On the rate of approximation for the Bézier variant of Kantorovich-Balazs operators

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Abstract

In the present paper we study the generalized Balazs-Kantorovich-Bézier operators $L_{n,\alpha}^*(f,x)$. The special cases of our operators reduce to some well known operators. Recently Gupta and Ispir [*Applied Mathematics Letter*] obtained the rate of convergence for function of bounded variation for the case when $\alpha \geq 1$. We now estimate the rate of convergence for functions of bounded variation for the other case when $0 < \alpha < 1$.

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1 Introduction

For a real valued function f defined on the interval $[0, \infty)$, Balazs [2] introduced the Bernstein type rational functions, which are defined by

$$
(1) \qquad R_n(f, x) = \frac{1}{(1 + a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right),
$$

where a_n and b_n are suitably chosen positive numbers independent of x.

The weighted estimates and uniform convergence for the case $a_n = n^{\beta-1}$, $b_n = n^{\beta}$, $0 < \beta \le 2/3$ were investigated in [3]. Recently Ispir and Atakut [5] introduced the generalization of the Balazs operators, which are defined by

$$
(2) \qquad L_n(f, x) = \frac{1}{\phi_n(a_n x)} \sum_{k=0}^{\infty} \frac{\phi_n^{(k)}(0)}{k!} (a_n x)^k f\left(\frac{k}{b_n}\right), \quad n \in \mathbb{N}, x \ge 0,
$$

where a_n and b_n are suitably chosen positive numbers independent of x and $\{\phi_n\}$ is a sequence of functions $\phi_n : \mathbb{C} \to \mathbb{C}$ satisfying the following conditions:

(i) $\phi_n(n = 1, 2, ...)$; is analytic in a domain D containing the disk $B = \{z \in C : |z - b| \le b\}$;

(ii) $\phi_n(0) = 1 (n = 1, 2, ...)$;

(iii) For any $x \ge 0, \phi_n(x) > 0$ and $\phi_n^{(k)}(0) \ge 0$ for any $n = 1, 2, ...$ and $k = 1, 2, ...$;

(iv) For every $n = 1, 2, ...$

$$
\frac{\phi_n^{(\nu)}(a_n x)}{n^{\nu} \phi_n(a_n x)} = 1 + O\left(\frac{1}{n a_n}\right), \quad \nu = 1, 2, 3, 4
$$

where $a_n \to 0, na_n \to \infty$ as $n \to \infty$.

The operators defined by (2) are summation type operators, which are not capable to approximate integrable functions. To approximate integrable functions on the interval $[0, \infty)$, the Kantorovich variant of the generalized Balazs type operators is defined as

$$(3) \qquad L_n^*(f, x) = na_n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{I_{n,k}} f(t)dt, \ n \in \mathbb{N}, x \geq 0,$$

where $I_{n,k} = [k/na_n, (k+1)/na_n]$, $p_{n,k}(x) = \sum_{k=0}^{\infty} \dfrac{\phi_n^{(k)}(0)(a_n x)^k}{k! \phi_n(a_n x)}$ and $x \geq 0$.

Some particular cases of the operators are defined as:

Case 1: If $a_n = 1$ and $\phi_n(x) = e^{nx}$, then we obtain the Szász-Kantorovich operators, which are defined by:

$$S_n^*(f, x) = ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{k/n}^{(k+1)/n} f(t)dt, x \in [0, \infty).$$

Case 2: If $\phi_n(x) = (1 + x)^n$, then we obtain the Bernstein-Balazs-Kantorovich operators, which are defined by:

$$K_n^*(f, x) = na_n \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k (1 + a_n x)^{-n} \int_{k/na_n}^{(k+1)/na_n} f(t)dt, \ x \in [0, \infty).$$

The second case was studied by Agratini [1], who obtained the rate of convergence for functions of bounded variation. Very recently Gupta and Ispir [4] estimated the rate of convergence for the Bézier variant of generalized Balazs-Kantorovitch- Bézier operators for the case when $\alpha \geq 1$. Bézier

basis functions play an important role in Computer Aided Geometric Design. This along with the recent work on some Bézier variants of some well known operators (see [6]), motivated us to study further on some different operators. The Bézier variant of the generalized Balazs type operators is defined as:

$$(4) \qquad L_{n,\alpha}^*(f,x) = na_n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{I_{n,k}} f(t)dt, \ n \in \mathbb{N}, x \geq 0,$$

where

$$Q_{n,k}^{(\alpha)}(x) = \left\{ \sum_{j=k}^{\infty} p_{n,j}(x) \right\}^{\alpha} - \left\{ \sum_{j=k+1}^{\infty} p_{n,j}(x) \right\}^{\alpha}, \alpha \geq 1 \text{ or } 0 < \alpha < 1.$$

It may be noted that the operators defined by (4) are linear positive operators and $L_{n,\alpha}^*(1,x) = 1$. If $\alpha = 1$, $L_{n,\alpha}^*(f,x)$ reduce to the operators $L_n(f,x)$, defined by (3).

