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A remark on the Hadamard products of n-starlike functions

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Abstract

In this paper we study the Hadamard products of two functions to the classes $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ of univalent functions with negative coefficients.

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1 Introduction

Let **U** denote the open unit disc: $\mathbf{U} = \{z\ ;\ z \in \mathbb{C}\ ,\ |z| < 1\}$, let **A** denote the class of functions

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1}$$

which are analytic in **U**, and let **S** denote the class of functions of the form (1) which are analytic and univalent in **U**.

A function $f(z) \in \mathbf{A}$ is said to be starlike of order $\alpha\,(0 \le \alpha < 1)$ in the unit disk $\mathbf{U}$ if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha$$

for all $z \in \mathbf{U}$.

For $f \in \mathbf{S}$ we define the differential operator $\mathbf{D}^n$ (Sălăgean [2])

$$\begin{aligned} \mathbf{D}^0 f(z) &= f(z) \\ \mathbf{D}^1 f(z) &= \mathbf{D}f(z) = zf'(z) \end{aligned}$$

and

$$\mathbf{D}^n f(z) = \mathbf{D}(\mathbf{D}^{n-1} f(z)) \quad ; \quad n \in \mathbb{N}^* = \{1, 2, 3, ...\}.$$

We note that if

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

then

$$\mathbf{D}^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \; ; \; z \in \mathbf{U}.$$

Let T denote the subclass of $\mathbf{S}$ which can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad ; \quad a_k \ge 0 \quad , \quad \text{for all } k \ge 2$$

We say that a function $f \in T$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$ if

$$\left| \frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1}{(B-A)\gamma\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - \alpha\right] - B\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1\right]} \right| < \beta \quad , \quad z \in \mathbf{U}$$

$$\frac{B}{B-A} < \gamma \le \begin{cases} \dfrac{B}{(B-A)\alpha} & ; \quad \alpha \ne 0 \\ \quad 1 & ; \quad \alpha = 0 \end{cases}$$

where

$$F_{n,\lambda}(z) = (1-\lambda)D^n f(z) + \lambda D^{n+1} f(z) \;\; ; \; \lambda \geq 0 \;\; ; \; f \in T.$$

For this class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, in Holhoş [1] is showed the following lemma:

Lemma 1. *Let* $f \in T$, $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$; $a_k \geq 0$, *for all* $k \geq 2$. *Then* $f(z)$ *is in the class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ *if and only if*

(2) $$\sum_{k=2}^{\infty} a_k k^n \left[1 + \lambda(k-1)\right]\left\{(k-1) + \beta\left[(B-A)\gamma(k-\alpha) - B(k-1)\right]\right\} \leq$$

$$\leq \beta\gamma(B-A)(1-\alpha)$$

and the result is sharp.

If we denote

$$D_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) =$$
$$= k^n\left[1 + \lambda\left(k-1\right)\right]\left\{(k-1) + \beta\left[(B-A)\,\gamma\left(k-\alpha\right) - B\left(k-1\right)\right]\right\}$$

then (2) can be rewritten

$$\sum_{k=2}^{\infty} a_k D_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B-A)(1-\alpha)$$

2 The Hadamard products

Let $f, g \in T$,

(3) $$f(z) = z - \sum_{k=2}^{\infty} a_k z^k; \;\; a_k \geq 0, \;\text{ for all } k \geq 2$$

and

(4) $$g(z) = z - \sum_{k=2}^{\infty} b_k z^k; \;\; b_k \geq 0, \;\text{ for all } k \geq 2,$$

then we define the Hadamard product of f and g by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

Theorem 1. *If the functions f and g defined by* (3) *and* (4) *belong to the same class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, then the Hadamard product belongs to $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.*

Proof. Since $f(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ by using Lemma we have

$$\sum_{k=2}^{\infty} a_k D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \le \beta\gamma(B - A)(1 - \alpha)$$

and

$$a_k \le \frac{\beta\gamma(B - A)(1 - \alpha)}{2^n(1 + \lambda)\left[1 - \beta B + \beta\gamma(B - A)(2 - \alpha)\right]} \;;\quad \text{for all } k \ge 2$$

where we used that

$$D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \le D_n(k + 1, A, B, \alpha, \beta, \gamma, \lambda) \;;\quad \text{for all } k \ge 2$$

and

$$D_n(2, A, B, \alpha, \beta, \gamma, \lambda) \le D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \;;\quad \text{for all } k \ge 2.$$

If $g(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ then

$$\sum_{k=2}^{\infty} b_k D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \le \beta\gamma(B - A)(1 - \alpha)$$

and

$$\sum_{k=2}^{\infty} a_k b_k D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \le$$

$$\le \frac{\beta^2\gamma^2(B - A)^2(1 - \alpha)^2}{2^n(1 + \lambda)\left[1 - \beta B + \beta\gamma(B - A)(2 - \alpha)\right]} \le \beta\gamma(B - A)(1 - \alpha)$$

because

$$\frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\left[1-\beta B+\beta\gamma(B-A)(2-\alpha)\right]} \leq \beta\gamma(B-A)(1-\alpha)$$

is equivalent to

$$\beta\gamma(B-A)(1-\alpha)\left\{\beta\gamma\left(B-A\right)\left(1-\alpha\right)-\right.$$

$$\left.-2^n(1+\lambda)\left[1-\beta B+\beta\gamma(B-A)(2-\alpha)\right]\right\} \leq 0.$$

