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An intermediate point property in some of the classical generalized formulas of quadrature

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Abstract

In this paper we study a property of the intermediate point (see [6]) from the classical generalized quadrature formulas of the rectangle, trapeze, Simpson and Newton.

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In [6] B. Jacobson studied a property of the intermediate point which appear in the mean-value theorem for integrals. This property has been studied in the articles [2], [3], and [7]. In this paper we will study this property for the classical generalized (composed) quadrature formulas of the rectangle, trapeze, Simpson and Newton.

1. Let us consider the generalized quadrature formula of the rectangle. If $f : [a, b] \to \mathbb{R}$, $f \in C^2[a, b]$, then for any $x \in (a, b]$ there is $c_x \in (a, x)$ such that:

$$(1)\quad \int_a^x f(t)dt = \frac{1}{n}(x-a)\sum_{k=1}^{n} f\left(a + \frac{2k-1}{2n}(x-a)\right) + \frac{(x-a)^3}{24n^2} f''(c_x),$$

(see [1], [4], [5]).

In the above conditions we have the following theorem:

Theorem 1. *If $f \in C^4[a, b]$ and $f'''(a) \neq 0$, then for the intermediate point c_x from (1), we have:*

$$\lim_{x\to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

Proof. Let $F, G : [a, b] \to \mathbb{R}$ defined as follows

$$F(x) = \int_a^x f(t)dt - \frac{(x-a)}{n}\sum_{k=1}^{n} f\left(a + \frac{2k+1}{2n}(x-a)\right) - \frac{(x-a)^3}{24n^2} f''(a),$$

$$G(x) = (x-a)^4.$$

Since F and G are four times derivable on $[a, b]$, and $G^{(i)}(x) \neq 0$, $i = \overline{1,4}$ for any $x \in (a, b)$, we have

$$F(a) = F'(a) = F''(a) = F'''(a) = 0,$$

$$G(a) = G'(a) = G''(a) = G'''(a) = 0.$$

By using successive l'Hospital rule, we obtain:

$$\lim_{x\to a} \frac{F(x)}{G(x)} = \lim_{x\to a} \frac{F^{(IV)}(x)}{G^{(IV)}(x)}.$$

Since

$$F^{(IV)}(x) = f'''(x) - \frac{1}{2n^4}\sum_{k=1}^{n}(2k-1)^3 f'''\left(a + \frac{2k-1}{2n}(x-a)\right) -$$

$$-\frac{(x-a)}{16n^5}\sum_{k=1}^{n}(2k-1)^4 f^{(IV)}\left(a + \frac{2k-1}{2n}(x-a)\right) \text{ and}$$

$G^{(IV)}(x) = 4!$, we obtain

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \frac{1}{48n^2}f'''(a). \tag{2}$$

On the other hand

$$\frac{F(x)}{G(x)} = \frac{\frac{(x-a)^3}{24n^2}[f''(c_x) - f''(a)]}{(x-a)^4} = \frac{1}{24n^2}\cdot\frac{f''(c_x) - f''(a)}{c_x - a}\cdot\frac{c_x - a}{x-a},$$

whence

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \frac{1}{24n^2}f'''(a)\cdot\lim_{x\to a}\frac{c_x - a}{x-a}. \tag{3}$$

From relations (2) and (3) we obtain

$$\lim_{x\to a}\frac{c_x - a}{x-a} = \frac{1}{2},$$

which is the conclusion of Theorem 1.

2. Now we will consider the generalized quadrature formula of the trapeze. If $f : [a,b] \to \mathbb{R}$, $f \in C^2[a,b]$, then for any $x \in [a,b]$ there is $c_x \in (a,x)$ such that:

$$\int_a^x f(t)dt = \frac{(x-a)}{n}\left[\frac{f(a)}{2} + \sum_{k=1}^{n-1} f\left(a + \frac{k(x-a)}{n}\right) + \frac{f(x)}{2}\right] - \tag{4}$$

$$-\frac{(x-a)^3}{12n^2}f''(c_x),$$

(see [1], [4], [5]).

We have the following result:

Theorem 2. *If $f \in C^4[a,b]$ and $f'''(a) \neq 0$, then for the intermediate point c_x from the formula (4), we have*

