General Mathematics Vol. 12, No. 2 (2004), 53–60

Weighted Markov inequalities for curved majorants

Ioan Popa

Abstract

We give exact estimations of certain weighted L^2-norms of the derivative of polynomial, which have a curved majorant. They are all obtained as applications of special quadrature formulae.

2000 Mathematical Subject Classification: 41A17, 41A05, 41A55, 65D30

Keywords: Bouzitat quadrature, Chebyshev polynomials, inequalities

1 Introduction

The following problem was raised by P.Turán.

Let $\varphi(x) \geq 0$ *for* $-1 \leq x \leq 1$ *and consider the class* $P_{n,\varphi}$ *of all polynomials of degree* n *such that*

$|p_n(x)| \leq \varphi(x)$ *for* $-1 \leq x \leq 1$.

How large can $max_{[-1,1]} \left| p_n^{(k)}(x) \right|$ *be if* p_n *is arbitrary in* $P_{n,\varphi}$ *?*

The aim of this paper is to consider the solution in the weighted L^2-norm for the majorants $\varphi(x)=\sqrt{\frac{1+x}{1-x}}$ and $\Omega_n(x)=\frac{n+1-nx}{(1-x)\sqrt{1-x^2}}$.
Let as denote by

$$x_i=\cos\frac{(2i-1)\pi}{2n},\ \ i=1,2,...,n, \text{the zeros of } T_n(x)=\cos n\theta, \tag{1}$$

$x=\cos\theta$ the Chebyshev polynomial of the first kind,

$$y_i \text{ the zeros of } U_{n-1}(x)\,,\ U_{n-1}(x)=\sin n\theta/\sin\theta, \tag{2}$$

$x=\cos\theta$, the Chebyshev polynomial of the second kind and

$$W_n(x)=\frac{\sin\left[(2n+1)\theta/2\right]}{\sin(\theta/2)}, x=\cos\theta. \tag{3}$$

Let H_1 be the class of real polynomials p_{n-1}, of degree $\le n-1$ such that

$$|p_{n-1}(x_i)|\le\sqrt{\frac{1+x_i}{1-x_i}}, i=1,2,...,n, \tag{4}$$

where the x_i's are given by (1).

Let H_2 be the class of real polynomials p_{n-1}, of degree $\le n-1$ such that

$$|p_{n-1}(x_i)|\le\frac{n+1-nx_i}{(1-x_i)\sqrt{1-x_i^2}}, i=1,2,...,n, \tag{5}$$

where the x_i's are given by (1). Note that $P_{n-1,\varphi}\subset H_1$, $P_{n-1,\Omega_n}\subset H_2$, $W_{n-1}\in H_1, W_{n-1}\notin P_{n-1,\varphi}, W_n'\in H_2, W_{n.}'\notin P_{n-1,\Omega_n}$.

2 Results

Theorem 1. *If* $p_{n-1} \in H_1$ *then we have*

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[p'_{n-1}(x)\right]^2 dx \le \frac{2\pi n(n-1)(2n-1)}{3} \tag{6}$$

with equality for $p_{n-1} = W_{n-1}$.

Theorem 2. *If* $p_{n-1} \in H_2$ *and*

$r(x) = b(b-2a)x^2 + 2c(b-a)x + a^2 + c^2$

with $0 < a < b, |c| < b-a, b \neq 2a$, *then we have*

$$\int_{-1}^{1} r(x) \sqrt{\frac{1-x}{1+x}} \left[p'_{n-1}(x)\right]^2 dx \tag{7}$$

$$\le \frac{\pi n\left(n^2-1\right)(n+2)(2n+1)\left[\left[7(a-b+c)^2+(a-b-c)^2\right]\left(n^2+n-2\right)-4(a-b-c)^2+28\left(a^2+c^2\right)\right]}{105}$$

with equality for $p_{n-1} = W'_n$.

Corollary 1. *If* $p_{n-1} \in H_2$ *then we have*

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[p'_{n-1}(x)\right]^2 dx \le \frac{8\pi n(n^2-1)(n+2)(2n+1)(n^2+n+1)}{105} \tag{8}$$

with equality for $p_{n-1} = W'_n$.

3 Lemmas

Here we state some lemmas which help us in proving the theorems.

