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A certain class of quadratures

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

Our aim is to investigate a quadrature of form:

$$(1) \int_0^1 f(x)dx = c_1 f(x_1)+c_2 f(x_2)+c_3 f(x_3)+c_4 f(x_4)+c_5 f(x_5)+R(f)$$

where $f : [0,1] \to \mathbb{R}$ is integrable, $R(f)$ is the remainder-term and the distinct knots x_j an supposed to be symmetric distributed in $[0,1]$. Under the additional hypothesis that all x_j an of rational type (see(4)), we are interested to find maximum degree of exactness of such quadrature.

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1 Introduction

Let $\prod_m$ be the linear space of all real polynomials of degree $\leq m$ and denote $e_j(t) = t^j$, $j \in \mathbb{N}$. A quadrature of form

$$(2) \qquad \int_0^1 f(x)dx = \sum_{k=0}^{n} c_k f(x_k) + R(f)$$

has degrees (of exactness) m if $R(h) = 0$ for any polynomial $h \in \prod_m$. If $R(h) = 0$ for all $h \in \prod_m$ and moreover $R(e_{m+1}) \neq 0$ it is said that (2) has the exact degree m. It is known that if (2) has degree m, then $m \leq 2n - 1$. Likewise, there exists only one formula (2) having maximum degree $2n - 1$.

The aim of this paper is to study the formulas like (2) for $n = 5$ having some practical properties. Let us note that in this case, the optimal formula having maximum degree $m = 9$ is

$$(3) \qquad \int_0^1 f(x)dx = \sum_{k=1}^{5} c_k f(x_k) + r(f)$$

$$x_k = \frac{1}{2} \pm \frac{1}{6}\sqrt{5 \pm 2\sqrt{\frac{10}{7}}}, 1 \leq k \leq 4, x_5 = \frac{1}{2}$$

It is clear that not all knots x_k are rational numbers.

Definition 1. *Formula (1) is said to be of "practical-type", if*

i) the knots x_j are of form

$$(4) \qquad x_1 = r_1, x_2 = r_2, x_3 = \frac{1}{2}, x_4 = 1 - r_2, x_5 = 1 - r_1$$

where r_1, r_2 distinct rational numbers from $\left[0, \frac{1}{2}\right)$

ii) all coefficients c_1, c_2, c_3, c_4, c_5 are rational numbers with $c_1 = c_5$ and $c_2 = c_4$.

iii) (1) is of order p, with $p \geq 1$. Therefore, in case $n = 5$ a practical-type formula has the form

$$(5)\quad \int_0^1 f(x)dx = A(f(r_1)+f(1-r_1))+B(f(r_2)+f(1-r_2))+C\cdot f\left(\frac{1}{2}\right)+R(f)$$

A, B being rational numbers, $C = 1-2(A+B)$, and when r_1, r_2 are distinct rational numbers from $\left[0, \frac{1}{2}\right)$.

Lemma 1. *Let s be a natural number and suppose in (5) we have $R(h) = 0$ for all $h \in \prod_{2s}$. Then $R(g) = 0$ for every g from $\prod_{2s+1}$.*

Proof. Let $H(x) = \left(x - \frac{1}{2}\right)^{2s+1}$. According to symmetry $\int_0^1 H(x)dx = 0$ and also $R(H) = 0$. Observe that $e_{2s+1}(x) \equiv x^{2s+1} = H(x)+$ $+ h_1(x)$ with $h_1 \in \prod_{2s}$. Therefore $R(e_{2s+1}) = 0$ and supposing $g \in \prod_{2s+1}$ with $g(x) = a_0x^{2s+1} + \ldots$, we have $R(g) = a_0 \cdot R(e_{2s+1}) + R(h_2)$, $h_2 \in R_{2s}$, that is $R(g) = 0$.