From here we have

$$\beta\gamma\left(B-A\right)\left(1-\alpha\right)-2^n(1+\lambda)\left(1-\beta B\right)-2^n(1+\lambda)\beta\gamma(B-A)(2-\alpha) \leq 0$$

where

$$\beta\gamma\left(B-A\right)\left(1-\alpha\right)-2^n(1+\lambda)\left(1-\beta B\right)-$$

$$-2^n(1+\lambda)\beta\gamma(B-A)-2^n(1+\lambda)\beta\gamma(B-A)(1-\alpha) \leq 0$$

which is equivalent to

$$\beta\gamma\left(B-A\right)\left(1-\alpha\right)\left[1-2^n(1+\lambda)\right]-$$

$$-2^n(1+\lambda)(1-\beta B)-2^n(1+\lambda)\beta\gamma(B-A) \leq 0.$$

Acording to Lemma we obtain $f * g \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Theorem 2. *Let* $\frac{1}{2^n(1+\lambda)} \leq \beta \leq 1$. *If the functions* f *and* g *defined by* (3) *and* (4) *belong to the same class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, *then the Hadamard product belongs to* $T_{n,\lambda}(A, B, \alpha, \beta^2, \gamma)$.

Since $f(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ by using Lemma we have

$$\sum_{k=2}^{\infty} a_k D_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B-A)(1-\alpha)$$

and

$$a_k \leq \frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\left[1-\beta B+\beta\gamma(B-A)(2-\alpha)\right]}\ ;\ \text{for all } k \geq 2.$$

If $g(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ then

$$\sum_{k=2}^{\infty} b_k D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \le \beta\gamma(B - A)(1 - \alpha).$$

Since $0 < \beta^2 \le \beta \le 1$ we have

$$\sum_{k=2}^{\infty} b_k D_n\left(k, A, B, \alpha, \beta^2, \gamma, \lambda\right) \le \sum_{k=2}^{\infty} b_k D_n(k, A, B, \alpha, \beta, \gamma, \lambda)$$

and then

$$\sum_{k=2}^{\infty} a_k b_k D_n\left(k, A, B, \alpha, \beta^2, \gamma, \lambda\right) \le \sum_{k=2}^{\infty} a_k b_k D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \le$$

$$\le \frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\left[1 - \beta B + \beta\gamma(B-A)(2-\alpha)\right]} \le \beta^2\gamma(B-A)(1-\alpha)$$

where we used that

$$\frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\left[1 - \beta B + \beta\gamma(B-A)(2-\alpha)\right]} \le \beta^2\gamma(B-A)(1-\alpha)$$

which is equivalent to

$$\beta^2\gamma(B-A)(1-\alpha)\left\{\gamma(B-A)(1-\alpha) - \right.$$

$$\left. -2^n(1+\lambda)\left[1 - \beta B + \beta\gamma(B-A)(2-\alpha)\right]\right\} \le 0.$$

Since $B^2 r(B-A)(1-\alpha) > 0$ we have:

$$\gamma(B-A)(1-\alpha) - 2^n(1+\lambda)(1-\beta B) - 2^n(1+\lambda)\beta\gamma(B-A)(2-\alpha) \le 0$$

where

$$\gamma(B-A)(1-\alpha) - 2^n(1+\lambda)(1-\beta B) - 2^n(1+\lambda)\beta\gamma(B-A) -$$

$$-2^n(1+\lambda)\beta\gamma(B-A)(1-\alpha) \le 0$$

which

$$\gamma(B-A)(1-\alpha)\left[1 - 2^n(1+\lambda)\beta\right] - 2^n(1+\lambda)(1-\beta B) -$$

$$-2^n(1+\lambda)\beta\gamma(B-A) \le 0.$$

Acording to Lemma we obtain $f * g \in T_{n,\lambda}(A, B, \alpha, \beta^2, \gamma)$.

