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Rational sequences converging to π

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

Our aim is to give sequences $(\mathbf{q}_n)_{n=0}^{\infty}$ and $(\mathbf{Q}_n)_{n=0}^{\infty}$ of rational numbers such that $\mathbf{q}_n < \mathbf{q}_{n+1} < \pi < \mathbf{Q}_{k+1} < \mathbf{Q}_k \ , \quad n,k \in \mathbb{N}$. It is shown that there exists positive constants C_1, C_2 such that for n large , $|\mathbf{q}_n - \pi| < \dfrac{C_1}{2^{5n}}$ and $|\mathbf{Q}_n - \pi| < \dfrac{C_2}{n \cdot 2^{5n}}$. Let us note that both sequences are constructed by means of the same three-term recurrence relation. Likewise, two series for π are given.

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1 Introduction

For $a \in \mathbb{C}\ ,\ k \in \mathbb{N}$, let us denote $(a)_k = a(a+1)\cdots(a+k-1)\ ,\ (a)_0 = 1$. Consider the Gauss hypergeometric series ${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \dfrac{(a)_k (b)_k}{(c)_k} \dfrac{z^k}{k!}$ where $(a)_k = a(a+1)\cdots(a+k-1)\ ,\ (a)_0 = 1, (a \in \mathbb{C}\ ,\ k \in \mathbb{N})$. For instance,

$R_n^{(\alpha,\beta)}(x) = {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2)$, $\alpha > -1, \beta > -1$, is Jacobi polynomial normalized by $R_n^{(\alpha,\beta)}(1) = 1$.

In the following $\gamma \in \{-\frac{1}{4}, \frac{1}{4}\}$ and

$$a_n(\gamma) = \frac{n^2 + (n-\gamma)(2n+3-4\gamma)}{2(n-\gamma)(n-\gamma+1)}$$

$$b_n(\gamma) = \frac{n^2(n-2\gamma)^2}{4(n-\gamma)^2(2n-1-2\gamma)(2n+1-2\gamma)} .$$

Lemma 1. *If* $(\mathbf{I}_n(\gamma))_{n=0}^{\infty}$, $(\mathbf{Y}_n(\gamma))_{n=0}^{\infty}$ *are the sequences*

$$(1) \qquad \begin{cases} \mathbf{I}_n(\gamma) = \dfrac{16\gamma}{\binom{2n-2\gamma}{n}} \displaystyle\int_0^1 \frac{(1-x^2)^n x^{2n-4\gamma+1}}{(1+x^2)^{n+1}}\, dx \\[2ex] \mathbf{Y}_n(\gamma) = \dfrac{\binom{n-2\gamma}{n}}{\binom{2n-2\gamma}{n}} \displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{(n+1-2\gamma)_k}{(1-2\gamma)_k} \end{cases}$$

then $(\mathbf{I}_n(\gamma))_{n=0}^{\infty}$, $(\mathbf{Y}_n(\gamma))_{n=0}^{\infty}$ *satisfy same three-term recurrence relation , namely*

$$(2) \qquad \begin{cases} \mathbf{I}_{n+1}(\gamma) = a_n(\gamma)\mathbf{I}_n(\gamma) - b_n(\gamma)\mathbf{I}_{n-1}(\gamma) \\ \mathbf{Y}_{n+1}(\gamma) = a_n(\gamma)\mathbf{Y}_n(\gamma) - b_n(\gamma)\mathbf{Y}_{n-1}(\gamma) \end{cases} , \ n \in \mathbb{N}^* ,$$

with initial values

$$\begin{pmatrix} \mathbf{I}_0(\gamma) & \mathbf{I}_1(\gamma) \\ \mathbf{Y}_0(\gamma) & \mathbf{Y}_1(\gamma) \end{pmatrix} = \begin{pmatrix} \pi + 2(4\gamma - 1) & \dfrac{8\gamma(4-\pi) + 2(11\pi - 34)}{15} \\ 1 & \dfrac{2(11-4\gamma)}{15} \end{pmatrix} .$$

Proof. In order to prove that $(\mathbf{I}_n(\gamma))_{n=0}^{\infty}$ verifies (2) it may be used repeated integration by parts. Recurrence (2) was put in evidence for $(\mathbf{Y}_n(\gamma))_{n=0}^{\infty}$ by means of three-term relation for Jacobi polynomials and equality $\mathbf{Y}_n(\gamma) = \frac{\binom{n-2\gamma}{n}}{\binom{2n-2\gamma}{n}} R_n^{(-2\gamma,0)}(3)$.