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

Proof. We consider the functions $F, G : [a,b] \to \mathbb{R}$ defined as follows:

$$F(x)=\int_a^x f(t)dt-\frac{(x-a)}{n}\left[\frac{f(a)}{2}+\sum_{k=1}^{n-1}f\left(a+\frac{k(x-a)}{n}\right)+\frac{f(x)}{2}\right]+$$

$$+\frac{(x-a)^3}{12n^2}f''(a),\quad G(x)=(x-a)^4.$$

Since F and G are four times derivable on $[a,b]$ and $G^{(i)}(x)\neq 0$, $i=\overline{1,4}$ for any $x\in(a,b)$, we have

$$F(a)=F'(a)=F''(a)=F'''(a)=0,$$

$$G(a)=G'(a)=G''(a)=G'''(a)=0.$$

By using successive l'Hospital rule, we will have:

$$\lim_{x\to a}\frac{F(x)}{G(x)}=\lim_{x\to a}\frac{F^{(IV)}(x)}{G^{(IV)}(x)}.$$

Since

$$F^{(IV)}(x)=\frac{n-2}{n}f'''(x)-\frac{4}{n^4}\sum_{k=1}^{n-1}k^3 f\left(a+\frac{k(x-a)}{n}\right)-$$

$$-\frac{(x-a)}{n^5}\sum_{k=1}^{n-1}k^4 f^{(IV)}\left(a+\frac{k(x-a)}{n}\right)-\frac{(x-a)}{2n}f^{(IV)}(x)\quad\text{and}$$

$G^{(IV)}(x)=4!$ we obtain:

$$\lim_{x\to a}\frac{F(x)}{G(x)}=-\frac{1}{24n^2}f'''(a).\tag{5}$$

On the other hand, we have:

$$\frac{F(x)}{G(x)} = \frac{-\frac{(x-a)}{12n^2}\left[f''(c_x)-f''(a)\right]}{(x-a)^4} = -\frac{1}{12n^2}\cdot\frac{f''(c_x)-f''(a)}{c_x-a}\cdot\frac{c_x-a}{x-a}$$

where we find:

$$\lim_{x\to a}\frac{F(x)}{G(x)} = -\frac{1}{12n^2}f'''(a)\cdot\lim_{x\to a}\frac{c_x-a}{x-a}. \tag{6}$$

From the relations (5) and (6), it follows

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2},$$

which is exactly the assertion Theorem 2.

3. We will continue with the generalized quadrature formula of Simpson. If $f:[a,b]\to\mathbb{R}$, $f\in C^4[a,b]$ then for any $x\in(a,b]$ there is $c_x\in(a,x)$ such that:

$$\int_a^x f(t)dt = \frac{(x-a)}{6n}\left[f(a)+2\sum_{k=1}^{n-1}f\left(a+\frac{k(x-a)}{n}\right)+\right. \tag{7}$$

$$\left.+4\sum_{k=1}^{n}f\left(a+\frac{2k-1}{2n}(x-a)\right)+f(x)\right]-\frac{1}{4\cdot 6!}\frac{(x-a)^5}{n^4}f^{(IV)}(c_x), \text{(see [1], [5])}.$$

For this formula it follows:

Theorem 3. *If $f\in C^6[a,b]$ and $f^{(V)}(a)\neq 0$, then for the intermediate point c_x form the formula (7), we have:*

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}$$

Proof. Let $F, G : [a,b] \to \mathbb{R}$ defined as follows:

$$F(x) = \int_a^x f(t)dt - \frac{x-a}{6n}\left[f(a) + 2\sum_{k=1}^{n-1} f\left(a + \frac{k(x-a)}{n}\right) + \right.$$

$$\left. +4\sum_{k=1}^{n} f\left(a + \frac{2k-1}{2n}(x-a)\right) + f(x)\right] + \frac{1}{4\cdot 6!}\frac{(x-a)^5}{n^4}f^{(IV)}(a),$$

$$G(x) = (x-a)^6.$$

We have that F and G are six times derivable on $[a,b]$, $G^{(i)}(x) \neq 0$, $i = \overline{1,5}$, for any $x \in (a,b)$ and $F^{(k)}(a) = 0$, $G^{(k)}(a) = 0$, $k = \overline{1,5}$.

By using successive l'Hospital rule, we obtain

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \lim_{x\to a}\frac{F^{(VI)}(x)}{G^{(VI)}(x)}.$$

Now, from

$$F^{(VI)}(x) = \frac{n-1}{n}f^{(V)}(x) - \frac{(x-a)}{6n}f^{(VI)}(x) -$$

$$-\frac{2}{n^6}\sum_{k=1}^{n-1} K^5 f^{(V)}\left(a + \frac{k(x-a)}{n}\right) -$$

$$-\frac{(x-a)}{3n^7}\sum_{k=1}^{n} K^6 f^{(VI)}\left(a + \frac{k(x-a)}{n}\right) -$$

$$-\frac{1}{8n^6}\sum_{k=1}^{n}(2k-1)^5 f^{(V)}\left(a + \frac{2k-1}{2n}(x-a)\right) -$$

$$-\frac{1}{8n^6}\sum_{k=1}^{n}(2k-1)^6 f^{(VI)}\left(a + \frac{2k-1}{2n}(x-a)\right),$$

$$G^{(VI)}(x) = 6!,$$

we obtain

$$\lim_{x\to a}\frac{F(x)}{G(x)}=-\frac{1}{8n^4\cdot 6!}f^{(V)}(a). \tag{8}$$

On the other hand, we have

$$\frac{F(x)}{G(x)}=-\frac{1}{4\cdot 6!n^4}\cdot\frac{f^{(IV)}(c_x)-f^{(IV)}(a)}{c_x-a}\cdot\frac{c_x-a}{x-a},$$

whence we obtain:

$$\lim_{x\to a}\frac{F(x)}{G(x)}=-\frac{1}{4\cdot 6!n^4}f^{(V)}(a)\cdot\lim_{x\to a}\frac{c_x-a}{x-a}. \tag{9}$$

