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About the Derivatives of Cauchy Type Integrals in the Polydisc

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In the paper the formulas for derivatives of the Cauchy type integral $K[f]$ of smooth functions f on the distinguished boundary of a polydisc are given. These formulas express the derivatives of order m from $K[f]$ through the derivatives of lower order (Theorem 1). We use them for the estimates of smoothness of derivatives of Cauchy type integral in terms of the Holder's ordinary scale (Theorem 2) and in terms of anisotropic Holder's classes (Theorem 5).

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1 Introduction

1.1. The well-known theorem of Privalov [1] states that if a function satisfies Holder's α-condition on the boundary of some domain, then its Cauchy type integral satisfies the same condition. This theorem is extended to the multidimensional case by B. Jöricke [2]. Namely, in [2] it is proved that the modulus of continuity of the Cauchy or of the Weyl type integral of a function $f(z)$ is bounded from above by const $\cdot\, \delta^\alpha (\log \frac{1}{\delta})^{n-1}$ provided $f(z)$ satisfies Holder's α-condition on the distinguished boundary of a polydisc or on the regular Weyl polyhedron of the space $\mathbb{C}^n$. Besides, [2] contains a relevant example which proves that this result cannot be improved.

In the paper [4] the anisotropic Holder's spaces $\widetilde{\Lambda}(\beta)$ have been introduced. It is proved that for these spaces Privalov's theorem is extended perfectly, i.e. no logarithmic multiplier arises.

The present paper is devoted to a natural problem related to the derivatives of the Cauchy type integral of functions $f(z)$. A formula representing derivatives of any order by derivatives of lower orders is established in Theorem 1 below. Using this formula and an induction, the above mentioned results are extended to the case of derivatives in Theorem 2 and Theorem 5 below.

1.2. We assume that

$U^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n\colon\ |z_k| < 1,\ k = 1, \ldots, n\}$ is the unit polydisc of $\mathbb{C}^n$ and $T^n = \{z \in \mathbb{C}^n\colon\ |z_k| = 1,\ k = 1, \ldots, n\}$ is the distinguished boundary of U^n.

For a function $f(z)$ given on T^n, $K[f]$ denotes the n-multiple Cauchy type integral of $f(z)$

$$K[f](z) = \frac{1}{(2\pi i)^n} \int\limits_{T^n} \frac{f(\zeta) d\zeta}{\prod\limits_{k=1}^{n} (\zeta_k - z_k)}.$$

Here $\zeta = (\zeta_1, \cdots, \zeta_n) \in T^n, \quad z = (z_1, \ldots, z_n) \in U^n, \quad d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n.$

Further we assume that:

$C^m(T^n)$ is the space of all functions which are m times continuously differentiable on T^n;

$C^{m,\alpha}(T^n)$ is the subset of those functions of $C^m(T^n)$, the m-th order derivatives of which belong to the Holder's α-class.

1.3. Below we apply the partial differential operators $\dfrac{\partial}{\partial \zeta_k}$ and $\dfrac{\partial}{\partial \bar{\zeta}_k}$ to functions which are in C^m on the distinguished boundary T^n of the unit polydisc. Due to the Whitney theorem (see for example, [3]) any function of such type can be extended to the neighbourhood of T^n with the same smoothness. Everywhere we deal with any continuation of a function g of $C^m(T^n)$, for which we shall use the same denotation g.

2 The formula for derivatives

We start with a preliminary lemma

Lemma 1. *Let* $g \in C^1(T^n), \quad 1 \leq k \leq n.$ *Then*

$$K\left[\frac{\partial g(\zeta)}{\partial \zeta_k}\right](z) = K\left[\bar{\zeta}_k^2 \frac{\partial g(\zeta)}{\partial \bar{\zeta}_k}\right](z) + \frac{\partial}{\partial z_k} K\left[g\right](z).$$

Proof. For a fixed $z \in U^n$, consider the following differential form

$$\omega = \frac{(-1)^{k-1} g(\zeta)}{\prod\limits_{j=1}^{n} (\zeta_j - z_j)} \bigwedge_{i \neq k} d\zeta_i.$$