Throughout the paper let

$$W_{n,\alpha}(x,t) = na_n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\chi_{n,k}(t),$$

where $\chi_{n,k}$ is the characteristic function of the interval $[k/na_n, (k+1)/na_n]$ with respect to $I \equiv [0,\infty)$. Thus with this definition it is obvious that

$$L_{n,\alpha}^*(f,x) = \int_0^{\infty} f(t)W_{n,\alpha}(x,t)dt.$$

In case $\alpha = 1, W_{n,1}(x,t) \equiv W_n(x,t) = na_n \sum_{k=0}^{\infty} p_{n,k}(x)\chi_{n,k}(t)$.

Gupta and Ispir [4] estimated the rate of convergence by the Bézier variant of generalized Balazs Kantorovich operators for functions of bounded

variation for the case $\alpha \geq 1$. In this paper we study the rate of convergence for the case $0 < \alpha < 1$ Our main theorem is as follows:

Theorem 1. *Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$ and $V_a^b(g_x)$ is the total variation of g_x on $[a, b]$. If $0 < \alpha < 1$, $x \in (0, \infty)$, $r > 1$ and $A_\alpha > 0$ be given and $f(t) = O(t^r)$, $t \to \infty$, then for n sufficiently large*

$$\left| L_{n,\alpha}^*(f, x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \leq E(n, x) +$$

$$+ \left[\frac{2M(1 + x + x^2) + a_n x^2}{n a_n x^2} + \frac{A_\alpha}{(n a_n)^m x^{2m}} \right] \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + O((n a_n)^{-r}),$$

where

$$E(n, x) \leq \begin{cases} \dfrac{\sqrt{1 + 3x}}{\sqrt{nx}} |f(x+) - f(x-)| + \dfrac{1}{\sqrt{2enx}} \varepsilon_n(x) |f(x) - f(x-)|, \\ \qquad\qquad\qquad\qquad \text{if } a_n = 1, \phi_n(x) = e^{nx}; \\[2mm] \dfrac{1 + a_n x}{\sqrt{2 e n a_n x}} |f(x+) - f(x-)| + \\ \quad + \dfrac{[1 + (a_n x)^2 + 0,5(1 + a_n x)^2]}{(1 + a_n x)[1 + \sqrt{n a_n x}]} \varepsilon_n(x) |f(x) - f(x-)|, \\ \qquad\qquad\qquad\qquad \text{if } \phi_n(x) = (1 + a_n x)^n \end{cases}$$

$$\varepsilon_n(x) = \begin{cases} 1, & \text{if } x = k'/n \text{ for some } k' \in \mathbb{N} \\ 0, & \text{if } x \neq k'/n \text{ for all } k \in \mathbb{N} \end{cases}$$

and

$$g_x = \begin{cases} f(t) - f(x-), 0 \leq 1 < x \\ 0, \ t = x \\ f(t) - f(x+), \ x < t < \infty \end{cases}$$

Some approximation properties for the special case $\phi_n(x) = (1 + x)^n$ and $\alpha = 1$, were recently studied by O. Agratini [1], he has also estimated the rate of convergence for bounded variation functions for this special case, but the author was not able to find the explicitly the sign term of the above estimate. This answer was given in the recent paper [4].

2   Auxiliary results

Lemma 1. *For $e_i(t) = t^i$, $i = 0, 1, 2, \ldots$ and for all $x \geq 0$, we have*

$$L_n^*(e_0, x) = 1, \quad L_n^*(e_1, x) = \frac{\phi_n(a_n x)}{n\phi_n'(a_n x)}x + \frac{1}{2na_n}$$

and

$$L_n^*(e_2, x) = \frac{\phi_n''(a_n x)}{n^2\phi_n(a_n x)}x^2 + \frac{2}{na_n}\frac{\phi_n'(a_n x)}{n\phi_n(a_n x)}x + \frac{1}{3n^2a_n^2}.$$

