General Mathematics Vol. 12, No. 4 (2004), 17–21

A note on the Gini means

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Abstract

We correct a proof given in [1] for the one-parameter family of Gini means, and point out general remarks on the general Gini means.

2000 Mathematical Subject Classification: 26D15, 26D99

Keywords: Gini means, power means, Stolarsky means

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In paper [1], the following two means are compared to each others: Let $0 < a < b$. The power mean of two arguments is defined by

$$M_p = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0 \\ \sqrt{ab}, & p = 0 \end{cases}, \tag{1}$$

while the Gini mean is defined as

$$S_p = \begin{cases} \left(\dfrac{a^{p-1} + b^{p-1}}{a+b}\right)^{1/(p-2)}, & p \neq 2 \\ S(a,b), & p = 2 \end{cases}, \tag{2}$$

where $S(a,b) = (a^a \cdot b^b)^{1/(a+b)}$. The properties of the special mean S have been extensively studied by us e.g. in [7], [8], [9], [10]. In paper [6] it is conjectured that

$$(3) \qquad \frac{S_p}{M_p} = \begin{cases} < 1, & \text{if } p \in (0,1) \\ = 1, & \text{if } p \in \{0,1\} \\ > 1, & \text{if } p \in (-\infty,0) \cup (1,\infty) \end{cases},$$

while in [1], (3) is corrected to the following:

$$(4) \qquad \frac{S_p}{M_p} = \begin{cases} < 1, & \text{if } p \in (0,1) \cup (1,2) \\ = 1, & \text{if } p \in \{0,1\} \\ > 1, & \text{if } p \in (-\infty,0) \cup [2,\infty) \end{cases},$$

For the proof of (4), for $p \notin \{0,1,2\}$, the author denotes $t = b/a > 1$, when $\log \frac{S_p}{M_p} = \frac{1}{p} f(t)$, where

$$f(t) = \frac{p}{p-2} \cdot \log \frac{1+t^{p-1}}{1+t} - \log \frac{1+t^p}{2}, \ t > 1.$$

Then $f'(t) = \frac{p}{p-2} \cdot \frac{g(t)}{(1+t)(1+t^{p-1})(1+t^p)}$, where

$$g(t) = t^{2p-2} - (p-1)t^p + (p-1)t^{p-2} - 1, \ t > 0.$$

It is immediate that $g'(t) = (p-1)t^{p-3}h(t)$, where $h(t) = 2t^p - pt^2 + p - 2$. Then the author **wrongly** writes $h'(t) = 2p(t^{p-1} - 1)$. In fact one has $h'(t) = 2pt(t^{p-2} - 1)$, and by analyzing the monotonicity properties, it follows easily that relations (3) are true (and not the corrected version (4)!).

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However, we want to show, that relations (3) are consequences of more general results, which are known in the literature.

In fact, Gini [2] introduced the two-parameter family of means

$$(5) \qquad S_{u,v}(a,b) = \begin{cases} \left(\dfrac{a^u + b^u}{a^v + b^v}\right)^{1/(u-v)}, & u \neq v \\ \exp\left(\dfrac{a^u \log a + b^u \log b}{a^u + b^u}\right), & u = v \neq 0 \\ \sqrt{ab}, & u = v = 0 \end{cases}$$

for any real numbers $u, v \in \mathbb{R}$. Clearly, $S_{0,-1} = H$ (harmonic mean), $S_{0,0} = G$ (geometric mean), $S_{1,0} = A$ (arithmetic mean), $S_{1,1} = S$ (denoted also by J. in [4], [10]), $S_{p-1,1} = S_p$, where S_p is introduced by (2). In 1988 Zs. Páles [5] proved the following result on the comparison of the Gini means (5).

Theorem 2.1 *Let $u, v, t, w \in \mathbb{R}$. Then the comparison inequality*

$$(6) \qquad S_{u,v}(a,b) \leq S_{t,w}(a,b)$$

is valid if and only if $u + v \leq t + w$, and

i) $\min\{u, v\} \leq \min\{t, w\}$, *if* $0 \leq \min\{u, v, t, w\}$,

ii) $k(u, v) \leq k(t, w)$, *if* $\min\{u, v, t, w\} < 0 < \max\{u, v, t, w\}$,

iii) $\max\{u, v\} \leq \max\{t, w\}$, *if* $\max\{u, v, t, w\} \leq 0$

Here $k(x,y) = \begin{cases} \dfrac{|x| - |y|}{x - y}, & x \neq y \\ sign(x), & x = y \end{cases}$

The cases of equality are trivial.

Now, remarking that $S_p = S_{p-1,1}$ and $M_p = S_{p,0}$, results (3) will be a consequence of this Theorem. In our case $u = p - 1$, $v = 1, t = p, w = 0$; so $u + v \leq t + w = p$, i.e. (6) is satisfied.

Now, it is easy to see that denoting $\min\{p - 1, 0, 1, p\} = a_p$, $\max\{p - 1, 0, 1, p\} = A_p$, the following cases are evident:

1) $p \leq 0 \Rightarrow p - 1 < p \leq 0 < 1$, so $a_p = p - 1, A_p = 1$

2) $p \in (0,1] \Rightarrow p-1 < 0 < p \le 1$, so $a_p = p-1, A_p = 1$

3) $p \in (1,2] \Rightarrow 0 < p-1 \le 1 < p$, so $a_p = 0, A_p = p$

4) $p > 2 \Rightarrow 0 < 1 < p-1 < p$, so $a_p = 0, A_p = p$.

In case 2) one has $\frac{|p-1|-1}{p-2} \le \frac{|p|}{p}$ if $p-1 < 0 < p$ only if $\frac{1-p-1}{p-2} \le 1$, i.e. $\frac{2(1-p)}{p-2} \le 0$, which is satisfied. The other cases are not possible.

Now, in case $p \notin (0,1)$ write $S_{p,0} < S_{p-1,1}$, and apply the same procedure.

For another two-parameter family of mean values, i.e. the Stolarsky means $D_{u,v}(a,b)$, and its comparison theorems, as well as inequalities involving these means see e.g. [11], [3], [4], [10], and the references.

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