It is evident that

$$(1)\quad 0=\int_{T^n} d_\zeta\omega=\int_{T^n}\frac{\partial}{\partial\zeta_k}\frac{g(\zeta)}{\prod\limits_{j=1}^{n}(\zeta_j-z_j)}d\zeta + \sum_{j=1}^{n}\int_{T^n}\frac{\partial}{\partial\overline{\zeta}_j}\frac{(-1)^{k-1}g(\zeta)}{\prod\limits_{j=1}^{n}(\zeta_j-z_j)}d\overline{\zeta}_j\wedge\Big(\underset{i\neq k}{\wedge}d\zeta_i\Big).$$

The first equality in (1) follows from Stokes' formula. Further, for $\zeta\in T^n$ we have $\zeta_j d\overline{\zeta}_j=-\overline{\zeta}_j d\zeta_j$, since $\zeta_j\overline{\zeta}_j=1$ $(j=1,\ldots,n)$. Therefore

$$(2)\qquad (-1)^{k-1}d\overline{\zeta}_j\wedge\Big(\underset{i\neq k}{\wedge}d\zeta_i\Big)=\begin{cases}-\overline{\zeta}_k^{\,2}d\zeta, & \text{if } j=k,\\ 0, & \text{if } j\neq k.\end{cases}$$

By (2) and (1)

$$(3)\quad \int_{T^n}\frac{\partial g(\zeta)}{\partial\zeta_k}\frac{d\zeta}{\prod\limits_{i=1}^{n}(\zeta_i-z_i)}-\int_{T^n}\frac{g(\zeta)d\zeta}{(\zeta_k-z_k)^2\prod\limits_{i\neq k}(\zeta_i-z_i)}- -\int_{T^n}\frac{\partial g(\zeta)}{\partial\overline{\zeta}_k}\frac{\overline{\zeta}_k^{\,2}d\zeta}{\prod\limits_{i=1}^{n}(\zeta_i-z_i)}=0$$

As

$$\frac{1}{(\zeta_k-z_k)^2}=\frac{\partial}{\partial z_k}\frac{1}{\zeta_k-z_k},$$

from (3) we have

$$\int_{T^n}\frac{\partial g(\zeta)}{\partial\zeta_k}\frac{d\zeta}{\prod\limits_{i=1}^{n}(\zeta_i-z_i)}=\int_{T^n}\frac{\partial g(\zeta)}{\partial\overline{\zeta}_k}\frac{\overline{\zeta}_k^{\,2}d\zeta}{\prod\limits_{i=1}^{n}(\zeta_i-z_i)}+\frac{\partial}{\partial z_k}\int_{T^n}\frac{g(\zeta)d\zeta}{\prod\limits_{i=1}^{n}(\zeta_i-z_i)}.$$

It remains to see that this formula coincides with (1) due to our notation.

The following theorem gives the formula, which expresses the derivatives of the order m from $K[f]$ through the derivatives of lower order.

Theorem 1. *Let $f \in C^m(T^n)$ and the multiindex $r = (r_1, \dots, r_n)$ satisfyes the condition $r_1 + \dots + r_n = m$. Then*

$$(4)\quad \frac{\partial^{r_1+\dots+r_n}}{\partial z_1^{r_1} \dots \partial z_n^{r_n}} K[f](z) = \\ = \sum_{j=1}^{n} \sum_{k_j=1}^{r_j} \frac{\partial^{r_1+\dots+r_j-k_j}}{\partial z_1^{r_1} \dots \partial z_{j-1}^{r_{j-1}} \partial z_j^{r_j-k_j}} K\left[\overline{\zeta}_j^2 \frac{\partial^{k_j+r_{j+1}+\dots+r_n} f(\zeta)}{\partial \overline{\zeta}_j \partial \zeta_j^{k_j-1} \partial \zeta_{j+1}^{r_{j+1}} \dots \partial \zeta_n^{r_n}}\right](z) - \\ - K\left[\frac{\partial^{r_1+\dots+r_n} f(\zeta)}{\partial \zeta_1^{r_1} \dots \partial \zeta_n^{r_n}}\right](z).$$

In (4) it is assumed that if the order $r_1 + \dots + r_j - k_j$ of differentiation is zero, then the differentiation is the identical operator.