Lemma 1. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \le \sqrt{\frac{1+x_i}{1-x_i}}, i=1,2,...,n,$ *where the* x_i*'s are given by (1). Then we have*

$$\left|p'_{n-1}(y_j)\right| \le \left|W'_{n-1}(y_j)\right|, j=1,...,n-1, \text{ and} \tag{9}$$

$$\left|p'_{n-1}(1)\right| \le \left|W'_{n-1}(1)\right|, \quad \left|p'_{n-1}(-1)\right| \le \left|W'_{n-1}(-1)\right| . \tag{10}$$

Proof. By the Lagrange interpolation formula based on the zeros of T_n and using $T'_n(x_i)=\frac{(-1)^{i+1}n}{\left(1-x_i^2\right)^{1/2}}$, we can represent any polynomial p_{n-1} by $p_{n-1}(x)=\frac{1}{n}\sum_{i=1}^{n}\frac{T_n(x)}{x-x_i}(-1)^{i+1}(1-x_i^2)^{1/2}p_{n-1}(x_i)$.

From $W_{n-1}(x_i)=(-1)^{i+1}\sqrt{\frac{1+x_i}{1-x_i}}$ we have $W_{n-1}(x)=\frac{1}{n}\sum_{i=1}^{n}\frac{T_n(x)}{x-x_i}(1+x_i)$.

Differentiating with respect to x we obtain

$p'_{n-1}(x)=\frac{1}{n}\sum_{i=1}^{n}\frac{T'_n(x)(x-x_i)-T_n(x)}{(x-x_i)^2}(-1)^{i+1}(1-x_i^2)^{1/2}p_{n-1}(x_i)$.

On the roots of $T'_n(x)=nU_{n-1}(x)$ and using (4) we find

$\left|p'_{n-1}(y_j)\right| \le \frac{1}{n}\sum_{i=1}^{n}\frac{|T_n(y_j)|}{(y_j-x_i)^2}(1+x_i)=\frac{|T_n(y_j)|}{n}\sum_{i=1}^{n}\frac{1+x_i}{(y_j-x_i)^2}=\left|W'_{n-1}(y_j)\right|$.

For $l_i(x)=\frac{T_n(x)}{x-x_i}$ taking into account that $l'_i(1)>0$ (see [5]) it folows

$\left|p'_{n-1}(1)\right| \le \frac{1}{n}\sum_{i=1}^{n}l'_i(1)(1+x_i)=\left|W'_{n-1}(1)\right|$.

Similarly $\left|p'_{n-1}(-1)\right| \le \left|W'_{n-1}(-1)\right|$.

Lemma 2. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \le \frac{n+1-nx_i}{(1-x_i)\sqrt{1-x_i^2}}, i=1,2,...,n,$ *where the* x_i*'s are given by* (1). *Then we have*

$$\left|p'_{n-1}(y_j)\right| \le |W''_n(y_j)|, j=1,...,n-1, \text{ and} \tag{11}$$

$$\left|p'_{n-1}(1)\right| \le |W''_n(1)|, \quad \left|p'_{n-1}(-1)\right| \le |W''_n(-1)| . \tag{12}$$

Proof. Using $W_n'(x_i) = (-1)^{i+1} \frac{n+1-nx_i}{(1-x_i)\sqrt{1-x_i^2}}$ the proof is along the same lines as in previous Lemma.

The following proposition was proved in [2]

Lemma 3. *A real polynomial* r *of exact degree 2 satisfies* $r(x) > 0$ *for* $-1 \le x \le 1$ *if and only if* $r(x) = b(b-2a)x^2 + 2c(b-a)x + a^2 + c^2$ *with* $0 < a < b, \ \ |c| < b - a, \ \ b \ne 2a.$

We need the following quadrature formulae:

Lemma 4. *For any given* n *we have the following formulae:*

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} f(x)\,dx = \frac{\pi}{n} f(-1) + \sum_{i=1}^{n-1} C_i f(y_i) \quad \text{of degree } 2n-2, \tag{13}$$

where the nodes are the roots of $T_n' = nU_{n-1}$ *and* $C_i > 0$,

$$\int_{-1}^{1} r(x) \sqrt{\frac{1-x}{1+x}} f(x)\,dx = \frac{\pi\,(a-b+c)^2}{n} f(-1) + \sum_{i=1}^{n-1} C_i r(y_i) f(y_i) \tag{14}$$

of degree 2n-4,

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} f(x)\,dx = \frac{\pi\,(2n-1)}{2n\,(n-1)} f(-1) + \frac{3\pi}{2n\,(n-1)\,(2n-1)} f(1) + \sum_{i=1}^{n-2} C_i f\left(x_i^{\left(\frac{3}{2},\frac{1}{2}\right)}\right), \tag{15}$$

of degree 2n-3, $x_i^{\left(\frac{3}{2},\frac{1}{2}\right)}$ *the roots of* $W_{n-1}'(x) = c \cdot P_{n-2}^{\left(\frac{3}{2},\frac{1}{2}\right)}(x)$,

