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Quasisymmetric and quasimöbius maps on locally convex spaces

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Abstract

We give the definitions of the quasisymmetric and quasimöbius maps, maps defined on a locally convex spaces and some properties of this applications.

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Let E be a locally convex spaces. We denote by $\mathcal{A}$, the family of continuous semi-norms on E.

Definition 1. *Let (x_1, x_2, x_3) be a triple of distinct points in E. For $\alpha \in \mathcal{A}$, the α–ration of (x_1, x_2, x_3) is the number $\rho_\alpha(x_1, x_2, x_3)$ defined by*

$$\rho_\alpha(x_1,x_2,x_3)=\begin{cases} \dfrac{|x_2-x_1|_\alpha}{|x_3-x_1|_\alpha}\ , & \textit{if} \quad |x_3-x_1|_\alpha \neq 0 \\ 0\ , & \textit{if} \quad |x_2-x_1|_\alpha = |x_3-x_1|_\alpha = 0 \\ \infty & \textit{if} \quad |x_2-x_1|_\alpha \neq 0\ ,\ |x_3-x_1|_\alpha = 0 \end{cases} .$$

Definition 2. *Let (x_1, x_2, x_3, x_4) be a quadruple of distinct points in E. For $\alpha \in \mathcal{A}$, the α–cross ratio of (x_1, x_2, x_3, x_4) is the number $\tau_\alpha(x_1, x_2, x_3, x_4)$ defined by*

$$\tau_\alpha(x_1,x_2,x_3,x_4) = \begin{cases} \dfrac{|x_1-x_3|_\alpha}{|x_1-x_4|_\alpha} \cdot \dfrac{|x_2-x_4|_\alpha}{|x_2-x_3|_\alpha} \ \textit{if}\ |x_1-x_4|_\alpha \cdot |x_2-x_3|_\alpha \neq 0 \\ 0 \ \textit{if} \begin{cases} |x_1-x_3|_\alpha \cdot |x_2-x_4|_\alpha = 0 \\ |x_1-x_4|_\alpha \cdot |x_2-x_3|_\alpha = 0 \end{cases} \\ \infty \ \textit{if} \begin{cases} |x_1-x_3|_\alpha \cdot |x_2-x_4|_\alpha \neq 0 \\ |x_1-x_4|_\alpha \cdot |x_2-x_3|_\alpha = 0 \end{cases} \end{cases}.$$

Proposition 1. *Lets E, F two locally convex spaces and $\mathcal{A}$, $\mathcal{B}$ the families of continuous semi-norms on E and F respectivelly. For $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$, and a homeomorphism $\eta : [0,\infty] \longrightarrow [0,\infty]$ $(\eta(\infty) = \infty)$, if $f : E \longrightarrow F$ verifies*

$$(1) \qquad |x' - x''|_{\varphi(\beta)} = 0 \textit{ is equivalent with } |f(x') - f(x'')|_\beta = 0$$

for any $x', x'' \in E$, then for

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) \in \{0, \infty\}$$

we have

$$\rho_\beta(f(x_1), f(x_2), f(x_3)) = \eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)).$$

Proof. If

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) = 0$$

then

$$\rho_{\varphi(\beta)}(x_1, x_2, x_3) = 0$$

and using the definition 1,

$$\begin{cases} |x_2 - x_1|_{\varphi(\beta)} = 0 \\ |x_3 - x_1|_{\varphi(\beta)} \geq 0. \end{cases}$$

From (1) we have

$$|f(x_2) - f(x_1)|_\beta = 0, \quad |f(x_3) - f(x_1)|_\beta \geq 0$$

and consequently,

$$\rho_\beta(f(x_1), f(x_2), f(x_3)) = 0.$$

If

$$\eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3)) = \infty$$

then

$$\rho_{\varphi(\beta)}(x_1, x_2, x_3) = \infty$$

and from the definition 2,

$$\begin{cases} |x_2 - x_1|_{\varphi(\beta)} \neq 0 \\ |x_3 - x_1|_{\varphi(\beta)} = 0. \end{cases}$$

