General Mathematics Vol. 14, No. 1 (2006), 87–93

On Univalence Criteria [1]

Dorin Blezu

Abstract

By means of a new univalence criterion for the analytic functions in the open unit disk U based upon the Becker's criterion, but which doesn't contain $|z|$, we give another criterion similar with the one given by Avhadiev F.G. and Aksentiev L.A.

Also using the above mentioned criterion, some univalence of the integral operators are prooved.

2000 Mathematical Subject Classification: 30C45.

Key words and phrases: Analytic functions, univalent functions, Becker's criterion

1 Introduction

Let $\mathcal{A}$ the class of functions $f(z)$ which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. Let S denote the subclass of A consisting of all functions $f(z)$ which are univalent in U. For $f \in A$ and $g \in A$, we say that the function $f(z)$ is subordinate to $g(z)$, written by $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$.

[1] *Received November 20, 2005*

Accepted for publication (in revised form) January 23, 2006

We need the following theorems due by Avhadiev F.G. and Aksentiev L.A. respectively, N.N.Pascu and V.Pescar.

Theorem A. [1] *Let* $f, g \in A$. *If*

$$(1) \qquad (1-|z|^2)\left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \qquad z \in U$$

and $\log f'(z) \prec \log g'(z)$, $\log f'(0) = \log g'(0) = 0$ *then the function* f *is in* S.

Theorem B. [4] *Let* $\alpha \in \mathbb{C}$, $Re\ (\alpha) \geq 0$. *If* $f \in A$ *and*

$$(2) \qquad \frac{1-|z|^{2Re\ (\alpha)}}{Re\ (\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

then the function

$$(3) \qquad F_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1}f'(u)du\right]^{1/\alpha}$$

belong to the class S.

Theorem C. [5] *Let* α, β, γ *be complex numbers and* $h \in S$. *If*

$$Re(\beta) \geq Re\ (\alpha) > 0$$

and

$$|\gamma| \leq \frac{Re\ (\alpha)}{2} \quad for \quad Re\ (\alpha) \in \left(0, \frac{1}{2}\right)$$

$$|\gamma| \leq \frac{1}{4} \quad for \quad Re\ (\alpha) \in \left[\frac{1}{2}, +\infty\right]$$

then the function

$$(4) \qquad G_{\beta,\gamma}(z) = \left[\beta\int_o^z u^{\beta-1}\left(\frac{h(u)}{u}\right)^\gamma du\right]^{1/\beta}$$

belong also to the class S.

Lemma D. (Caratheodory) *Let* $g \in A$, *and let,* $M > 0$.

If Re $(g(z)) \leq M$*, for any* $z \in U$ *then*

$$(1-|z|)|g(z)| \leq 2M|z| \qquad z \in U$$

Proof. Let us define the function $h(z)$ by

$$h(z) = \frac{g(z)}{2M - g(z)}$$

Then $h(z) \in A$ and $|h(z)| \leq 1$, $z \in U$ because

$$|g(z)| \leq |2M - g(z)|$$

According to the Schwarz's Lemma we have

$$|h(z)| \leq |z| \qquad (\forall\ z \in U)$$

that is

$$|g(z)| \leq |z|\ |2M - g(z)| \leq |z|(2M + |g(z)|)$$

This implies that

$$(1-|z|)|g(z)| \leq 2M|z|$$

2 Main Results

First we give a univalence criterion based on the Becker's criterion but which doesn't use the modulus of z. For this reason it is easily used for practical applications. In the second part, the Lemma D and this criterion are used to obtain several univalence criteria analogous to those given by Avhadiev and Aksentiev [1], Pascu and Pescar [5].

Theorem 1. [3] *If* $f \in A$ *satisfies for some* $\theta \in [0, 2\pi]$ *the inequality*

$$Re\left[e^{i\theta}\frac{zf''(z)}{f'(z)}\right] \leq \frac{1}{4},$$

then $f \in S$

Proof. If we take

$$g(z) = e^{i\theta} \frac{zf''(z)}{f'(z)}$$

in Lemma D, then we have

$$(1 - |z|) \left| \frac{zf''(z)}{f'(z)} \right| \leq 2 \cdot \frac{1}{4}|z| = \frac{|z|}{2}$$

In addition, we see that

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| = (1 + |z|)(1 - |z|) \left| \frac{zf''(z)}{f'(z)} \right| \leq (1 + |z|)\frac{|z|}{2} \leq 1 \tag{5}$$

According to Becker's univalence criterion [2] we conclude that $f \in S$.

