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A preserving property of a generalized Libera integral operator

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Abstract

In this paper we prove that the logarithmically n-spirallike of type γ functions are preserved by a generalized Libera integral operator.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U and $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$.

Let consider the integral operator $L_a : A \to A$ defined as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt \ , \quad a \in \mathbb{C} \ , \quad Re\ a \geq 0.$$

If we consider $a = 1$ we obtain the Libera integral operator and for $a = 0$ we obtain the Alexander integral operator. In the case $a = 1, 2, 3, ...$ this operator was introduced by S. D. Bernardi and it was studied by many authors in different general cases.

Let D^n be the Sălăgean differential operator (see [7]) defined as:

$$D^n : A \to A \;, \quad n \in \mathbb{N} \;\text{ and }\; D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z) \;, \quad D^n f(z) = D(D^{n-1} f(z)).$$

2 Preliminary results

Definition 2.1. *Let $f \in A$ and $n \in \mathbb{N}$. We say that f is a n-starlike function if:*

$$Re \frac{D^{n+1} f(z)}{D^n f(z)} > 0 \;,\; z \in U.$$

We denote this class with ${S^*}_n$.

Definition 2.2. *Let $f \in A$ and $n \in \mathbb{N}$. We say that f is logarithmically n-spirallike of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $D^n f(z) \neq 0$, $z \in U$ and*

$$Re \left[e^{i\gamma} \frac{D^{n+1} f(z)}{D^n f(z)} \right] > 0 \,,\; z \in U \,.$$

We denote this class with $S_{\gamma,n}$.

Remark 2.1. *If we consider $\gamma = 0$ we obtain the concept of n-starlike functions and for $n = 0$ we obtain the classical spirallike functions. We denote the set of all spirallike functions with S_γ.*

The next theorem is result of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [2], [3], [4]).

Theorem 2.1. *Let h convex in U and $Re\,[\beta h(z)+\delta] > 0$. If $q \in \mathcal{H}(U)$ with $q(0)=h(0)$ and q satisfied $q(z)+\dfrac{zq'(z)}{\beta q(z)+\delta} \prec h(z)$, then $q(z) \prec h(z)$.*

3 Main results

Theorem 3.1. *If $F(z) \in S_{\gamma,n}$ then $f(z) = L_a F(z) \in S_{\gamma,n}$.*

Proof. By differentiating (1) we obtain

$$(1+a)F(z) = af(z) + zf'(z)\,.$$

By means of the applications of the linear operator D^{n+1} we obtain:

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf'(z))$$

or

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z)\,.$$

It is easy to see that in the conditions of the hypothesis we have $D^n f(z) \neq 0\,,\ z \in U$.

With notation $\dfrac{D^{n+1}f(z)}{D^n f(z)} = p(z)$, where $p(z) = 1 + p_1 z + ...$, by simple calculations we obtain

$$\frac{D^{n+1}F(z)}{D^nF(z)} = p(z) + \frac{1}{p(z)+a}\cdot zp'(z)\,.$$

From here we have

$$e^{i\gamma}\frac{D^{n+1}F(z)}{D^nF(z)} = e^{i\gamma}p(z) + \frac{e^{i\gamma}}{p(z)+a}\cdot zp'(z)\,.$$

If we denote $e^{i\gamma}p(z) = q(z)$ we obtain

$$(2)\qquad e^{i\gamma}\frac{D^{n+1}F(z)}{D^nF(z)} = q(z) + \frac{1}{e^{-i\gamma}q(z)+a}\cdot zq'(z)\,.$$

If we consider $h(z) = \dfrac{1+z}{1-z}e^{i\gamma}$ which is convex in U and maps the unit disc into a convex domain included in the right half plane, then using the hypothesis from (2) we obtain:

$$q(z) + \frac{1}{e^{-i\gamma}q(z)+a} \cdot zq'(z) \prec h(z)\,.$$

In this conditions, using $Re\, a \geq 0$, we obtain $Re\left[e^{-i\gamma}h(z)+a\right] > 0$. From Theorem (2.1), with $\beta = e^{-i\gamma}$ and $\delta = a$, we have $q(z) \prec h(z)$ or

$$e^{i\gamma}p(z) = e^{i\gamma}\frac{D^{n+1}f(z)}{D^n f(z)} \prec h(z) \prec \frac{1+z}{1-z}\,.$$

Thus we obtain $Re\left[e^{i\gamma}\dfrac{D^{n+1}f(z)}{D^n f(z)}\right] > 0\,,\ z \in U$ or $f(z) = L_aF(z) \in S_{\gamma,n}$.

If we take $\gamma = 0$ in Theorem (3.1) we obtain:

Corollary 3.1. *If* $F(z) \in {S_n}^*$ *then* $f(z) = L_aF(z) \in {S_n}^*$.

Remark 3.1. *In the case* $n = 0$ *from Theorem (3.1) we obtain:*

If $F(z) \in S_\gamma$ *then* $f(z) = L_aF(z) \in S_\gamma$.

This result is a particular case of the more general results given by P.T. Mocanu and S.S. Miller in [5] and [6].

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