Other forms for $\mathbf{I}_n(\gamma)$ and $\mathbf{Y}_n(\gamma)$ are

$$\mathbf{Y}_n(\gamma) = \frac{\binom{n-2\gamma}{n}}{\binom{2n-2\gamma}{n}} \sum_{k=0}^{n} \binom{n}{k}^2 \frac{2^{n-k} k!}{(1-2\gamma)_k} = \frac{1}{\binom{2n-2\gamma}{n}} \sum_{k=0}^{n} \binom{n}{k} \binom{n-2\gamma}{k} 2^k =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{n-2\gamma}{k}}{\binom{2n-2\gamma}{k}} = 2^n \frac{\binom{n-2\gamma}{n}}{\binom{2n-2\gamma}{n}} {}_2F_1\left(-n, -n; 1-2\gamma; \frac{1}{2}\right)$$

$$\mathbf{I}_n(\gamma) = \frac{8\gamma}{(2n-2\gamma+1)\binom{2n-2\gamma}{n}^2} \cdot {}_2F_1(n-2\gamma+1, n+1; n-2\gamma+2; -1) =$$

$$= \frac{8\gamma}{2^n(2n-2\gamma+1)\binom{2n-2\gamma}{n}^2} \cdot {}_2F_1(1, 1-2\gamma; n-2\gamma+2; -1) =$$

$$= \frac{16\gamma\binom{n-2\gamma}{n}}{\binom{2n-2\gamma}{n}} \int_0^1 \frac{x^{-2\gamma} R_n^{(-2\gamma,0)}(1-2x^2)}{1+x^2}\, dx.$$

Using theory of hypergeometric functions (see [1]), it may be proved that

$$\mathbf{I}_n(\gamma) = \frac{8\gamma \cdot \zeta_n(\gamma)}{2^n\,(2n-2\gamma+1)\binom{2n-2\gamma}{n}^2} \tag{3}$$

where $\zeta_n(\gamma) = 1 - \frac{1-2\gamma}{n} + \frac{8(1-2\gamma)(1-\gamma)}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \quad , \quad (n \to \infty).$

Lemma 2. *If* $\mathbf{E}_n(\gamma) := \left|\frac{\mathbf{I}_n(\gamma)}{\mathbf{Y}_n(\gamma)}\right|$, *then there exists* $n \in \mathbb{N}$ *and positive constants* C_1, C_2 *such that*

$$\mathbf{E}_n(\gamma) < \begin{cases} \dfrac{C_2}{n \cdot 2^{5n}} \quad , \quad \gamma = -\frac{1}{4} \\ \dfrac{C_1}{2^{5n}} \quad , \quad \gamma = \frac{1}{4}\ . \end{cases} \qquad \textit{for} \quad n \geq n_0.$$

Proof. Suppose that $\mathbf{I}_n(\gamma)$ and $\mathbf{Y}_n(\gamma)$, are as in (1) . Because

$$(4)\quad R_n^{(-2\gamma,0)}(3) \geq \frac{\Gamma(2n-2\gamma+1)\Gamma(1-2\gamma)}{|\Gamma(n+1-2\gamma)|^2} > \begin{cases} \dfrac{4^n}{3n} & , \quad \gamma = -\frac{1}{4} \\ \\ \dfrac{4^n}{3} & , \quad \gamma = \frac{1}{4} \,, \end{cases}$$

from (3) we find

$$\mathbf{E}_n(\gamma) = \frac{2|\zeta_n(\gamma)|}{2^n(2n-2\gamma+1)\binom{2n-2\gamma}{n}\binom{n-2\gamma}{n}R_n^{(-2\gamma,0)}(3)} \,.$$