Proof. For simplicity we consider the case $n = 2$. Putting $g(z) = \dfrac{\partial^{r_1-1+r_2} f(z)}{\partial z_1^{r_1-1} \partial z_2^{r_2}}$ and $k = 1$ in Lemma 1, we get

$$K\left[\frac{\partial^{r_1+r_2} f(\zeta)}{\partial \zeta_1^{r_1} \partial \zeta_2^{r_2}}\right](z) = K\left[\overline{\zeta}_1^2 \frac{\partial^{r_1+r_2} f(\zeta)}{\partial \overline{\zeta}_1 \partial \zeta_1^{r_1-1} \partial \zeta_2^{r_2}}\right](z) + \frac{\partial}{\partial z_1} K\left[\frac{\partial^{r_1-1+r_2} f(\zeta)}{\partial \zeta_1^{r_1-1} \partial \zeta_2^{r_2}}\right](z).$$

Note that the order of derivatives by ζ_1, contained in the integrand of the second summand on the right-hand side, is less by 1 than the order of the derivatives at the left-hand side. Thus, the obtained formula is of recurrent character. Successively applying this formula r_1 times, we get

$$(5)\quad K\left[\frac{\partial^{r_1+r_2} f(\zeta)}{\partial \zeta_1^{r_1} \partial \zeta_2^{r_2}}\right](z) = K\left[\overline{\zeta}_1^2 \frac{\partial^{r_1+r_2} f(\zeta)}{\partial \overline{\zeta}_1 \partial \zeta_1^{r_1-1} \partial \zeta_2^{r_2}}\right](z) \\ + \frac{\partial}{\partial z_1}\left\{K\left[\overline{\zeta}_1^2 \frac{\partial^{r_1-1+r_2} f(\zeta)}{\partial \overline{\zeta}_1 \partial \zeta_1^{r_1-2} \partial \zeta_2^{r_2}}\right](z) + \frac{\partial}{\partial z_1} K\left[\frac{\partial^{r_1-2+r_2} f(\zeta)}{\partial \zeta_1^{r_1-2} \partial \zeta_2^{r_2}}\right](z)\right\} \\ \dots = \sum_{k_1=1}^{r_1} \frac{\partial^{r_1-k_1}}{\partial z_1^{r_1-k_1}} K\left[\overline{\zeta}_1^2 \frac{\partial^{k_1+r_2} f(\zeta)}{\partial \overline{\zeta}_1 \partial \zeta_1^{k_1-1} \partial \zeta_2^{r_2}}\right](z) + \frac{\partial^{r_1}}{\partial z_1^{r_1}} K\left[\frac{\partial^{r_2} f(\zeta)}{\partial \zeta_2^{r_2}}\right](z).$$

One can see that the last summand of the right-hand side now does not contain derivative with respect to ζ_1 under the integral K. For deleting of the remaining derivatives, it suffices to apply formula (1) to the variable ζ_2. We get

$$(6)\quad K\left[\frac{\partial^{r_2} f(\zeta)}{\partial\zeta_2^{r_2}}\right](z) = \sum_{k_2=1}^{r_2} \frac{\partial^{r_2-k_2}}{\partial z_1^{r_2-k_2}} K\left[\overline{\zeta}_2^2 \frac{\partial^{k_2} f(\zeta)}{\partial\overline{\zeta}_2 \partial\zeta_2^{k_2-1}}\right](z) + \frac{\partial^{r_2}}{\partial z_2^{r_2}} K[f](z).$$

From (5) and (6) we get

$$\frac{\partial^{r_1+r_2}}{\partial z_1^{r_1}\partial z_2^{r_2}} K[f](z) = \sum_{k_1=1}^{r_1} \frac{\partial^{r_1-k_1}}{\partial z_1^{r_1-k_1}} K\left[\overline{\zeta}_1^2 \frac{\partial^{k_1+r_2} f(\zeta)}{\partial\overline{\zeta}_1 \partial\zeta_1^{k_1-1}\partial\zeta_2^{r_2}}\right](z)$$
$$+ \sum_{k_2=1}^{r_2} \frac{\partial^{r_1+r_2-k_2}}{\partial z_1^{r_1}\partial z_2^{r_2-k_2}} K\left[\overline{\zeta}_2^2 \frac{\partial^{k_2} f(\zeta)}{\partial\overline{\zeta}_2 \partial\zeta_2^{k_2-1}}\right](z) - K\left[\frac{\partial^{r_1+r_2} f(\zeta)}{\partial\zeta_1^{r_1}\partial\zeta_2^{r_2}}\right](z).$$

This is the formula (4) for the case $n = 2$. For getting (4) for arbitrarily n, it suffices to apply formula (1) successively to the variables $\zeta_1 \ldots, \zeta_n$, appropriately choosing k and the function $g(z)$.

