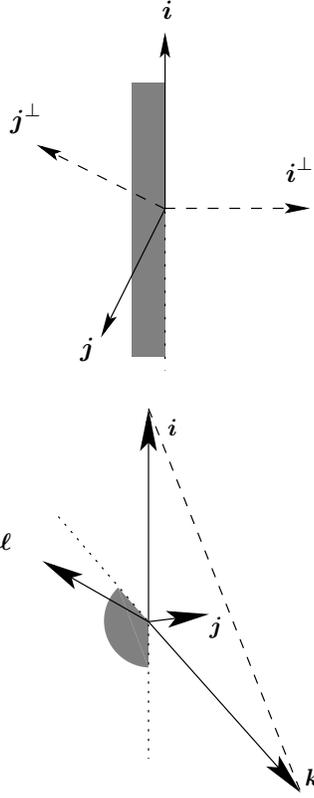**FIGURE 4**. Deducing sign patterns.
Top: If $i, i + j, j \in \mathbb{R}^2$ come in counterclockwise order around 0, then $ij^\perp = +$.
Bottom: In this situation, if $[ijk] = +$, then $[i\ell j] = [j\ell k] = +$.

(b) *If in a configuration of four vectors $i, j, k, \ell \in \mathbb{R}^2 \setminus \{0\}$ the vectors i, j, k are ordered clockwise around 0, $j \in \mathrm{relint}\,\mathrm{cone}(i, k)$, $[ijk] = +$, and $\ell \in \mathrm{relint}\,\mathrm{cone}(-i, -k)$, then $[i\ell j] = [j\ell k] = +$.*

Convention 5.3. The vertices of $C_d(d + 3)$ are labeled by $[d + 3]$ in the natural order, so that the facets are given by certain d-subsets of $[d + 3]$ according to Gale's evenness criterion. The vectors in any Gale transform are then labeled so that $\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{7}, \ldots, \mathbf{2}, \mathbf{4}, \mathbf{6}, \ldots$ come in clockwise order around the origin. We identify each facet of $C_d(d + 3)$ with the indices of the three vertices it misses, so that ordering this index set yields a triangle with counterclockwise orientation that encloses the origin (see Figure 5).

Now we polarize. Correspondingly, we label each vertex $\boldsymbol{p}$ of $C_d(d+3)^\Delta$ by the three-element set $N_{\boldsymbol{p}}$ of (indices of) the facets it does *not* lie on.
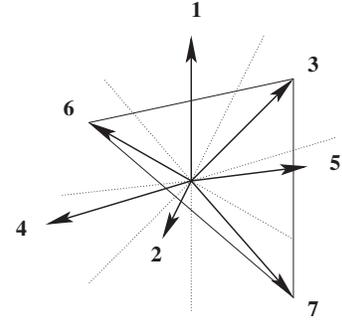

FIGURE 5. A Gale transform of $C_4(7)$. The set $N_{\boldsymbol{p}} = \{3, 6, 7\}$ corresponds to the vertex $\boldsymbol{p}$ of $C_4(7)^\Delta$ not on those facets, and $3 < 6 < 7$ is a counterclockwise orientation of the triangle 367.

Lemma 5.4. *Let $\mathcal{G}(P)$ be an extended Gale diagram of $P = C_d(d+3)^\Delta$, and let $N_{\boldsymbol{p}} = \{i, j, k\}$ index a vertex $\boldsymbol{p}$ of P. With the assumptions of Observation 3.2 and Convention 5.1, the intersection height $z_{\boldsymbol{p}} = z_{\{i,j,k\}}$ is given by*

$$z_{\{i,j,k\}} = \frac{ij^\perp h_k + ki^\perp h_j + jk^\perp h_i}{[ijk]}. \qquad (5\text{–}1)$$

Proof: Expand the third row of the determinant in the equation

$$\begin{vmatrix} 0 & x_i & x_j & x_k \\ 0 & y_i & y_j & y_k \\ z_{\{i,j,k\}} & h_i & h_j & h_k \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

As a consistency check, note that (5–1) is symmetric under any permutation of the indices. $\qquad \square$

By Proposition 3.3, the total ordering of the vertices $\boldsymbol{p}$ of $C_d(d+3)^\Delta$ induced by the linear objective function $\boldsymbol{g}^T$ induces a total ordering of the intersection heights $z_{\boldsymbol{p}}$, that is, of the heights of the intersections of the affine hyperplanes $H_{\boldsymbol{p}}$ in $\mathbb{R}^3$ with the z-axis. If two vertices of $C_d(d+3)^\Delta$ span an edge, then the corresponding facets of $C_d(d+3)$ share a ridge, which in turn means that the corresponding triangles have two points $\boldsymbol{w}_i^*$ and $\boldsymbol{w}_j^*$ in common. This permits us to relate the intersection heights of two adjacent vertices in the graph of $C_d(d+3)^\Delta$ in the following way.