If $n_0 \in \mathbb{N}$ is such that $|\zeta_n(\gamma)| < 2$ for $n \geq n_0$, according to (4) we find $\mathbf{E}_n(\gamma) < \delta_n(\gamma)$ where $\delta_n(\gamma) = \dfrac{4 \cdot |\Gamma(n+1)\Gamma(n+1-2\gamma)|^2}{2^n\Gamma(2n-2\gamma+1)\Gamma(2n-2\gamma+1)}$. Using log-convexity of Gamma function we have

$$\delta_n(1/4) < \frac{2\pi\sqrt{2}}{\sqrt{n}\, 2^{5n}} \quad , \quad \delta_n(-1/4) < \frac{20\pi}{2^{5n}} \quad , \quad (n \geq n_0),$$

which completes the proof.

Define $(\mathbf{X}_n(\gamma))_{n=0}^{\infty}$ by $\mathbf{X}_n(\gamma) = \pi \cdot \mathbf{Y}_n(\gamma) - \mathbf{I}_n(\gamma)$, then

$$\mathbf{X}_{n+1}(\gamma) = a_n(\gamma)\mathbf{X}_n(\gamma) - b_n(\gamma)\mathbf{X}_{n-1}(\gamma) \quad , \quad n \in \mathbb{N}^*,$$

with

$$\mathbf{X}_0(\gamma) = 2(1-4\gamma) \quad , \quad \mathbf{X}_1(\gamma) = \frac{4(17-8\gamma)}{15} \,.$$

Using the fact that above recurrences (2) are linear, we give

Lemma 3. *If* $\mathbf{Z}_n(\gamma) = \dfrac{\mathbf{X}_n(\gamma)}{\mathbf{Y}_n(\gamma)}$, *then*

$$\mathbf{Z}_{k+1}(\gamma) - \mathbf{Z}_k(\gamma) = b_k(\gamma)\frac{\mathbf{Y}_{k-1}(\gamma)}{\mathbf{Y}_{k+1}(\gamma)}\left(\mathbf{Z}_k(\gamma) - \mathbf{Z}_{k-1}(\gamma)\right) \;,\; k \in \mathbb{N}^*$$

and

$$(5)\quad \mathbf{Z}_{n+1}(\gamma) - \mathbf{Z}_n(\gamma) = \frac{16\gamma(n-\gamma+1)}{(n-2\gamma+1)^2\binom{n-2\gamma}{n}^2 R_n^{(-2\gamma,0)}(3)R_{n+1}^{(-2\gamma,0)}(3)} \,.$$

Proof. It may be seen that equalities

$$\mathbf{Z}_{n+1}(\gamma) - \mathbf{Z}_n(\gamma) = \frac{4(8\gamma+1)}{3\mathbf{Y}_n(\gamma)\mathbf{Y}_{n+1}(\gamma)} \prod_{k=1}^{n} b_k(\gamma)\ ,\ n \geq 1.$$

and $\prod_{k=1}^{n} b_k(\gamma) = \frac{1-2\gamma}{(2n-2\gamma+1)\binom{2n-2\gamma}{n}^2}$ are verified.
Note that $\mathbf{Z}_0(\gamma) = 2(1-4\gamma)$, $\mathbf{Z}_1(\gamma) = 3 + \frac{1}{12}(1-4\gamma)$, and for $n \geq 1$

$$\mathbf{Z}_{n+1}(\gamma) = (1 + c_n(\gamma))\,\mathbf{Z}_n(\gamma) - c_n(\gamma)\mathbf{Z}_{n-1}(\gamma) \quad , \quad c_n(\gamma) := b_n(\gamma)\frac{\mathbf{Y}_{n-1}(\gamma)}{\mathbf{Y}_{n+1}(\gamma)}\ .$$

