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Certain functions with positive real part

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

We find conditions on the complex-valued functions $A, B : U \to \mathbb{C}$ defined in the unit disc U and the real constants α, β, γ such that the differential inequality

$$\text{Re}\ [A(z)p^{4k}(z)+B(z)p^{4k-2}(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}+\gamma] > 0$$

implies Re $p(z) > 0$, where $p \in \mathcal{H}[1, n]$, and n, k are two positive integers.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}[U],\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots,\ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i [1, p. 35].

Lemma A. [1, p. 35]. *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be a function which satisfies*

$$\text{Re}\ \psi(\rho i, \sigma; z) \le 0,$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \le -\frac{n}{2}(1+\rho^2)$, $z \in U$ and $n \ge 1$.

If $p \in \mathcal{H}[1,n]$ and

$$\text{Re}\ \psi(p(z), zp'(z); z) > 0$$

then

$$\text{Re}\ p(z) > 0.$$

2 Main results

Theorem. *Let $\alpha \ge 0$, $\beta \ge 0$, $\gamma \le \alpha \left(\frac{n}{2}\right)^{2k} + \beta \left(\frac{n}{2}\right)^{2k-1}$ and n, k be two positive integers.*

Suppose that the functions $A, B : U \to \mathbb{C}$ satisfy:

$$\text{(1)} \quad \begin{aligned} &i)\ \ \text{Re}\ A(z) \le \alpha \left(\frac{n}{2}\right)^{2k} \\ &ii)\ \ \text{Re}\ B(z) \ge -2k\alpha \left(\frac{n}{2}\right)^{2k} - \beta \left(\frac{n}{2}\right)^{2k-1}. \end{aligned}$$

If $p \in \mathcal{H}[1,n]$ *and*

(2) $\quad \text{Re}\,[A(z)p^{4k}(z)+B)z)p^{4k-2}(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}+\gamma]>0$

then

$$\text{Re}\ p(z)>0.$$

Proof. We let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ be defined by

(3) $$\psi(p(z),zp'(z);z)=$$

$$=A(z)p^{4k}(z)+B(z)p^{4k-2}(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}+\gamma$$

From (2) we have

(4) $$\text{Re}\ \psi(p(z),zp'(z);z)>0 \text{ for } z\in U.$$

For $\sigma,\rho\in\mathbb{R}$ satisfying $\sigma\le -\dfrac{n}{2}(1+\rho^2)$ and $z\in U$, by using (1) we obtain:

$$\text{Re}\ \psi(\rho i,\sigma;z)=\text{Re}\,[A(z)(\rho i)^{4k}+B(z)(\rho i)^{4k-2}(z)-\alpha\sigma^{2k}+\beta\sigma^{2k-1}+\gamma]\le$$

$$\le\rho^{4k}\text{Re}\ A(z)-\rho^{4k-2}\text{Re}\ B(z)-\alpha\left(\frac{n}{2}\right)^{2k}(1+\rho^2)^{2k}-$$

$$-\beta\left(\frac{n}{2}\right)^{2k-1}(1+\rho^2)^{2k-1}+\gamma\le\rho^{4k}\text{Re}\ A(z)-\rho^{4k-2}\text{Re}\ B(z)-$$

$$-\alpha\left(\frac{n}{2}\right)^{2k}[1+C_{2k}^{1}\rho^2+C_{2k}^{2}\rho^4+\ldots+C_{2k}^{2k-1}(\rho^2)^{2k-1}+C_{2k}^{2k}(\rho^2)^{2k}]-$$

$$-\beta\left(\frac{n}{2}\right)^{2k-1}(1+C_{2k-1}^{1}\rho^2+C_{2k-1}^{2}(\rho^2)^2+\ldots+C_{2k-1}^{2k-2}(\rho^2)^{2k-2}+$$

$$+C_{2k-1}^{2k-1}(\rho^2)^{2k-1}]\le\rho^{4k}\left[\text{Re}\ A(z)-\alpha\left(\frac{n}{2}\right)^{2k}\right]-$$

$$-\rho^{4k-2}\left[\text{Re}\ B(z)+2k\cdot\alpha\left(\frac{n}{2}\right)^{2k}+\beta\left(\frac{n}{2}\right)^{2k-1}\right]-$$

$$-\rho^{4k-4}\left[\alpha\left(\frac{n}{2}\right)^{2k}(2k-1)k+\beta\left(\frac{n}{2}\right)^{2k-1}(2k-1)\right]-$$

$$-\rho^{4k-6}\left[\alpha\left(\frac{n}{2}\right)^{2k}\cdot\frac{k(2k-2)(2k-1)}{3}+\beta\left(\frac{n}{2}\right)^{2k-1}\cdot(k-1)(2k-1)\right]-\ldots-$$

$$-\rho^{4}\left[\alpha\left(\frac{n}{2}\right)^{2k}k(2k-1)+\beta\left(\frac{n}{2}\right)^{2k-1}\cdot\frac{2k-1}{2}\right]-$$

$$-\rho^{2}\left[\alpha\left(\frac{n}{2}\right)^{2k}\cdot 2k+\beta\left(\frac{n}{2}\right)^{2k-1}(2k-1)\right]-$$

$$-\alpha\left(\frac{n}{2}\right)^{2k}-\beta\left(\frac{n}{2}\right)^{2k-1}+\gamma\le 0.$$

By using Lemma A we have that Re $p(z)>0$.