Lemma 5.5. *Suppose that the vertices $\{i, j, k\}$ and $\{i, j, \ell\}$ span an edge of $C_d(d + 3)^\Delta$. Then the following relation*
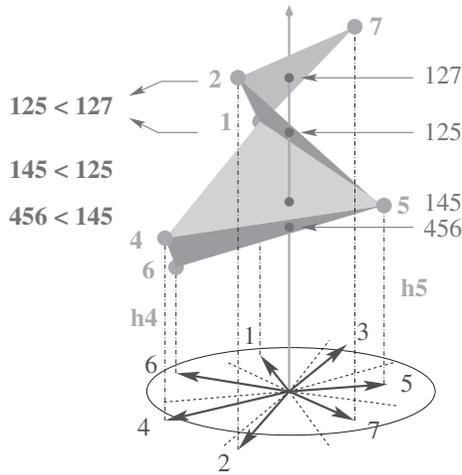
FIGURE 6. Intersection heights encode values of the objective function. Suppose that the objective function $\tilde{g}$ orders four vertices of $C_4(7)$ by $1237 < 2367 < 3467 < 3456$. Then the heights of the intersections between $\mathbb{R}\tilde{g}^*$ and the lifted triangles corresponding to the complements of these labels are ordered $z_{456} < z_{145} < z_{125} < z_{127}$.

holds between the corresponding intersection heights:

$$z_{\{i,j,k\}} - z_{\{i,j,\ell\}} = \frac{(ij^\perp)[jk\ell]}{[ijk][ij\ell]}h_i + \frac{(ij^\perp)[ki\ell]}{[ijk][ij\ell]}h_j$$
$$+ \frac{ij^\perp}{[ijk]}h_k + \frac{-ij^\perp}{[ij\ell]}h_\ell.$$

If $[ijk] = [ij\ell]$, then the signs of the coefficients of the h's are, in this order,

$$(ij^\perp)[jk\ell], \quad (ij^\perp)[ki\ell], \quad +, \quad and \quad -.$$

Proof: The first statement follows via direct calculation from (5–1), using the straightforward identity

$$(ij^\perp)(k\ell^\perp) = (\ell i^\perp)(jk^\perp) + (j\ell^\perp)(ik^\perp).$$

The second statement is a consequence of Lemma 5.2 and Convention 5.3. $\qquad\square$

5.2 Contradictions via a Combinatorial Farkas Lemma

We will use a combinatorial version of the following Farkas Lemma [Schrijver 86, Section 7.8]:

Lemma 5.6. *For any matrix $A \in \mathbb{R}^{m \times d}$, exactly one of the following is true:*

- *There exists an $h \in \mathbb{R}^d$ such that $Ah < 0$.*

- *There exists a $c \in \mathbb{R}^m$ such that $c \geq 0$, $c^T A = 0$, and $c \neq 0$.*

Given a d-dimensional polytope P with $d + 3$ facets and an orientation $\mathcal{O}$ on P's graph G, we assume that we have a realization of P and a linear objective function g^T that induces $\mathcal{O}$ on G. We would like to apply Lemma 5.6 to prove the infeasibility of the system $Ah < 0$ of $m = $ # edges of $C_d(d+3)^\Delta = \frac{1}{4}\binom{d+4}{3}$ linear inequalities on the lifting heights $h_1, h_2, \ldots, h_{d+3}$ given by

$$z_{\{i,j,k\}} - z_{\{i,j,\ell\}} < 0 \quad \text{for all oriented edges} \atop (\{i,j,k\},\{i,j,\ell\}) \text{ of } \mathcal{O}. \tag{5–2}$$

However, the only information we have available about A are sign patterns of determinants as given by Lemma 5.5. Therefore, to show infeasibility of (5–2), we must produce a Farkas certificate c that shows already at the level of signs ("using only oriented matroid information") that some positive combination of the rows of A sums to zero.

Proposition 5.7. *The orientation*

$$NR_1^4: \quad z_{145} < z_{147} < z_{127} < z_{125} < z_{123} < z_{236}$$
$$< z_{234} < z_{345} < z_{347} < z_{367} < z_{167}$$
$$< z_{567} < z_{256} < z_{456}$$

of the graph of $C_4(7)^\Delta$ is not realizable.