Proof.   From [5], it follows that

$$L_n(e_0, x) = 1, \quad L_n(e_1, x) = \frac{\phi_n'(a_n x)}{n\phi_n(a_n x)}x,$$

and

$$L_n(e_2, x) = \frac{\phi_n''(a_n x)}{n^2\phi_n(a_n x)}x^2 + \frac{1}{na_n}\frac{\phi_n'(a_n x)}{n\phi_n(a_n x)}x.$$

Using the above estimates, we have

$$L_n^*(e_2, x) = na_n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{I_{n,k}} t^2 dt = na_n \sum_{k=0}^{\infty} p_{n,k}(x)\frac{3k^2 + 3k + 1}{3n^3a_n^3} =$$

$$= \sum_{k=0}^{\infty} \left(\frac{k}{na_n}\right)^2 p_{n,k}(x) + \frac{1}{na_n}\sum_{k=0}^{\infty}\frac{k}{na_n}p_{n,k}(x) + \frac{1}{3n^2a_n^2}\sum_{k=0}^{\infty}p_{n,k}(x) =$$

$$= L_n(e_2, x) + \frac{1}{na_n}L_n(e_1, x) + \frac{1}{3n^2a_n^2}L_n(e_0, x).$$

Substituting the values of $L_n(e_i, x), i = 0, 1, 2$, we get the desired result. The proofs of $L_n^*(e_i, x), i = 0, 1$ are obvious.

Remark 1. *Note that for sufficiently large n, there exists a constant $M > 0$ such that*

$$\mu_{n,2}(x) = L_n^*((e_1 - xe_0)^2, x) \leq \frac{M(1 + x + x^2)}{na_n}.$$

Lemma 2. *Let $x \in (0, \infty)$ and $0 < \alpha < 1$, then for sufficiently large n, we have*

$$(5) \qquad \beta_{n,\alpha}(x, y) = \int_0^y W_{n,\alpha}(x, t)dt \leq \frac{M(1 + x + x^2)}{na_n(x - y)}, \quad 0 \leq y < x$$

and

$$(6) \quad 1 - \beta_{n,\alpha}(x, z) = \int_z^\infty W_{n,\alpha}(x, t)dt \leq \frac{A_\alpha}{(na_n)^m(z - x)^{2m}}, \quad x < z < \infty.$$

Proof. We first prove (5). By Remark 1, there holds

$$\int_0^y W_{n,\alpha}(x, t)dt \leq \int_0^y W_{n,\alpha}(x, t)\frac{(x - t)^2}{(x - y)^2}dt \leq (x - y)^{-2}L_n^*((t - x)^2, x)$$

$$\leq \frac{M(1 + x + x^2)}{na_n(x - y)^2}, \quad 0 \leq y < x$$

where we have applied Lemma 1. this completes the proof of (5).

Next we prove (6). For $0 < \alpha < 1$,it is easily verified that

$$\int_z^\infty W_{n,\alpha}(x, t)dt \leq \left(\int_z^\infty W_n(x, t)\frac{(t - x)^{2m/\alpha}}{(z - x)^{2m/\alpha}}dt\right)^\alpha \leq$$

$$\leq (z - x)^{-2m}\left(\int_0^\infty W_n(x, t)(t - x)^{2m/\alpha}dt\right)^\alpha.$$

For all conjugate $p, q \geq 1$ i.e. $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left(\int_0^\infty \{W_n(x,t)\}^{1/p} \{W_n(x,t)\}^{1/q} |t - x|^{2m/\alpha} dt \right)^\alpha =$$

$$= \left(\int_0^\infty W_n(x,t) |t - x|^{2mp/\alpha} dt \right)^{\alpha/p} \left(\int_0^\infty W_n(x,t) dt \right)^{\alpha/q}$$

Also since

$$\left(\int_0^\infty W_n(x,t) dt \right)^{\alpha/q} = 1$$

Choosing $p = \frac{\alpha}{m} \left[\frac{m}{\alpha} + 1 \right]$ we have that $2mp/\alpha$ is an even positive integer. By the well known results $L_{n,1}^*((t - x)^{2r}, x) = O((na_n)^{-r})$ as $n \to \infty$ $(r = 1, 2, 3, ...)$ we obtain

$$\left(\int_0^\infty W_n(x,t) |t - x|^{2mp/\alpha} dt \right)^{\alpha/p} =$$

$$= (L_{n,1}^*((t - x)^{2mp/\alpha}, x))^{\alpha/p} = O((na_n)^{-m})$$

as $n \to \infty$.