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} f(x)\,dx = \frac{M\,(2n+1)\,(6n^2+6n-11)}{5} f(-1) \tag{16}$$

$+\frac{5M\left(10n^2+10n-11\right)}{7(2n+1)}f\left(1\right)+M\left(2n+1\right)f'\left(-1\right)-\frac{15M}{2n+1}f'\left(1\right)+\sum_{i=1}^{n-2}c_if\left(x_i^{\left(\frac{5}{2},\frac{3}{2}\right)}\right),$
of degree $2n-1$, $M=\frac{3\pi}{4(n^3-n)(n+2)}$ *and the nodes are the roots of* $W''_n\left(x\right)=c\cdot P_{n-2}^{\left(\frac{5}{2},\frac{3}{2}\right)}\left(x\right)$,

$$(17)\qquad \int_{-1}^{1} r\left(x\right)\sqrt{\frac{1-x}{1+x}}f\left(x\right)dx=Af\left(-1\right)+Bf\left(1\right)+Cf'\left(-1\right)-Df'\left(1\right)$$

$+\sum_{i=1}^{n-2}c_ir\left(x_i^{\left(\frac{5}{2},\frac{3}{2}\right)}\right)f\left(x_i^{\left(\frac{5}{2},\frac{3}{2}\right)}\right)$ *of degree* $2n-3$, *where*
$A=\frac{M(2n+1)(\left(6n^2+6n-11\right)(a-b+c)^2+10d)}{5}$, $B=\frac{M(5\left(10n^2+10n-11\right)(a-b-c)^2-210e)}{7(2n+1)}$
$C=M\left(2n+1\right)\left(a-b+c\right)^2$, $D=\frac{-15M(a-b-c)^2}{2n+1}$,
$d=2ab+bc-ac-b^2, e=b^2-2ab+bc-ac.$

Proof. (13) is the Bouzitat formula of the first kind [3, *formula* (4.7.1)] on the zeroes of $U_{n-1}=c\cdot P_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}$. If in formula (13) we replace $f\left(x\right)$ with $r\left(x\right)f\left(x\right)$ we get (14). (15) is the Bouzitat formula of the second kind [3, *formula* (4.8.1)] on the zeroes of $W'_{n-1}\left(x\right)=c\cdot P_{n-2}^{\left(\frac{3}{2},\frac{1}{2}\right)}\left(x\right)$.

We will write the formula (16) in the following way
$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f\left(x\right)dx=Af\left(-1\right)+Bf\left(1\right)+Cf'\left(-1\right)+Df'\left(1\right)+\sum_{i=1}^{n-2}c_if\left(x_i^{\left(\frac{5}{2},\frac{3}{2}\right)}\right).$

If in above formula we put $f\left(x\right)=\left(1-x\right)\left(1+x\right)^2P_{n-2}^{\left(\frac{5}{2},\frac{3}{2}\right)}\left(x\right)$ we obtain D, for $f\left(x\right)=\left(1-x\right)^2\left(1+x\right)P_{n-2}^{\left(\frac{5}{2},\frac{3}{2}\right)}\left(x\right)$ we get C, for $f\left(x\right)=\left(1+x\right)^2P_{n-2}^{\left(\frac{5}{2},\frac{3}{2}\right)}\left(x\right)$ we find B and for $f\left(x\right)=\left(1-x\right)^2P_{n-2}^{\left(\frac{5}{2},\frac{3}{2}\right)}\left(x\right)$ we find A.
If in formula (16) we replace $f\left(x\right)$ with $r\left(x\right)f\left(x\right)$ we get (17).

4 Proofs of the theorems

Since $W_n\left(x\right)=\frac{(2n)!!}{(2n-1)!!}P_n^{\left(\frac{1}{2},\frac{-1}{2}\right)}\left(x\right)$ we recall the formulae:

$$(18) \qquad \frac{d}{dx}P_m^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+m+1}{2}P_{m-1}^{(\alpha+1,\beta+1)}(x),$$

$$P_m^{(\alpha,\beta)}(1) = \frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)\Gamma(m+1)}, P_m^{(\alpha,\beta)}(-1) = \frac{(-1)^m\Gamma(m+\beta+1)}{\Gamma(\beta+1)\Gamma(m+1)}$$

.