If $\delta=\alpha\left(\frac{n}{2}\right)^{2k}+\beta\left(\frac{n}{2}\right)^{2k-1}$, then the Theorem can be rewritten as follows:

Corollary 1. *Let $\alpha\ge 0$, $\beta\ge 0$ and n,k be two positive integers.*

Suppose that the functions $A,B:U\to\mathbb{C}$ satisfy:

i) Re $A(z)\le\alpha\left(\frac{n}{2}\right)^{2k}$

ii) Re $B(z)\ge -2\alpha k\left(\frac{n}{2}\right)^{2k}-\beta\left(\frac{n}{2}\right)^{2k-1}$.

If $p\in\mathcal{H}[1,n]$ and

$$\text{Re }\left[A(z)p^{4k}(z)+B(z)p^{4k-2}(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}+\right.$$

$$\left.+\alpha\left(\frac{n}{2}\right)^{2k}+\beta\left(\frac{n}{2}\right)^{2k-1}\right]>0$$

then

$$\text{Re }p(z)>0.$$

If $\alpha\equiv 0$, then the Theorem can be rewritten as follows:

Corollary 2. *Let $\beta\ge 0$, $\gamma\le\beta\left(\frac{n}{2}\right)^{2k-1}$, and n,k be two positive integers.*

Suppose that the functions $A,B:U\to\mathbb{C}$ satisfy:

i) Re $A(z)\le 0$

ii) Re $B(z) \geq -\beta \left(\frac{n}{2}\right)^{2k-1}$.

If $p \in \mathcal{H}[1, n]$ *and*

$$\text{Re } [A(z)p^{4k}(z) + B(z)p^{4k-2}(z) + \beta(zp'(z))^{2k-1} + \gamma] > 0$$

then

$$\text{Re } p(z) > 0.$$

If $\beta \equiv 0$, then the Theorem can be rewritten as follows:

Corollary 3. *Let* $\alpha \geq 0$, $\gamma \leq \alpha \left(\frac{n}{2}\right)^{2k}$ *and* n, k *be two positive integers.*

Suppose that the functions $A, B : U \to \mathbb{C}$ *satisfy*

i) Re $A(z) \leq \alpha \left(\frac{n}{2}\right)^{2k}$

ii) Re $B(z) \geq -2\alpha k \left(\frac{n}{2}\right)^{2k}$.

If $p \in \mathcal{H}[1, n]$ *and*

$$\text{Re } [A(z)p^{4k}(z) + B(z)p^{4k-2}(z) - \alpha(zp'(z))^{2k} + \gamma] > 0$$

then Re $p(z) > 0$.

If $\gamma = 0$, then the Theorem can be rewritten as follows:

Corollary 4. *Let* $\alpha \geq 0$, $\beta \geq 0$, *and* n, k *be two positive integers.*

Suppose that the functions $A, B : U \to \mathbb{C}$ *satisfy:*

i) Re $A(z) \leq \alpha \left(\frac{n}{2}\right)^{2k}$

ii) Re $B(z) \geq -2\alpha k \left(\frac{n}{2}\right)^{2k} - \beta \left(\frac{n}{2}\right)^{2k-1}$.

If $p \in \mathcal{H}[1, n]$ *and*

$$\text{Re } [A(z)p^{4k}(z) + B(z)p^{4k-2}(z) - \alpha(zp'(z))^{2k} + \beta(zp'(z))^{2k-1}] > 0$$

then

$$\text{Re } p(z) > 0.$$

If $n=1$, $\alpha=1$, $\beta=2$, $\gamma=\frac{5}{2^k}$, $A(z)=-1+\frac{z}{2}$, $B(z)=1+2z$, then in this case from Corollary 1 we deduce:

Example 1. If $p\in\mathcal{H}[1,n]$ and

$$\mathrm{Re}\left[\left(-1+\frac{z}{2}\right)p^{4k}(z)+(1+2z)p^{4k-2}(z)-(zp'(z))^{2k}+2(zp'(z))^{2k-1}+\frac{5}{2^k}\right]>0$$

then

$$\mathrm{Re}\, p(z)>0.$$

If $n=2$, $\alpha=0$, $\beta=3$, $\gamma=3$, $A(z)=-3+z$, $B(z)=1+z$, then in this case from Corollary 2 we deduce:

Example 2. If $p\in\mathcal{H}[1,2]$ and

$$\mathrm{Re}\ [(-3+z)p^{4k}(z)+(1+z)p^{4k-2}(z)+3(zp'(z))^{2k-1}+3]>0$$

then

$$\mathrm{Re}\, p(z)>0.$$

If $n=3$, $\alpha=4$, $\beta=0$, $\gamma=4\left(\frac{3}{2}\right)^{2k}$, $A(z)=-2+\frac{z}{2}$, $B(z)=2-z$, then in this case from Corollary 3 we deduce:

Example 3. If $p\in\mathcal{H}[1,3]$ and

$$\mathrm{Re}\ \left[\left(-2+\frac{z}{2}\right)p^{4k}(z)+(2-z)p^{4k-2}(z)-4(zp'(z))^{2k}+4\left(\frac{3}{2}\right)^{2k}\right]>0$$

then

$$\mathrm{Re}\, p(z)>0.$$

If $n=4$, $\alpha=\frac{1}{2}$, $\beta=\frac{3}{4}$, $A(z)=-4+2z$, $B(z)=3-\frac{z}{2}$, then in this case from Corollary 4 we deduce:

Example 4. If $p \in \mathcal{H}[1,4]$ and

$$\text{Re}\left[(-4+2z)p^{4k}(z)+\left(3-\frac{z}{2}\right)p^{4k-2}(z)-\frac{1}{2}(zp'(z))^{2k}+\frac{3}{4}(zp'(z))^{2k-1}\right]>0$$

then

$$\text{Re } p(z) > 0.$$

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

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