Remark 1. *The multipliers $\overline{\zeta}_j^2$ under the Cauchy integrals $K[\overline{\zeta}_j^2 ...]$ of the right-hand side of (4) are of purely technical character, and they are not essential in applications.*

3 The Holder's estimates of derivatives

3.1. For some $\alpha, 0 < \alpha < 1$, we assume that

$A^m(U^n)$ is the space of all functions holomorphic in U^n which are m times continuously differentiable on $\overline{U}^n$;

$A^{m,\alpha}(U^n)$ is the subset of those functions of $A^m(U^n)$, the m-th order derivatives of which belong to the Holder's α-class on $\overline{U}^n$;

$A^{m,\alpha}_{(\log)}(U^n)$ is the subset of those functions from $A^m(U^n)$, the modulus of continuity of all m-th order derivatives of which are bounded above by $\mathrm{const}\cdot\delta^\alpha\left(\log\frac{1}{\delta}\right)^{\mathrm{n}-1}$ on $\overline{U}^n$.

We omit m in these notations if $m=0$. Note that $A^0(U^n)=A(U^n)$ is the usual polydisc-algebra.

3.2. The following theorem is the extension to the case of derivatives of the result in [2].

Theorem 2. *Let $f\in C^{m,\alpha}(T^n),\ 0<\alpha<1$. Then $K[f]\in A^{m,\alpha}_{(\log)}(U^n)$.*

Proof. The proof is based on an induction argument. If $m=0$, then our statement is the theorem of Jöricke [2].

The initial assumption of induction is that the desired statement is true for all orders of smoothness, which are smaller than m, and we use the identity (4) of Lemma 1. Under our requirements, the function

$$\overline{\zeta}_j^2\frac{\partial^{k_j+r_{j+1}+\cdots+r_n}f(\zeta)}{\partial\overline{\zeta}_j\partial\zeta_j^{k_j-1}\partial\zeta_{j+1}^{r_{j+1}}\dots\partial\zeta_n^{r_n}}$$

belongs to $C^{r_1+\cdots+r_j-k_j,\alpha}(T^n)$. According to our assumption,

$$K\left[\overline{\zeta}_j^2\frac{\partial^{k_j+r_{j+1}+\cdots+r_n}f(\zeta)}{\partial\overline{\zeta}_j\partial\zeta_j^{k_j-1}\partial\zeta_{j+1}^{r_{j+1}}\dots\partial\zeta_n^{r_n}}\right](z)\in A^{r_1+\cdots+r_j-k_j,\alpha}_{(\log)}(U^n).$$

Therefore, the summand under the double sum belongs to $A^\alpha_{(\log)}(U^n)$, i.e.

$$\frac{\partial^{r_1+\cdots+r_j-k_j}}{\partial z_1^{r_1}\dots\partial z_{j-1}^{r_{j-1}}\partial z_j^{r_j-k_j}}K\left[\overline{\zeta}_j^2\frac{\partial^{k_j+r_{j+1}+\cdots+r_n}f(\zeta)}{\partial\overline{\zeta}_j\partial\zeta_j^{k_j-1}\partial\zeta_{j+1}^{r_{j+1}}\dots\partial\zeta_n^{r_n}}\right](z)\in A^\alpha_{(\log)}(U^n). \tag{7}$$

Further, one can see that

$$\frac{\partial^{r_1+\cdots+r_n}f(\zeta)}{\partial\zeta_1^{r_1}\dots\partial\zeta_n^{r_n}}\in C^\alpha(T^n),$$

and

$$K\left[\frac{\partial^{r_1+\cdots+r_n} f(\zeta)}{\partial\zeta_1^{r_1}\ldots\partial\zeta_n^{r_n}}\right](z) \in A^{\alpha}_{(\log)}(U^n) \tag{8}$$

for the last summand in (4). From (7), (8) and (4) it follows that

$$\frac{\partial^{r_1+\cdots+r_n}}{\partial z_1^{r_1}\ldots\partial z_n^{r_n}} K[f](z) \in A^{\alpha}_{(\log)}(U^n).$$

One can be convinced that this inclusion means that $K[f] \in A^{m,\alpha}_{(\log)}(U^n)$. This proves our theorem.