From (1) we have

$$\begin{cases} |f(x_2) - f(x_1)|_\beta \neq 0 \\ |f(x_3) - f(x_1)|_\beta = 0 \end{cases}$$

and

$$\rho_\beta(f(x_1), f(x_2), f(x_3)) = \infty.$$

Proposition 2. *Lets E, F two locally convex spaces and $\mathcal{A}$, $\mathcal{B}$ the families of continuous semi-norms on E and F respectivelly. For $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$, and a homeomorphism $\eta : [0, \infty] \longrightarrow [0, \infty]$ $(\eta(\infty) = \infty)$, if $f : E \longrightarrow F$ verifies*

$$|x' - x''|_{\varphi(\beta)} = 0 \textit{ is equivalent with } |f(x') - f(x'')|_\beta = 0$$

for any $x', x'' \in E$, *then for*

$$\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) \in \{0, \infty\}$$

we have

$$\tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) = \eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)).$$

Proof. We have the implications

$$\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) = 0 \text{ which implies } \tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4) = 0$$

and from the definition 2,

$$\begin{cases} |x_1 - x_3|_{\varphi(\beta)} \cdot |x_2 - x_4|_{\varphi(\beta)} = 0 \\ |x_1 - x_4|_{\varphi(\beta)} \cdot |x_2 - x_3|_{\varphi(\beta)} \geq 0. \end{cases}$$

Using (1) we have

$$\begin{cases} |f(x_1) - f(x_3)|_\beta \cdot |f(x_2) - f(x_4)|_\beta = 0 \\ |f(x_1) - f(x_4)|_\beta \cdot |f(x_2) - f(x_3)|_\beta \geq 0 \end{cases}$$

and

$$\tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) = 0.$$

Also if

$$\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)) = \infty$$

then

$$\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4) = \infty$$

and

$$\begin{cases} |x_1 - x_3|_{\varphi(\beta)} \cdot |x_2 - x_4|_{\varphi(\beta)} \neq 0 \\ |x_1 - x_4|_{\varphi(\beta)} \cdot |x_2 - x_3|_{\varphi(\beta)} = 0 \end{cases} \text{ which implies}$$

$$\begin{cases} |f(x_1)-f(x_3)|_\beta \cdot |f(x_2)-f(x_4)|_\beta \neq 0 \\ |f(x_1)-f(x_4)|_\beta \cdot |f(x_2)-f(x_3)|_\beta = 0. \end{cases}$$

Finally,

$$\tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) = \infty.$$

Definition 3. *Lets E, F two locally convex spaces, $\mathcal{A}$, $\mathcal{B}$ the families of continuous semi-norms on E and F respectively and $D \subset E$ a open set. We say that $f : D \longrightarrow F$ is a quasisymmetric map if there exists $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ and a homeomorphism $\eta : [0,\infty] \longrightarrow [0,\infty]$ $(\eta(\infty) = \infty)$ such that*

(i) $|x_1 - x_2|_{\varphi(\beta)} = 0$ *is equivalent with* $|f(x_1) - f(x_2)|_\beta = 0$

(ii) $\rho_\beta(f(x_1), f(x_2), f(x_3)) \leq \eta(\rho_{\varphi(\beta)}(x_1, x_2, x_3))$

for any $x_1, x_2, x_3 \in D$.

Definition 4.*Lets E, F two locally convex spaces, $\mathcal{A}$, $\mathcal{B}$ the families of continuous semi-norms on E and F, respectively and $D \subset E$ a open set. We say that $f : D \longrightarrow F$ is a quasiimöbius map if there exists $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ and a homeomorphism $\eta : [0,\infty] \longrightarrow [0,\infty]$ $(\eta(\infty) = \infty)$ such that*

(i) $|x_1 - x_2|_{\varphi(\beta)} = 0$ *is equivalent with* $|f(x_1) - f(x_2)|_\beta = 0$

(ii) $\tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) \leq \eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4))$

for any $x_1, x_2, x_3, x_4 \in D$.

Proposition 3.*If $T \in Isom\,(E, F)$, then there exists $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$ so that T is quasisymmetric and quasimöbius for $\eta : [0,\infty] \longrightarrow [0,\infty]$, $\eta(t) = t$.*

Proof. Let $\beta \in \mathcal{B}$. The application $x \longrightarrow |Tx|_\beta$ is a continuous semi-norm on E. There exists $\alpha \in \mathcal{A}$ so that $|x|_\alpha = |Tx|_\beta$. We define $\varphi : \mathcal{B} \longrightarrow \mathcal{A}$, $\varphi(\beta) = \alpha$.