Theorem 2. *Let* $f, g \in A$. *If for some* $\theta \in [0, 2\pi]$ *the inequality*

$$Re \left[e^{i\theta} \frac{zg''(z)}{g'(z)} \right] \leq \frac{1}{4} \qquad z \in U$$

is valid and $\log f'(z) \prec \log g'(z)$, $\log f'(0) = \log g'(0) = 0$ *then* f *is in* S, $\forall\ \theta \in [0, 2\pi]$.

Proof. If we take $g(z) = e^{i\theta} \frac{zf''(z)}{f'(z)}$ in Lemma D and using a similar way as in Theorem 1 we obtain the condition (1). According to Theorem A, the conclusion of Theorem 2 follows immediately.

Theorem 3. *Let* $f \in A$, $\alpha \in C$, $Re\ (\alpha) > 0$. *If for some* $\theta \in [0, 2\pi]$ *the inequality*

$$Re \left[e^{i\theta} \frac{zf''(z)}{f'(z)} \right] \leq \begin{cases} Re\dfrac{(\alpha)}{2} & for \quad 0 < Re\ (\alpha) < 1 \\ \frac{1}{4} & for \quad Re\ (\alpha) \geq 1 \end{cases} \quad z \in U \tag{6}$$

is valid, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{1/\alpha}$$

is in S, for all $\theta \in [0, 2\pi]$.

Proof. We consider two cases:

a) $Re\ (\alpha) \geq 1$.

It is easy to observe that the function $h : (0, \infty) \to \mathbb{R}$

$$h(x) = \frac{1 - a^{2x}}{x} \qquad (0 < a < 1)$$

is a decreasing function, and that, if we take $z \in U$, $a = |z|$ then

$$\frac{1 - |z|^{2Re\ (\alpha)}}{Re\ (\alpha)} \leq 1 - |z|^2 \tag{7}$$

If we put $g(z) = e^{i\theta} \frac{zf''(z)}{f'(z)}$, and $M = \frac{1}{4}$ in Lemma D, then we obtain the inequality (5). According to (7) we have

$$\frac{1 - |z|^{2Re\ (\alpha)}}{Re\ (\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{8}$$

b) $0 < Re\ (\alpha) < 1$. The function $q(x) = 1 - a^{2x}$, $0 < a < 1$ is a increasing function, and for $a = |z|$, $z \in U$ one obtains

$$1 - |z|^{2Re\ (\alpha)} \leq 1 - |z|^2 \qquad (0 < Re\ (\alpha) \leq 1) \tag{9}$$

Now if we take $M = \frac{Re\ (\alpha)}{4}$ in Lemma D, then

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq Re\ (\alpha)$$

According to (9) we have

$$(1 - |z|^{2Re\ (\alpha)}) \left| \frac{zf''(z)}{f'(z)} \right| \leq (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq Re\ (\alpha)$$

In the conclusion for all $\alpha \in C$ with $Re\ (\alpha) > 0$ the condition (6) implies the inequality (2) from Theorem B, that is the function F_α from (3) it is univalent. This completes the proof of Theorem 3.

Theorem 4. *Let be α, β, γ complex numbers so that*

$$Re\ (\beta) \geq Re\ (\alpha) > 0$$

and

$$|\gamma| \leq \frac{Re\ (\alpha)}{2} \quad \text{for } Re\ (\alpha) \in \left(0, \frac{1}{2}\right)$$

$$|\gamma| \leq \frac{1}{4} \quad \text{for } Re\ (\alpha) \in \left[\frac{1}{2}, +\infty\right)$$

If $h \in A$ and for some $\theta \in [0, 2\pi]$

$$Re\left[e^{i\theta}\frac{zh''(z)}{h'(z)}\right] \leq \frac{1}{4} \quad (z \in U),$$

then the function

$$G_{\beta,\alpha}(z) = \left[\beta \int_0^z u^{\beta-1}\left(\frac{h(u)}{u}\right)^{\gamma} du\right]^{1/\beta} \tag{10}$$

belong to the class S.

Proof. For the function $\left(\frac{h(z)}{z}\right)^{\gamma}$ in (10), we can choose the regular branch which is equal to 1 at the origin. According to the Theorem 1 and Theorem C imply the conclusion of the Theorem 4.

Remark. *In all above univalence criteria the hypothesis have conditions which do not contain $|z|$ that is, these are more practical than other similar criteria.*

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Dorin Blezu,
Department of Mathematics
"Lucian Blaga" University
550012 Sibiu, str.Dr.I.Raţiu 5–7, Romania
E–MAIL dorin.blezu@ulbsibiu.ro