3.3. In the paper [4] for the functions on the distinguished boundary T^n the anisotropic Holder's spaces $\widetilde{\Lambda}(\beta)$ have been introduced as follows. Let $\beta = (\beta_1, ...\beta_n), \quad 0 < \beta_j < 1.$

Definition 1. *The function $f(\zeta)$ is said to be in $\widetilde{\Lambda}(\beta) = \widetilde{\Lambda}(\beta_1, ...\beta_n)$, if $f \in C(T^n)$ and for $h = (h_{j_1}, \ldots, h_{j_k}) \in \mathbb{R}^k$*

$$|\Delta_{h_{j_k}\ldots h_{j_1}} f(\zeta)| \le C_{j_k\ldots j_1}|h_{j_1}|^{\beta_{j_1}}\ldots|h_{j_k}|^{\beta_{j_k}}, \tag{9}$$

where $\quad 1 \le k \le n, \quad 1 \le j_l \le n, \quad 1 \le l \le n,$ *and*

$$\Delta_{h_j} f(\zeta) = f(e^{i\vartheta_1}, ..., e^{i\vartheta_{j-1}}, e^{i\vartheta_j + ih_j}, e^{i\vartheta_{j+1}}, ..., e^{i\vartheta_n}) - f(e^{i\vartheta_1}, ..., e^{i\vartheta_n}),$$

$$\Delta_{h_{j_k}\ldots h_{j_1}} = \Delta_{h_{j_k}}(\Delta_{h_{j_{k-1}}}\ldots(\Delta_{h_{j_1}})).$$

Definition 2. *The function $f(z)$ is said to be in $\widetilde{\Lambda}_a(\beta)$, if $f \in A(U^n)$ and it's boundary value belongs to $\widetilde{\Lambda}(\beta)$, i.e. $\widetilde{\Lambda}_a(\beta) = A^m(U^n) \cap \widetilde{\Lambda}(\beta)$.*

Note that in the case $k = n$ the classes $\Lambda(\beta)$ and $\Lambda_a(\beta)$ are introduced by Nikolśkii (see [5]). The classes $\widetilde{\Lambda}(\beta)$ have several interesting properties. So, for example, unlike ordinary Holder's classes they form an algebra and also are invariant with respect to multiplication by monomials. Besides for them the full analog of Privalov's theorem is valid, namely, in [4] the following theorem is proved

Theorem 3. *If* $f \in \widetilde{\Lambda}(\beta)$, *then* $K[f] \in \widetilde{\Lambda}_a(\beta)$.

In [4] an example is given, which shows that the analogue of this theorem for the classes $\Lambda(\beta)$ is not valid. More precisely, the following is proved.

Theorem 4. *There exist a function* $f \in \Lambda(\beta)$, *such that* $K[f] \notin \Lambda_a(\beta)$.

Hence from the point of view of Privalov's theorem the classes $\widetilde{\Lambda}(\beta)$ are more natural not only in comparison with ordinary Holder's, but also in comparison with the $\Lambda(\beta)$ classes.

Now we define the classes $\widetilde{\Lambda}^m(\beta)$ and $\widetilde{\Lambda}_a^m(\beta)$. Let m be a nonnegative integer.

Definition 3. *A function* f *is said to be in* $\widetilde{\Lambda}^m(\beta)$ *if* $f \in C^m(T^n)$ *and for any multiindex* $k = (k_1, \dots, k_n)$ *such that* $k_1 + \cdots + k_n = m$,

$$\frac{\partial^{k_1+\cdots+k_n}}{\partial\theta_1^{k_1}\dots\partial\theta_n^{k_n}} f(e^{i\theta_1}, \dots, e^{i\theta_n}) \in \widetilde{\Lambda}(\beta)$$

is valid.

Definition 4. *The function* $f(z)$ *is said to be in* $\widetilde{\Lambda}_a^m(\beta)$, *if* $f \in A^m(U^n)$ *and it's boundary value belongs to* $\widetilde{\Lambda}^m(\beta)$, *i.e.* $\widetilde{\Lambda}_a^m(\beta) = A^m(U^n) \cap \widetilde{\Lambda}^m(\beta)$. *If* $m = 0$, *we identify* $\widetilde{\Lambda}_a^0$ *with* $\widetilde{\Lambda}_a$.

The following theorem is the extension of Theorem 3 to the case of any m.

Theorem 5. *If* $f \in \widetilde{\Lambda}^m(\beta)$, *then* $K[f] \in \widetilde{\Lambda}_a^m(\beta)$.

The performence of the proof uses an inductive reasoning analogous to the one for Theorem 2.

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