Lemma 2. *If in (5) we have $R(h) = 0$ for every polynomial of degree ≤ 4, then*

$$(6)\qquad A = \frac{10r_2^2 - 10r_2 + 1}{60(1-2r_1)^2(r_1-r_2)(1-r_1-r_2)}$$

$$B = \frac{10r_2^2 - 10r_1 + 1}{60(1-2r_2)^2(r_2-r_1)(1-r_1-r_2)}$$

$$C = \frac{8 + 40(r_1^2 + r_2^2) - 40(r_1 - r_2) + 240r_1r_2(1 - r_1 - r_2 + r_1r_2)}{15(1-2r_1)^2(1-2r_2)^2}$$

Proof. We use standard method, namely by considering polynomials

$$l_j = \frac{\omega(x)}{(x-x_j)\omega'(x_j)},\ j \in \{1,2,3,4,5\},\ \omega(x) = \prod_{k=1}^{5}(x - x_k)$$

For instance, taking into account that

$$\omega'(x) = -\frac{1}{4}(1-2r_1)^2(r_1-r_2),\ \text{with } \delta = \frac{1}{2}$$

are found

$$0 = R(l_1) = \int_0^1 l_1(x)dx - Al_1(x_1)$$

and we conclude with

$$A = \frac{1}{\omega'(x_1)} \int_{-\frac{1}{2}}^{\frac{1}{2}} t[t-(1-2r_1)h][t^2-(1-2r_2)^2h^2]dt =$$

$$= \frac{10r_2^2 - 10r_2 + 1}{60(1-2r_1)^2(r_1-r_2)(1-r_1-r_2)}$$

In a similar way are found coefficients B and C. Taking into account that (5) is symmetric, we give:

Corollary 1. *Quadrature formula (5) has order,* $m \geq 5$*, if and only if the coefficients are given by (6).*

Lemma 3. *If (5) has order m,* $m \geq 6$*, then* r_1, r_2 *must be distinct rational numbers from* $(0,1]$ *such that*

(7) $560r_1^2r_2^2 + 56(r_1^2+r_2^2) - 56(r_1+r_2) + 560r_1r_2(1-r_1-r_2) + 5 = 0.$

Proof. It is sufficient to impose condition $R(e_6) = 0$, $e_6(x) = x^6$. By considering $[a,b] = [-1,1]$, are found $R(e_6) = \frac{1}{7} - 2Ar_1^6 - 2Br_2^6 = 0$. Using Lemma 2, see (6) we obtain condition (7).

Corollary 2. *Suppose that (5) is of practical-type. If* r_1, r_2 *are distinct rational numbers from* $(0,1]$ *such that equalities (6) and (7) are verified, then (5) has order* $m = 7$.

Let us remark, that the above proposition implies that

$$r_1 + r_2 - 2r_1r_2 \geq \frac{2}{7}$$

Corollary 3. *The maximum order of* m *of practical-type quadratus formula at 5-knots satisfied* $m \leq 7$.

Proof. Formulas like (7) having order $m = 8$ does not exist. The reason is that by assuming $m \geq 8$, then according to Lemma 1 we must have $m = 9$. But in this case numbers r_1 and r_2 are not rational (see (3)).

Lemma 4. *Then does not exist pairs of rational numbers* (r_1, r_2) *which satisfy*

$$560r_1^2r_2^2 + 56(r_1^2 + r_2^2) - 56(r_1 + r_2) + 560r_1r_2(1 - r_1 - r_2) + 5 = 0.$$

Proof. The case $(1 - 2r_1)(1 - 2r_2) = 0$ is impossible. Further, consider

$$(1 - 2r_1)(1 - 2r_2) \neq 0$$

and let $1 - 2r_1 = \frac{p}{2}, 1 - 2r_2 = \frac{x}{y},\ p, q, x, y, \in \mathbb{Z},\ q > 0, y > 0$, with $(p,q) = 1$, $(x,y) = 1$.