Proof: We abbreviate '$z_{\{i,j,k\}} < z_{\{i,j,\ell\}}$' by '$ijk < ij\ell$'.

To any extended Gale diagram corresponding to a realization of NR_1^4 we may apply an affine transformation that fixes the z-axis and moves the plane spanned by w_3^*, w_4^*, and w_5^* to the $\mathbb{R}^2$-plane given by $z = 0$; that is, we may assume that $h_3 = h_4 = h_5 = 0$. This affine transformation does not change the projection along the z-axis, which still yields the same Gale transform of $C_4(7)$. The resulting configuration is the extended Gale diagram for $C_4(7)^\Delta$ with the objective function $\tilde{g} = cg$ scaled such that the level hyperplane $\tilde{g}^T x = 1$ contains $p = \{3,4,5\}$. Thus, at this point we have modified Step (1) in the construction of Section 3.

We proceed to write down the sign patterns of the inequalities $Ah < 0$ for $h = (h_1, h_2, h_6, h_7)$ implied by Lemmas 5.4 and 5.5:

	h_1	h_2	h_6	h_7	$i\ j\ k\ \ell$
$567 < 256$:	0	$-$	$-[257]$	$+$	5 6 7 2
$234 < 345$:	0	$+$	0	0	3 4 2 5
$345 < 456$:	0	0	$-$	0	4 5 3 6
$345 < 347$:	0	0	0	$-$	3 4 5 7

If $[257] = -$ or $[257] = 0$, we can find a positive combination of the rows of this matrix that sums to zero, regardless of the actual values of the entries. Therefore, $[257] = +$ in any realization of NR_1^4. By Lemma 5.2, we deduce that, therefore, $[157] = -$.

Now consider the rows

	h_1	h_2	h_6	h_7	$i\ j\ k\ \ell$
$127 < 125$:	$-[257] = -$	$[157] = -$	0	$+$	$1\ 2\ 7\ 5$
$145 < 345$:	$+$	0	0	0	$4\ 5\ 1\ 3$
$234 < 345$:	0	$+$	0	0	$3\ 4\ 2\ 5$
$345 < 347$:	0	0	0	$-$	$3\ 4\ 5\ 7$,

which admit a positive combination that sums to zero and, therefore, prove the nonrealizability of the orientation NR_1^4. □

Remark 5.8. Proposition 5.7 provides an example of a nonrealizable abstract objective function that satisfies the Holt–Klee conditions, on a simple 4-polytope with only 7 facets. The first examples for this were obtained on a 7-dimensional polytope with 9 facets, by Gärtner et al. [Gärtner et al. 01]; Morris [Morris 02] obtained examples on the 4-cube (with 8 facets). No such examples of dimension $d \le 3$ exist ([Mihalisin and Klee 00]).

Proposition 5.9. *No Hamilton HK-AOF of $C_6(9)^\Delta$ is realizable.*

Proof: The reasoning is analogous to the proof of Proposition 5.7; we will give the details only for NR_1^6, and sketch the proof for the rest of the orientations.

Suppose that we are given a realization of the polytope $C_6(9)^\Delta$ along with a linear objective function that induces NR_1^6 on its graph. After an affine transformation of the extended Gale diagram, we may suppose that $h_3 = h_4 = h_5 = 0$, where we consider the lifting heights numbered as $h_0,\ h_1,\ \ldots, h_8$.

Now consider the rows

	h_0	h_1	h_2	h_6	h_7	h_8	$i\ j\ k\ \ell$
$567 < 056$:	$-$	0	0	$-[057]$	$+$	0	$5\ 6\ 7\ 0$
$034 < 345$:	$+$	0	0	0	0	0	$3\ 4\ 0\ 5$
$345 < 456$:	0	0	0	$-$	0	0	$4\ 5\ 3\ 6$
$345 < 347$:	0	0	0	0	$-$	0	$3\ 4\ 5\ 7$,

from which we deduce, as above, that $[057] = +$; and, via Lemma 5.2, that $[578] = -$. But, now we reach a contradiction via

	h_0	h_1	h_2	h_6	h_7	h_8	$i\ j\ k\ \ell$
$078 < 058$:	$[578] = -$	0	0	0	$+$	$-[057] = -$	$0\ 8\ 7\ 5$
$034 < 345$:	$+$	0	0	0	0	0	$3\ 4\ 0\ 5$
$345 < 347$:	0	0	0	0	$-$	0	$3\ 4\ 5\ 7$
$458 < 345$:	0	0	0	0	0	$+$	$4\ 5\ 8\ 3$,

which proves the claim. Some "good" sets of vanishing heights for the remaining orientations are as follows:

Orientation:	NR_2^6	NR_3^6	NR_4^6	NR_5^6	NR_6^6
Height indices:	0,1,2	0,5,6	0,5,6	0,1,6	0,1,2

This concludes the proof. □

Proposition 5.10. *The Hamilton HK-AOFs NR_2^4 and NR_3^4 are not realizable.*

Proof: The method used in the proof of Propositions 5.7 and 5.9 does not directly apply here, as no choice of vanishing heights immediately yields a Farkas contradiction for these orientations. Therefore, we prove the nonrealizability of NR_2^4 in the following way:

Suppose that in a realization of NR_2^4, we have $[136] = +$, and therefore $[137] = -$ by Lemma 5.2. This leads to a contradiction by the following table for $h_1 = h_4 = h_5 = 0$:

	h_2	h_3	h_6	h_7
$367 < 167$:	0	$+$	$-[137] = +$	$[136] = +$
$145 < 345$:	0	$-$	0	0
$145 < 456$:	0	0	$-$	0
$145 < 147$:	0	0	0	$-$

We deduce that $[136] = -$ or $[136] = 0$ must hold in any realization of NR_2^4. But, setting $h_2 = h_3 = h_6 = 0$ then yields the following table,

	h_1	h_4	h_5	h_7
$367 < 167$:	$-$	0	0	$[136]$
$123 < 236$:	$+$	0	0	0
$367 < 236$:	0	0	0	$+$,

and a global contradiction.

The same argument proves that NR_3^4 is nonrealizable. The only difference between this orientation and NR_2^4 is that $345 < 347$ in NR_3^4, whereas $347 < 345$ in NR_2^4, but the proof of the nonrealizability of NR_2^4 did not use this inequality. □

Remark 5.11. The short nonrealizability proofs above were found by computer, though they can be checked by hand. Propositions 5.7 and 5.9 were found by trying to eliminate signs from all minors of A obtained by successively deleting triples of columns, while the proof of Proposition 5.10 was obtained by moreover assuming various signs to be positive resp. negative. We presented instances of the shortest proofs found.

6. SUMMARY AND OPEN PROBLEMS

Our strategy for proving bounds on $M(d, n)$ depends on combining theoretical concepts with computer generation/enumeration. For *lower bounds* on $M(d, n)$, we use

(1) information on the *combinatorial type* of a candidate polytope—e.g., the combinatorial type of a six-dimensional 9-facet polytope with maximally many vertices is uniquely that of $C_6(9)^\Delta$;

(2) *reduction of the dimension* via extended Gale diagrams—to dimension 3 in the case of $C_6(9)^\Delta$; and

(3) *random generation* of instances of the extended Gale diagram, and finding the longest ascending path on the induced orientation (this step involves the computer).

Our proof of the *upper bound* $M(6, 9) < 30$ uses (1), (2), and additionally

(4) the Holt-Klee and AOF properties of orientations induced by linear functions;

(5) *exhaustive enumeration* (up to symmetry) of all orientations that admit a Hamilton path and satisfy the HK and AOF conditions;

(6) a combinatorial version of the Farkas Lemma; and

(7) computer assistance to find short nonrealizability proofs.

Step (5) of our present proof depends crucially on computer assistance. Finding short proofs in step (7) also requires computer assistance, but the result can be checked by hand.

Our methods were successful for small dimensions and coranks, but they do not yield (non)existence statements or asymptotics for large d and $n - d$. Thus, we leave the following problems open for now:

- Does $C_d(d + 3)^\Delta$ have *any* Hamilton HK AOFs for even $d > 6$? If not, this would give a purely combinatorial proof that, for some parameters, $M(d, n) < M_{\text{ubt}}(d, n)$.

  What happens for odd $d \geq 7$?

- Is it true that $M(d, n) \ll M_{\text{ubt}}(d, n)$ for large $n \geq d + 3$, $d \geq 6$?

To demonstrate that the gaps in our asymptotic knowledge are substantial, we note that in the "diagonal" case of $n = 2d$ all we know is

$$\tfrac{5}{4}\, 2^d \;\leq\; M(d, 2d) \;\leq\; M_{\text{ubt}}(d, 2d) \;\approx\; 2.6^d.$$

(The lower bound follows from $M(4, 8) = 20$ by taking deformed prisms.)

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