This completes the proof of Lemma 2.

Lemma 3. *For $x \in (0, \infty)$, we have*

$$p_{n,k}(x) \leq \begin{cases} \dfrac{1}{\sqrt{2enx}}, & \textit{if } a_n = 1, \ \phi_n(x) = e^{nx} \\ \dfrac{1 + a_n x}{\sqrt{2ena_n x}}, & \textit{if } \phi_n(x) = (1 + x)^n \end{cases}.$$

These bounds can be found in [7] and [8], we just have to replace the variable for the second inequality.

Lemma 4. *For $x \in (0, \infty)$, we have:*

(i) *For $a_n = 1, \phi_n(x) = e^{nx}$, we have*

$$\left| \sum_{k>na_nx} p_{n,k}(x) - \frac{1}{2} \right| \le \frac{\sqrt{1+3x}}{\sqrt{nx}}.$$

(ii) *For $\phi_n(x) = (1+x)^n$, we have*

$$\left| \sum_{k>na_nx/(1+a_nx)} p_{n,k}(x) - \frac{1}{2} \right| \le \frac{[1 + (a_nx)^2 + 0,5(1+a_nx)^2]}{(1+a_nx)[1+\sqrt{na_nx}]}.$$

For the proof of above Lemma, we refer to [4].

Lemma 5. *For $x \in (0, \infty)$, we have:*

(i) *For $a_n = 1, \phi_n(x) = e^{nx}$, we have*

$$\left| \left(\sum_{k>na_nx} p_{n,k}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| \le \frac{\sqrt{1+3x}}{\sqrt{nx}};$$

(ii) *For $\phi_n(x) = (1+x)^n$, we have*

$$\left| \left(\sum_{k>na_nx/(1+a_nx)} p_{n,k}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| \le \frac{[1 + (a_nx)^2 + 0,5(1+a_nx)^2]}{(1+a_nx)[1+\sqrt{na_nx}]}.$$

Proof. We prove (i), by mean value theorem, we have

$$\left| \left(\sum_{j>na_nx} p_{n,j}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| = \alpha(\zeta_{n,j}(x))^{\alpha-1} \left| \sum_{j>na_nx} p_{n,j}(x) - \frac{1}{2} \right|$$

where $\zeta_{n,j}(x)$ lies between $\frac{1}{2}$ and $\sum_{j>na_nx} p_{n,j}(x)$. It is observed that for na_n sufficiently large, the intermediate point $\varsigma_{n,j}$ is arbitrary close to $1/2$ i.e.

$$\varsigma_{n,j} = \frac{1}{2+\varepsilon}$$

with an arbitrary small $|\varepsilon|$. Then we have

$$\alpha(\varsigma_{n,j}(x))^{\alpha-1} \leq \alpha(2+\varepsilon)^{1-\alpha}.$$

The latter expression is positive and strictly increasing for $\alpha \in (0,1)$, since

$$\frac{\partial}{\partial \alpha}\alpha(2+\varepsilon)^{1-\alpha} = (2+\varepsilon)^{1-\alpha}[1 - \alpha \log(2+\varepsilon)] > 0,$$

for sufficiently small $|\varepsilon|$. Thus it takes maximum value at $\alpha = 1$. This implies

$$\alpha(\zeta_{n,j}(x))^{\alpha-1} \leq 1.$$

Hence by using Lemma 4, we have

$$\left|\left(\sum_{j>na_n x} p_{n,j}(x)\right)^{\alpha} - \frac{1}{2^{\alpha}}\right| \leq \frac{\sqrt[3]{(1+x)}}{\sqrt{nx}}.$$

The proof (ii) is similar.

3   Proof of theorem

Proof.   Making use of the following for all n, we have

$$f(t) = \frac{1}{2^{\alpha}}f(x+) + \left(1 - \frac{1}{2^{\alpha}}\right)f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^{\alpha}}sign_x(t)+$$

$$+\delta_x(t)\left[f(t) - \frac{1}{2^{\alpha}}f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right)f(x-)\right]$$

where

$$sign_x(t) = \begin{cases} 2^{\alpha} - 1, & t > x \\ 0, & t = x \\ -1, & t < x \end{cases}$$

and

$$\delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases}.$$

It follows that

(7)
$$\left| L_{n,\alpha}^*(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \leq$$

$$\leq |L_{n,\alpha}^*(g_x,x)| + \left| \frac{f(x+) - f(x-)}{2^\alpha} L_{n,\alpha}^*(sign(t-x),x) + \right.$$

$$\left. + \left[f(x) - \frac{1}{2^\alpha} - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] L_{n,\alpha}^*(\delta_x,x) \right|.$$