Further, the sequences $(\mathbf{q}_n)_{n=1}^{\infty}$, $(\mathbf{Q}_n)_{n=1}^{\infty}$ are defined by

(6) $$\mathbf{q}_n = \mathbf{Z}_n(1/4) = \frac{\mathbf{X}_n(1/4)}{\mathbf{Y}_n(1/4)} \quad , \quad \mathbf{Q}_n = \mathbf{Z}_n(-1/4) = \frac{\mathbf{X}_n(-1/4)}{\mathbf{Y}_n(-1/4)}\ .$$

If $\mathbf{r}_n = \frac{\mathbf{I}_n(1/4)}{\mathbf{Y}_n(1/4)} \quad , \quad \mathbf{R}_n = \frac{\mathbf{I}_n(-1/4)}{\mathbf{Y}_n(-1/4)}$, we have

$$\pi = \mathbf{q}_n + \mathbf{r}_n \qquad \text{and} \qquad \pi = \mathbf{Q}_n + \mathbf{R}_n\ .$$

where

$$\mathbf{r}_n = \frac{4}{\binom{n-\frac{1}{2}}{n} R_n^{(-\frac{1}{2},0)}(3)} \int_0^1 \frac{(1-x^2)^n x^{2n}}{(1+x^2)^{n+1}}\, dx\ = \mathcal{O}\left(\frac{1}{32^n}\right)$$

$$\mathbf{R}_n = -\frac{4}{\binom{n+\frac{1}{2}}{n} R_n^{(\frac{1}{2},0)}(3)} \int_0^1 \frac{(1-x^2)^n x^{2n+2}}{(1+x^2)^{n+1}}\, dx\ = \mathcal{O}\left(\frac{1}{\sqrt{n}\ \cdot 32^n}\right).$$

From above remarks we find

Propozition 1. *Suppose that* $(\mathbf{q}_n)_{n=1}^{\infty}, (\mathbf{Q}_n)_{n=0}^{\infty}$ *are as in (6) . Then*

$$3 = \mathbf{q}_1 < \cdots < \mathbf{q}_n < \mathbf{q}_{n+1} < \cdots < \pi < \cdots < \mathbf{Q}_{k+1} < \mathbf{Q}_k < \cdots < \mathbf{Q}_1 = 19/6\ .$$

For instance $\mathbf{Q}_4 = 3763456/1197945 = \mathbf{3.14159}33118799...$.

Propozition 2. *If* $P_n^{(\alpha,\beta)}(z) = \binom{n+\alpha}{n} R_n^{(\alpha,\beta)}(z)$, *then*

$$\sum_{k=0}^{n} \frac{2(4k+3)}{(k+1)(2k+1)P_k^{(-1/2,0)}(3)P_{k+1}^{(-1/2,0)}(3)} < \pi <$$
$$< 4 - \sum_{k=0}^{n} \frac{2(4k+5)}{(k+1)(2k+3)P_k^{(1/2,0)}(3)P_{k+1}^{(1/2,0)}(3)}.$$

Proof. From (5) we find

$$\mathbf{Z}_{n+1}(\gamma) = 2(1-4\gamma) + 16\gamma \sum_{k=0}^{n} \frac{k+1-\gamma}{(k+1)(k+1-2\gamma)P_k^{(-2\gamma,0)}(3)P_{k+1}^{(-2\gamma,0)}(3)} .$$

For $\gamma \in \{-\frac{1}{4}, \frac{1}{4}\}$ and we conclude with desired inequalities.

For $n \to \infty$ we give

Corollary 1. *The following equalities are valid*

$$\pi = 2\sum_{k=0}^{\infty} \frac{4k+3}{(k+1)(2k+1)P_k^{(-1/2,0)}(3)P_{k+1}^{(-1/2,0)}(3)}$$
$$\pi = 4 - 2\sum_{k=0}^{\infty} \frac{4k+5}{(k+1)(2k+3)P_k^{(1/2,0)}(3)P_{k+1}^{(1/2,0)}(3)}.$$

References

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