If $x_1, x_2 \in E$

$$|T(x_1) - T_{x_2}|_\beta = |T(x_1 - x_2)|_\beta = |x_1 - x_2|_{\varphi(\beta)}$$

and (i) is satisfied.

Also, we have,

$$\rho_\beta(T(x_1), T(x_2), T(x_3)) = \rho_{\varphi(\beta)}(x_1, x_2, x_3)$$

$$\tau_\beta(T(x_1), T(x_2), T(x_3), T(x_4)) = \tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)$$

for any $x_1, x_2, x_3, x_4 \in E$ and the conditions (ii) from the two definitions are true for $\eta(t) = t$.

Proposition 4.*If*

$$|x_1 - x_3|_\alpha \cdot |x_1 - x_4|_\alpha + |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha \neq 0 \ ,$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \frac{1}{\tau_\alpha(x_1, x_2, x_4, x_3)} \ .$$

Proof. If

$$\begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha \neq 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha \neq 0 \end{cases}$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \frac{|x_1 - x_3|_\alpha}{|x_1 - x_4|_\alpha} \cdot \frac{|x_2 - x_4|_\alpha}{|x_2 - x_3|_\alpha} =$$

$$= \left(\frac{|x_1 - x_4|_\alpha}{|x_1 - x_3|_\alpha} \cdot \frac{|x_2 - x_3|_\alpha}{|x_2 - x_4|_\alpha} \right)^{-1} = \frac{1}{\tau_\alpha(x_1, x_2, x_4, x_3)} \ .$$

If

$$\begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha = 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha \neq 0 \end{cases}$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = 0$$

and

$$\tau_\alpha(x_1, x_2, x_4, x_3) = \infty \ ,$$

and if

$$\begin{cases} |x_1 - x_3|_\alpha \cdot |x_2 - x_4|_\alpha \neq 0 \\ |x_1 - x_4|_\alpha \cdot |x_2 - x_3|_\alpha = 0 \end{cases}$$

then

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \infty$$

and

$$\tau_\alpha(x_1, x_2, x_4, x_3) = 0.$$

In the last two cases, we have also

$$\tau_\alpha(x_1, x_2, x_3, x_4) = \frac{1}{\tau_\alpha(x_1, x_2, x_4, x_3)} \ .$$

Proposition 5. *If $f : D \longrightarrow F$ is a quasimöbius map then we have*

$$\eta^{-1}(\tau^{-1}_{\varphi(\beta)}(x_1, x_2, x_3, x_4) \leq \tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)) \leq \eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)),$$

for any $x_1, x_2, x_3, x_4 \in D$ with

$$|x_1 - x_3|_{\varphi(\beta)} \cdot |x_2 - x_4|_{\varphi(\beta)} + |x_1 - x_4|_{\varphi(\beta)} \cdot |x_2 - x_3|_{\varphi(\beta)} \neq 0 \ .$$

Proof. Firstly, we have

$$|x_1 - x_3|_{\varphi(\beta)} \cdot |x_2 - x_4|_{\varphi(\beta)} + |x_1 - x_4|_{\varphi(\beta)} \cdot |x_2 - x_3|_{\varphi(\beta)} \neq 0$$

is equivalent with

$$|f(x_1) - f(x_3)|_\beta \cdot |f(x_2) - f(x_4)|_\beta + |f(x_1) - f(x_4)|_\beta \cdot |f(x_2) - f(x_3)|_\beta \neq 0$$

and using the previous proposition, we can write:

$$\eta^{-1}(\tau^{-1}_{\varphi(\beta)}(x_1, x_2, x_3, x_4) = \frac{1}{\eta\left(\dfrac{1}{\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4)}\right)} = \frac{1}{\eta(\tau_{\varphi(\beta)}(x_1, x_2, x_4, x_3))} \leq$$

$$\leq \frac{1}{\tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4))} = \tau_\beta(f(x_1), f(x_2), f(x_3), f(x_4)).$$
$$\leq \eta(\tau_{\varphi(\beta)}(x_1, x_2, x_3, x_4).$$

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