Because $(1 - 2r_2)^2 = \frac{3[5 - 7(1 - 2r_1)^2]}{7[3 - 5(1 - 2r_1)^2]}$, we obtain $7x^2(3q^2 - 5p^2) = 3y^2(5q^2 - 7p^2)$. It follows that $x^2 \equiv 0$ $(mod\ 3)$ or $p^2 \equiv 0$ $(mod\ 3)$. Therefore x or p is divisible by 3, $x = 0$ $(mod\ 3)$, $x = 3k$ with $k \in \mathbb{Z}$. Then after dividing by 3, are finds $y^2(5q^2 - 7p^2) = 3 \cdot 7(3q^2 - 5p^2)$, which means that $5q^2 - 7p^2$ must be divisible by 3. From $(x, y) = 1$ it is clear that y is not divisible by 3. Now

$$5q^2 - 7p^2 = 6(q^2 - p^2) - (q^2 + p^2) \equiv -(q^2 + p^2) \equiv 0 \ (mod\ 3)$$

implies $p^2 + q^2 \equiv 0 \ (mod\ 3)$ which is impossible unless $p \equiv q \equiv 0 \ (mod\ 3)$, which can't happen because $(p, g) = 1$.

Theorem 1. *The practical quadratures at five knots, having maximal degree of exactness $m = 5$ are those of form*

$$(8) \int_0^1 f(x)dx = A[f(r_1)+f(1-r_1)]+B[f(r_2)+f(1-r_2)]+Cf\left(\frac{1}{2}\right)+R(f)$$

where $R(f)$ is remainder, r_1, r_2 are distinct rational numbers from $(0, 1]$ and

$$A = \frac{10r_2^2 - 10r_2 + 1}{60(1-2r_1)^2(r_2 - r_1)(1 - r_1 - r_2)}$$

$$B = \frac{10r_1^2 - 10r_1 + 1}{60(1-2r_2)^2(r_2 - r_1)(1 - r_1 - r_2)}$$

$$C = \frac{8 + 40(r_1^2 + r_2^2) - 40(r_1 - r_2) + 240r_1r_2(1 - r_1 - r_2 + r_1r_2)}{15(1-2r_1)^2(1-2r_2)^2}$$

Let us note that in quadrature formula from (8) we have

$$R(e_6) = \frac{560r_1^2r_2^2 + 56(r_1^2 + r_2^2) - 56(r_1 + r_2) + 560r_1r_2(1 - r_1 - r_2) + 5}{105} \cdot \frac{1}{2^6}$$

If by $[z_0, z_1, ..., z_k; f]$ is denoted the difference of a function $f : [0, 1] \to \mathbb{R}$ at a system of distinct points $\{z_0, z_1, ..., z_k\} \subset [0, 1]$, it may be shown that.

Theorem 2. *Any partial quadratures at five knots, having maximal degree $m = 5$ may be written as*

$$(9) \qquad \int_0^1 f(x)dx = f\left(\frac{1}{2}\right) + \frac{1}{12}\left[r_1, \frac{1}{2}, 1 - r_1; f\right] + \frac{3 - 5(1-2r_1)^2}{240} \cdot$$

$$\cdot\left[r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right] + R(f),$$

where r_1, r_2 *are distinct rational numbers from* $(0,1]$

2 Examples

In the following of $R_j(f), j \in \mathbb{N}^*$, we shall denote the remainders terms in certain quadratures formulas.

Example 1. The closed formulas like (8) are obtained in case $r_2 = 1$, namely

$$(10)\int_0^1 f(x)dx = A_0[f(0)+f(1)] + C_0 f(\frac{1}{2}) + B_0[f(r)+f(1-r)] + R_1(f)$$

where $r \in \mathbb{Q}, r \in (0,1), R_1(e_6) = \dfrac{14(1-2r)^6 - 6}{105 \cdot 2^6}$ and

$$A_0 = \frac{1}{6} - \frac{1}{15(1-2r)^2}; \quad B_0 = \frac{1}{60r(1-2r)^2(1-r)}; \quad C_0 = \frac{3}{2} - \frac{2}{15(1-2r)^2}.$$

