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Some results for modified Bernstein polynomials

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Abstract

Our aim is to find some properties for polynomials $L_n f$, defined as:

$$(L_n f)(x) = (B_n f)(x) - \frac{x(1-x)}{2(n-1)}(B_n f)''(x),\ f \in C[0,1],\ n = 2, 3, ...$$

where $B_n f$ is Bernstein polynomial corresponding to f.

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1. Introduction

The *Bernstein polynomials* of $f : [0,1] \to \mathbb{R}$ is

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad n = 1, 2, \ldots$$

We consider the polynomials (see [2])

$$(L_nf)(x) = (B_nf)(x) - \frac{x(1-x)}{2(n-1)}(B_nf)''(x),\ n \geq 2, \tag{1}$$

which satisfy $L_ne_0 = e_0$, $L_ne_1 = e_1$, $L_ne_2 = e_2$, with $e_j(t) = t^j$, $j = 0, 1, \ldots$.

We use the notations

$$K = [a, b], \quad \infty < a < b < +\infty,$$

$$\Omega_j(t, x) = \Omega_j(t) = |t - x|^j, \quad j = 0, 1, \ldots, \quad x \in K, \tag{2}$$

$$\omega(f;\delta) = \sup_{|t-x|<\delta} |f(t) - f(x)|, \quad t, x \in K,\ \delta \in [0, b-a).$$

A. Lupaş [3] has proved following proposition:

Theorem 1. *If $L : C(K) \to C(K_1)$, $K_1 = [a_1, b_1] \subseteq K$, is a linear positive operator, then for all $f \in C(K)$ and $\delta > 0$ we have*

$$||f - Lf||_{K_1} \leq ||f|| \cdot ||e_0 - Le_0||_{K_1} + \inf_{m=1,2,\ldots} \{||Le_0||_{K_1} + \delta^{-m}||L\Omega_m||_{K_1}\}\omega(f;\delta),$$

where $|| \cdot || = \max_K | \cdot |$ and Ω_m is defined in (2).

2. The main results

Theorem 2. *If L_nf is defined by (1), then for $f \in C[0, 1]$,*

$$||f - L_nf|| \leq \frac{19}{16}\omega\left(f; \frac{1}{\sqrt{n}}\right) + \frac{n}{4}\omega\left(f; \frac{1}{n}\right)$$

Proof. We consider $m = 4$ and $K = [0, 1]$ in Theorem 1, $\Omega_4(t) = (t - x)^4$.

Taking into account that

$$(B_n\Omega_4)(x) = \frac{1}{n^3}\left[3(n-2)x^2(1-x)^2 + x(1-x)\right],$$

$$(B_nf)''(x) =$$

(3) $$= \frac{2n(n-1)}{n^2} \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k} \left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; f\right],$$

it follows

$$||f - B_nf|| \le \frac{19}{16}\omega\left(f; \frac{1}{\sqrt{n}}\right).$$

Further, if $\alpha(x) = \dfrac{x(1-x)}{2(n-1)}$ we have

$$||f - L_nf|| = \left|\left|f - L_nf - \alpha(B_nf)'' + \alpha(B_nf)''\right|\right| \le$$

$$\le ||f - B_nf|| + \left|\left|\alpha(B_nf)''\right|\right|.$$

But from (3) the theorem is proved because

$$\alpha(x)(B_nf)''(x) = \frac{x(1-x)}{2(n-1)}(B_nf)''(x) =$$

$$= \frac{x(1-x)}{2(n-1)} \frac{2!n(n-1)}{n^2} \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k} \cdot$$

$$\cdot \frac{\left[\frac{k+1}{n}, \frac{k+2}{n}; f\right] - \left[\frac{k}{n}, \frac{k+1}{n}; f\right]}{\frac{2}{n}} =$$

$$= \frac{x(1-x)}{2} \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k} \cdot$$

$$\cdot \left(\frac{f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)}{\frac{1}{n}} - \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{\frac{1}{n}} \right) \le$$

$$\le \frac{x(1-x)}{2} n \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k} \left(\omega\left(f; \frac{1}{n}\right) + \omega\left(f; \frac{1}{n}\right)\right) =$$

$$= nx(1-x)\omega\left(f; \frac{1}{n}\right) \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{n-2-k} =$$

$$= nx(1-x)\omega\left(f;\frac{1}{n}\right) \le \frac{n}{4}\omega\left(f;\frac{1}{n}\right).$$

Corollary 1. *If $L_n f$ is defined by (1), then for $f \in C[0,1]$ we have*

$$||f - L_n f|| \le \frac{19}{16}\omega\left(f;\frac{1}{\sqrt{n}}\right) + \frac{n}{8}\omega_2\left(f;\frac{1}{n}\right)$$

Proof. We observe

$$\left[\frac{k}{n},\frac{k+1}{n},\frac{k+2}{n};f\right] = \frac{f\left(\frac{k}{n}\right)}{\frac{2}{n^2}} - \frac{f\left(\frac{k+1}{n}\right)}{\frac{1}{n^2}} + \frac{f\left(\frac{k+2}{n}\right)}{\frac{2}{n^2}} =$$

$$= \frac{n^2}{2}\left[f\left(\frac{k}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k+2}{n}\right)\right] = \frac{n^2}{2}\Delta^2_{\frac{1}{n}}\left(f;\frac{k}{n}\right),$$

where

$$\Delta^r_h(f;x) = \sum_{k=0}^{r}\binom{r}{k}(-r)^{r-k}f(x+kh), \quad r=1,2,\ldots, \quad h \in \mathbb{R}.$$

Using the following definition

$$\omega_r(f,\delta) = \sup_{0<h\le\delta}||\Delta^r_h(f;\cdot)||, \quad \text{with} \quad \delta > 0,$$

results

$$\left[\frac{k}{n},\frac{k+1}{n},\frac{k+2}{n};f\right] \le \frac{n^2}{2}\omega_2\left(f;\frac{1}{n}\right),$$

and

$$\alpha(x)(B_n f)''(x) \le \frac{x(1-x)}{2(n-1)}\frac{2n(n-1)}{n^2}\frac{n^2}{2}\omega_2\left(f;\frac{1}{n}\right) =$$

$$= \frac{nx(1-x)}{2}\omega_2\left(f;\frac{1}{n}\right) \le \frac{n}{8}\omega_2\left(f;\frac{1}{n}\right).$$

from above formula the proposition is proved.

References

[1] Bernstein S.N., *Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités*, Comm. Soc. Math. Charkow Sér. 2 t. 13, (1912) 1-2.

[2] Franchetti C., *Sull'innalzamento dell'ordine di aprossimazione dei polinomi di Bernstein*, Rend. Sem. Mat. Univ. Politec. Torino, 28 (1968-69) 1-7.

[3] Lupaş A., *Contribuţii la teoria aproximării prin operatori liniari*, Teză de doctorat (Cluj), (1976) (in Romanian).

[4] Sofonea F., *Remainder in approximation by means of certain linear operators*, Mathematical Analysis and Aproximation Theory, Burg Verlag (Sibiu), (2002), 255-258.

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