First we estimate $L_{n,\alpha}^*(sign(t-x),x)$, as follows

$$|L_{n,\alpha}^*(sign(t-x),x)| \leq \left| \sum_{k=k'+1}^\infty 2^\alpha Q_{n,k}^{(\alpha)}(x) - 1 \right| + \varepsilon_n(x) Q_{n,k}^{(\alpha)}(x),$$

where

$$\varepsilon_n(x) = \begin{cases} 1, & \text{if } x = k'/n \text{ for some } k' \in \mathbb{N} \\ 0, & \text{if } x \neq k'/n \text{ for all } k \in \mathbb{N}. \end{cases}$$

Also by direct calculation, we have

$$L_{n,\alpha}^*(\delta_x,x) = \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x)$$

Thus

$$\left| \frac{f(x+) - f(x-)}{2^\alpha} L_{n,\alpha}^*(sign(t-x),x) + \right.$$

$$\left. + \left[f(x) - \frac{1}{2^\alpha} - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] L_{n,\alpha}^*(\delta_x,x) \right| =$$

$$= \left| \frac{f(x+) - f(x-)}{2^\alpha} \left[2^\alpha \left(\sum_{k=k'+1}^\infty p_{n,k}(x) \right)^\alpha - 1 \right] + \right.$$

$$+[f(x+) - f(x-)]\varepsilon_n(x)Q_{n,k'}^{(\alpha)}(x)\Big|$$

Now using Lemma 5 and Lemma 3, we obtain

(8)
$$\left| \frac{f(x+) - f(x-)}{2^\alpha} \left[2^\alpha \left(\sum_{k=k'+1}^\infty p_{n,k}^{(x)} \right)^\alpha - 1 \right] + \right.$$

$$\left. + [f(x) - f(x-)]\varepsilon_n(x)Q_{n,k'}^{(\alpha)}(x) \right| \leq$$

$$\leq \begin{cases} \dfrac{\sqrt{1+3x}}{\sqrt{nx}}|f(x+) - f(x-)| + \dfrac{1}{\sqrt{2enx}}\varepsilon_n(x)|f(x) - f(x-)| \\ \qquad\qquad if \ a_n = 1, \ \phi_n(x) = e^{nx} \\ \dfrac{1 + a_n x}{\sqrt{2ena_n x}}|f(x+) - f(x-)| + \\ \quad + \dfrac{[1 + (a_n x)^2 + 0,5(1 + a_n x)^2]}{(1 + a_n x)[1 + \sqrt{na_n x}]}\varepsilon_n(x)|f(x) - f(x-)| \\ \qquad\qquad if \ \phi_n(x) = (1 + a_n x)^n \end{cases}$$

Now we estimate $L_{n,\alpha}^*(g_x, x)$ as follows:

$$L_{n,\alpha}^*(g_x, x) = \int_0^\infty g_x(t)W_{n,\alpha}(x,t)dt =$$

(9)
$$= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{2x} + \int_{2x}^\infty \right) W_{n,\alpha}(x,t)g_x(t)dt =$$

$$= E_1 + E_2 + E_3 + E_4, \ \text{say.}$$

We start with E_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|g_x(t)| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leq \frac{1}{n}\sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$

and thus

(10)
$$|E_2| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leq \frac{1}{n}\sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x).$$

Next we estimate E_1. Setting $y = x - x/\sqrt{n}$ and integrating by parts, we have

$$E_1 = \int_0^y g_x(t)d_t(\beta_{n,\alpha}(x,t)) = g_x(y)\beta_{n,\alpha}(x,y) - \int_0^y \beta_{n,\alpha}(x,t)d_t(g_x(t))$$

Since $|g_x(y)| \leq V_y^x(g_x)$, we conclude

$$|E_1| \leq V_y^x(g_x)\beta_{n,\alpha}(x,y) + \int_0^y \beta_{n,\alpha}(x,t)d_t(-V_t^x(g_x))$$

Also $y = x - x/\sqrt{n} \leq x$, therefore (5) of Lemma 2 implies for n sufficiently large

$$|E_1| \leq \frac{M(1+x+x^2)}{na_n(x-y)^2}V_y^x(g_x) + \frac{M(1+x+x^2)}{na_n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)).$$