Theorem 3. *Let $p > 0$ and $\frac{p+2-\sqrt{p^2+4p}}{2} \le \alpha \le \frac{1}{1+p}$. If the functions f and g defined by* (3) *and* (4) *belong to the same class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, then the Hadamard product belongs to the class $T_{n,\lambda}(A, B, 1 - p\alpha, \beta, \gamma)$.*

We have

$$D_n\left(k, A, B, 1 - p\alpha, \beta, \gamma, \lambda\right) \le D_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right);\ \text{when}\ \ \alpha \in \left[0, \frac{1}{p+1}\right]$$

because

$$k^n\left[1 + \lambda\left(k - 1\right)\right]\left\{\left(k - 1\right) + \beta\left[\left(B - A\right)\gamma\left(k - 1 + p\alpha\right) - B\left(k - 1\right)\right]\right\} \le$$
$$\le k^n\left[1 + \lambda\left(k - 1\right)\right]\left\{\left(k - 1\right) + \beta\left[\left(B - A\right)\gamma\left(k - \alpha\right) - B\left(k - 1\right)\right]\right\}$$

if and only if

$$\beta\left(B - A\right)\gamma\left(k - 1 + p\alpha - k + \alpha\right) \le 0,$$

if and only if

$$\alpha\left(1 + p\right) \le 1 \text{ if anf only if } \alpha \le \frac{1}{1 + p}$$

and

$$\sum_{k=2}^{\infty} a_k b_k D_n\left(k, A, B, 1 - p\alpha, \beta, \gamma, \lambda\right) \le \sum_{k=2}^{\infty} a_k b_k D_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \le$$
$$\le \frac{\beta^2\gamma^2(B - A)^2(1 - \alpha)^2}{2^n(1 + \lambda)\left[1 - \beta B + \beta\gamma(B - A)(2 - \alpha)\right]} \le$$
$$\le \beta\gamma(B - A)(1 - \alpha)^2 \le \beta\gamma(B - A)p\alpha,$$

because

$$\left(1 - \alpha\right)^2 \le p\alpha \Leftrightarrow \alpha \in \left[\frac{p + 2 - \sqrt{p^2 + 4p}}{2}, \frac{p + 2 + \sqrt{p^2 + 4p}}{2}\right].$$

We note that $\frac{p + 2 - \sqrt{p^2 + 4p}}{2} < \frac{1}{p + 1} < \frac{p + 2 + \sqrt{p^2 + 4p}}{2}$ when $p > 0$.

Corollary 1. *Let* $\frac{3-\sqrt{5}}{2} \le \alpha \le \frac{1}{2}$ *and the functions* f *defined by* (3) *and* g *defined by* (4) *be in the same class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. *Then the Hadamard product belongs to the class* $T_{n,\lambda}(A,B,1-\alpha,\beta,\gamma)$.

Corollary 2. *Let* $2-\sqrt{3} \le \alpha \le \frac{1}{3}$ *and let the functions* f *and* g *defined by* (3) *and* (4) *be in the same class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. *Then the Hadamard product belongs to the class* $T_{n,\lambda}(A,B,1-2\alpha,\beta,\gamma)$.

Theorem 4. *Let* $p > 0$, $\frac{p+2-\sqrt{p^2+4p}}{2} \le \alpha \le \frac{1}{1+p}$ *and* $\frac{1}{2^n(1+\lambda)} \le \beta \le 1$. *If the functions* f *and* g *defined by* (3) *and* (4) *belong to the same class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, *then the Hadamard product belongs to the class* $T_{n,\lambda}(A,B,1-p\alpha,\beta^2,\gamma)$.

Proof. By using

$$a_k \le \frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\left[1-\beta B+\beta\gamma(B-A)(2-\alpha)\right]} \text{ ; for all } k \ge 2$$

and

$$\sum_{k=2}^{\infty} b_k D_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right) \le \beta\gamma(B-A)(1-\alpha)$$

we obtain

$$\sum_{k=2}^{\infty} a_k b_k D_n\left(k,A,B,1-p\alpha,\beta^2,\gamma,\lambda\right) \le \sum_{k=2}^{\infty} a_k b_k D_n\left(k,A,B,\alpha,\beta^2,\gamma,\lambda\right) \le$$

$$\le \sum_{k=2}^{\infty} a_k b_k D_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right) \le$$

$$\le \frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\left[1-\beta B+\beta\gamma(B-A)(2-\alpha)\right]} \le$$

$$\le \beta^2\gamma(B-A)(1-\alpha)^2 \le \beta^2\gamma(B-A)p\alpha$$

which implies that $f*g \in T_{n,\lambda}(A,B,1-p\alpha,\beta^2,\gamma)$.

Corollary 3. *Let* $\frac{3-\sqrt{5}}{2} \leq \alpha \leq \frac{1}{2}$, $\frac{1}{2^n(1+\lambda)} \leq \beta \leq 1$ *and let the functions* f *and* g *defined by* (3) *and* (4) *be in the same class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. *Then the Hadamard product belongs to the class* $T_{n,\lambda}(A, B, 1-\alpha, \beta^2, \gamma)$.

Corollary 4. *Let* $2-\sqrt{3} \leq \alpha \leq \frac{1}{3}$, $\frac{1}{2^n(1+\lambda)} \leq \beta \leq 1$ *and let the functions* f *and* g *defined by* (3) *and* (4) *be in the same class* $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. *Then the Hadamard product belongs to the class* $T_{n,\lambda}(A, B, 1-2\alpha, \beta^2, \gamma)$.

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