From the relations (8) and (9) we find

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

4. In the final part of this paper we will consider the generalized quadrature formula of Newton ([1]).

If $f:[a,b]\to\mathbb{R}$, $f\in C^4[a,b]$ then for any $x\in(a,b]$ there is $c_x\in(a,x)$ such that:

$$\int_a^x f(t)dt=\frac{(x-a)}{8n}\left[f(a)+2\sum_{k=1}^{n-1}f\left(a+\frac{k(x-a)}{n}\right)+\right.$$
$$\left.+3\sum_{k=1}^{n}f\left(a+\frac{3k-2}{3n}(x-a)\right)+3\sum_{k=1}^{n}f\left(a+\frac{3k-1}{3n}(x-a)\right)+f(x)\right]-$$
$$-\frac{1}{9\cdot 6!}\frac{(x-a)^5}{n^4}f^{(IV)}(c_x). \tag{10}$$

The result obtained consist in following:

Theorem 4. *If $f\in C^6[a,b]$ and $f^{(V)}(a)\neq 0$ then for the intermediate point c_x from (10) we have:*

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

Proof. Let $F, G : [a,b] \to \mathbb{R}$ defined as follows:

$$F(x) = \int_a^x f(t)dt - \frac{(x-a)}{8n}\left[f(a) + 2\sum_{k=1}^{n-1} f\left(a + \frac{k(x-a)}{n}\right) + \right.$$

$$\left. +3\sum_{k=1}^{n} f\left(a + \frac{3k-2}{3n}(x-a)\right) + 3\sum_{k=1}^{n} f\left(a + \frac{3k-1}{3n}(x-a)\right) + f(x)\right] +$$

$$+\frac{1}{9\cdot 6!}\frac{(x-a)^5}{n^4} f^{(IV)}(a),$$

$$G(x) = (x-a)^6.$$

Since F and G are seven times derivable on $[a,b]$, $G^{(i)}(x) \neq 0$, $i = \overline{1,6}$ for any $x \in (a,b)$ and $F^{(k)}(a) = 0$, $G^{(k)}(a) = 0$, $k = \overline{0,5}$, by using successive l'Hospital rule, we obtain:

$$\lim_{x\to a} \frac{F(x)}{G(x)} = \lim_{x\to a} \frac{F^{VI}(x)}{G^{(VI)}(x)}.$$

We have that:

$$F^{(VI)}(x) = \frac{4n-3}{4n} f^{(V)}(x) - \frac{(x-a)}{8n} f^{(VI)}(x) -$$

$$-\frac{3}{2n^6}\sum_{k=1}^{n-1} k^5 f^{(V)}\left(a + \frac{k(x-a)}{n}\right) - \frac{(x-a)}{4n^7}\sum_{k=1}^{n-1} k^6 f^{(VI)}\left(a + \frac{k(x-a)}{n}\right) -$$

$$-\frac{1}{108n^6}\sum_{k=1}^{n}(3k-2)^5 f^V\left(a + \frac{3k-2}{3n}(x-a)\right) -$$

$$-\frac{(x-a)}{3\cdot 648}\sum_{k=1}^{n}(3k-2)^6 f^{(VI)}\left(a + \frac{3k-2}{3n}(x-a)\right) -$$

$$-\frac{1}{108n^6}\sum_{k=1}^{n}(3k-1)^5 f^{(V)}\left(a + \frac{3k-1}{3n}(x-a)\right) -$$

$$-\frac{(x-a)}{3\cdot 648}\sum_{k=1}^{n}(3k-1)^6 f^{(VI)}\left(a+\frac{3k-1}{3n}(x-a)\right),$$

$$G^{(VI)}(x)=6!.$$

Therefore:

$$\lim_{x\to a}\frac{F(x)}{G(x)}=-\frac{1}{18\cdot 6!n^4}f^{(V)}(a). \tag{11}$$

On the other hand, we have

$$\frac{F(x)}{G(x)}=-\frac{1}{9\cdot 6!n^4}\frac{f^{(IV)}(c_x)-f^{(IV)}(a)}{c_x-a}\cdot\frac{c_x-a}{x-a},$$

whence:

$$\lim_{x\to a}\frac{F(x)}{G(x)}=-\frac{1}{9\cdot 6!n^4}f^{(V)}(a)\cdot\lim_{x\to a}\frac{c_x-a}{x-a}. \tag{12}$$

From the relations (11) and (12) we obtain

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

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