Integrating by parts the last integral, we obtain

$$|E_1| \leq \frac{M(1+x+x^2)}{na_n}\left(x^{-2}V_0^x(g_x) + 2\int_0^y \frac{V_t^x(g_x)dt}{(x-t)^3}\right).$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\int_0^{x-x/\sqrt{n}} V_t^x(g_x)(x-t)^{-3}dt = \sum_{k=1}^{n-1}\int_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} V_{x-t}^x(g_x)t^{-3}dt \leq$$

$$\leq \frac{1}{2x^2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x).$$

Hence

(11) $$|E_1| \leq \frac{2M(1+x+x^2)}{na_n x}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x)$$

Next we estimate E_3, choosing $y = x + x/\sqrt{n}$, we have

$$E_3 = \lim_{R \to -\infty} \left\{ g_x(y)[1 - \beta_{n,\alpha}(x,y)] + \widehat{g}_x(R)[\beta_{n,\alpha}(x,R) - 1] + \right.$$

$$\left. + \int\limits_y^R [1 - \beta_{n,\alpha}(x,t)] d_t \widehat{g}_x(t) \right\}.$$

By equation (6) of Lemma 2, we conclude for each $\lambda > 1$ and n sufficiently large

$$|E_3| \le \frac{A_\alpha}{(na_n)^m} \lim_{R \to +\infty} \left\{ \frac{V_x^y(g_x)}{(y-x)^2} + \frac{\widehat{g}_x(R)}{(R-x)^2} + \int\limits_0^x \frac{1}{(t-x)^2} d_t(V_x^t(\widehat{g}_x)) \right\} =$$

$$= \frac{A_\alpha}{(na_n)^m} \left\{ \frac{V_x^y(g_x)}{(y-x)^{2m}} + \int\limits_0^{2x} \frac{1}{(t-x)^{2m}} d_t(V_x^t(g_x)) \right\}$$

Using the similar method as above, we get

$$\int\limits_y^{2x} \frac{1}{(t-x)^{2m}} d_t(V_x^t(g_x)) \le x^{-2m} V_x^{2x}(g_x) - \frac{V_x^y(g_x)}{(y-x)^{2m}} + x^{-2m} \sum_{k=1}^{n-1} V_x^{x+x/\sqrt{k}}(g_x)$$

which implies the estimate

$$(12) \qquad\qquad |E_3| \le \frac{2A_\alpha}{(na_n)^m x^{2m}} \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x)$$

Lastly we estimate E_4. By assumption there exists an integer $r > 1$ such that $f(t) = O(t^{2r})$, $t \to \infty$. Thus for certain constant $M > 0$ depending only on f, x, r, we have

$$|E_4| \le M_1 n a_n \sum_{k=0}^\infty Q_{n,k}^{(\alpha)}(x) \int\limits_{2x}^\infty \chi_{n,k}(t) t^{2r} dt \le$$

$$\leq M_1 n a_n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} \chi_{n,k}(t) t^{2r} dt$$

By Lemma 1, we have

(13) $$|E_4| \leq 2^r M L_n^*((t-x)^{2r}, x) = O((na_n)^{-r}), n \to \infty$$

Finally collecting the estimates of (7) - (13), we get the required result. This completes the proof of the theorem.

References

[1] O. Agartini, *An approximation process of Kantorovich type*, Math. Notes, Miskolc 2(1), (2001), 3-10.

[2] K. Balasz, *Approximation by Bernstein type rational functions*, Acta Math. Acad. Sci. Hungar, 26 (1-2), (1975), 123-134.

[3] C. Balasz, J. Szabados, *Approximation by Bernstein type rational functions II*, Acta Math. Acad. Sci. Hungar, 40 (3-4), (1982), 331-337.

[4] V. Gupta, N. Ispir, *On the Bezier variant of generalized Kantorovich type Balasz operators*, Applied Math. Letters, (to appear).

[5] N. Ispir, C. Atakut, *Approximation by generalized Balasz type rational functions*, Int. J. Comput. Numer. Anal. Appl., 4 (3), 2003, 297-316

[6] P. Pych Taberska, *On the rate of convergence of the Feller operators*, Ann. Soc. Math. Pol. Comment. Math., 31 (1991), 147-156.

[7] X. M. Zeng, *Bounds for Bernstein basis functions and Meyer-König and Zeller basis functions*, J. Math. Anal. Appl., 219, (1998), 364-376.

[8] X. M. Zeng, J. N. Zhao, *Exact bounds for some basis functions of approximation operators*, J. Inequal Appl., 6 (2001), 563-575.

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