4.1 Proof of Theorem 1

Proof. According to quadrature formula (13), $C_i > 0$ and using (9) and (10) we have

$$\int_{-1}^{1}\sqrt{\tfrac{1-x}{1+x}}\left[p'_{n-1}(x)\right]^2 dx = \tfrac{\pi}{n}\left(p'_{n-1}(-1)\right)^2 + \sum_{i=1}^{n-1} C_i\left(p'_{n-1}(y_i)\right)^2$$

$$\leq \tfrac{\pi}{n}\left(W'_{n-1}(-1)\right)^2 + \sum_{i=1}^{n-1} C_i\left(W'_{n-1}(y_i)\right)^2 = \int_{-1}^{1}\sqrt{\tfrac{1-x}{1+x}}\left[W'_{n-1}(x)\right]^2 dx.$$

In order to complete the proof we apply formula (15) to $f = \left[W'_{n-1}(x)\right]^2$.

Using (18) we get $W'_{n-1}(-1) = (-1)^{n-2}n(n-1)$ and $W'_{n-1}(1) = \frac{n(n-1)(2n-1)}{3}$.

From (15) and having in mind that $W'_{n-1}\left(x_i^{\left(\frac{3}{2},\frac{1}{2}\right)}\right) = 0$, we find

$$\int_{-1}^{1}\sqrt{\tfrac{1-x}{1+x}}\left[W'_{n-1}(x)\right]^2 dx = \tfrac{\pi(2n-1)}{2n(n-1)}\left[W'_{n-1}(-1)\right]^2$$

$$+\tfrac{3\pi}{2n(n-1)(2n-1)}\left[W'_{n-1}(1)\right]^2 = \tfrac{2\pi n(n-1)(2n-1)}{3}$$

4.2 Proof of Theorem 2

Proof. According to quadrature formula (14), $C_i > 0$ and using (11) and (12) we have

$$\int_{-1}^{1} r(x)\sqrt{\tfrac{1-x}{1+x}}\left[p'_{n-1}(x)\right]^2 dx \leq \int_{-1}^{1} r(x)\sqrt{\tfrac{1-x}{1+x}}\left[W''_n(x)\right]^2 dx.$$

To complete the proof we apply formula (17) to $f = \left[W''_n(x)\right]^2$. From (18), $W''_n(-1) = \frac{(-1)^{n-2}\left(n^3-n\right)(n+2)}{3}$, $W''_n(1) = \frac{\left(n^3-n\right)(n+2)(2n+1)}{15}$,

$W_n^{(3)}(-1) = \frac{-(n-2)(n+3)}{5} W_n''(-1)$, $W_n^{(3)}(1) = \frac{(n-2)(n+3)}{7} W_n''(1)$.

Having in mind $W_n''\left(x_i^{\left(\frac{5}{2},\frac{3}{2}\right)}\right) = 0$ and using (17) we find

$$\int_{-1}^{1} r(x)\sqrt{\frac{1-x}{1+x}}\left[W_n''(x)\right]^2 dx = A\left(W_n''(-1)\right)^2 + B\left(W_n''(1)\right)^2 +$$
$$+2CW_n''(-1)\,W_n^{(3)}(-1) + 2DW_n''(1)\,W_n^{(3)}(1)$$
$$= \frac{\pi n\left(n^2-1\right)(n+2)(2n+1)\left[\left[7(a-b+c)^2+(a-b-c)^2\right]\left(n^2+n-2\right)-4(a-b-c)^2+28\left(a^2+c^2\right)\right]}{105}.$$

References

[1] D. K. Dimitrov:*Markov Inequalities for Weight Functions of Chebyshev Type*. J.Approx.Theory 83 (1995), 175–181.

[2] W. Gautschi and S. E. Notaris: *Gauss-Kronrod quadrature formulae for weight function of Bernstein-Szegö type*. J.Comput.Appl.Math. 25 (2)(1989), 199–224.

[3] A. Ghizzetti and A. Ossicini: *Quadrature formulae*, Akademie-Verlag, Berlin, 1970.

[4] A. Lupaş: *Numerical Methods*, Constant Verlag, Sibiu, 2001.

[5] R. Pierre and Q. I. Rahman: *On polynomials with curved majorants.* in Studies in Pure Mathematics,Budapest,(1983), 543–549.

Str. Ciortea 9/ 43

3400 Cluj-Napoca, Romania

E-mail address: *ioanpopa.cluj@personal.ro*