Example 2. For instance, when $(r_1, r_2) = \left(1; \frac{1}{2}\right)$, (10) gives

$$(11) \qquad \int_0^1 f(x)dx = \frac{7}{90}[f(0)+f(1)] +$$

$$+\frac{16}{25}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + \frac{2}{15} f\left(\frac{1}{2}\right) + R_2(f)\right]$$

$R_2(e_6) = \dfrac{1}{21 \cdot 2^7}$

Example 3. In case $(r_1, r_2) = \left(\frac{1}{2}; \frac{1}{4}\right)$ are found

$$(12) \quad \int_0^1 f(x)dx = \frac{86}{45}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right] - \frac{224}{45}\left[f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right)\right] +$$

$$+\frac{107}{15}f\left(\frac{1}{2}\right)+R_3(f)$$

$R_3(e_6)=\dfrac{115}{21\cdot 2^{12}}$

3 The remainder term

In order to investigate the remainder we use same methods as in [1] – [6].

Theorem 3. *Let* $m=\frac{1}{2}, h=\frac{1}{2}, x_1=r_1, x_2=r_2, x_3=\frac{1}{2}, x_4=1-r_2, x_5=1-r_1$.

If $\Omega(t)=\left[t^2-(1-2r_1)^2\cdot\frac{1}{4}\right]\left[t^2-(1-2r_2)^2\cdot\frac{1}{4}\right]$.

$$R(f)=\int_{-\frac{1}{2}}^{\frac{1}{2}} t^2\Omega(t)\left[\frac{1}{2}-t,r_1,r_2,\frac{1}{2},1-r_2,1-r_1,\frac{1}{2}+t;f\right]dt \tag{13}$$

Proof. Let $\omega(x)=\prod_{j=1}^{5}(x-x_j)$. Because our formula (8) is of interpolatory type, it follows that we have

$$\int_0^1 f(x)dx=\int_0^1 L_4(x_1,x_2,x_3,x_4,x_5;f)dx+R(f)$$

where $R(f)=\int_0^1\omega(x)[x,x_1,x_2,x_3,x_4,x_5;f]dx$.

But $\int_0^1 f(1-x)dx=\int_0^1 f(x)dx$ and using the symmetry of knots $\{x_1,x_2,...,x_5\}$ we have

$$L_4\left(r_1,r_2,\frac{1}{2},1-r_2,1-r_1;f|1-x\right)=L_4\left(r_1,r_2,\frac{1}{2},1-r_2,1-r_1;f|x\right).$$

Further, the equality $\omega(1-x) = -\omega(x)$ gives

$$R(f) = -\int_0^1 \omega(x) \left[1-x, r_1, r_2, \frac{1}{2}; 1-r_2, 1-r_1; f\right] dx$$

Therefore the remainder from (8) may be written as $R(f) = \frac{1}{2}\int_0^1 \omega(x) D(f;x) dx$ with

$$D(f;x) = \left[x, r_1, r_2, \frac{1}{2}; 1-r_2, 1-r_1; f\right] - \left[1-x; r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right] =$$

$$= 2\left(x-\frac{1}{2}\right)\left[x, r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right]$$

In this manner

$$R(f) = \int_0^1 \left(x-\frac{1}{2}\right)\omega(x)\left[x, r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right] dx$$

which is the same with (13).

Further for $g \in C[0,1]$ we use the uniform norm $||g|| = \max\limits_{x \in [a,b]} |g(x)|$.

Corollary 4. *Let us denote*

$$\omega(x) = (x-r_1)(x-r_2)(x-1+r_1)(x-1+r_2), J(r_1,r_2) = \int_0^1 \left(x-\frac{1}{2}\right)^2 |\omega(x)| dx$$

If $R(f)$, is the remainder in (8), then for $f \in C^6[0,1]$

$$|R(f)| \le \frac{1}{46080} J(r_1,r_2) ||f^{((6)}||. \tag{14}$$

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