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A Note on Ostrowski Like Inequalities in $\mathbf{L}_1(a,b)$ Spaces [1]

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Abstract

The main aim of this paper is to establish Ostrowski like inequalities for product of two continuous functions whose derivatives are in $\mathrm{L}_1(a,b)$ spaces and provide new estimates on these inequalities.

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1 Introduction

In 1938, A. M. Ostrowski [6] proved the following inequality(see also[4, P. 468]):

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Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{I}$ (interior of I), and let $a, b \in \overset{0}{I}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f\,'\|_{\infty} := \sup\limits_{t\in(a,b)} |f\,'(t)| < \infty$, then we have:*

$$(1.1) \qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\,\|f\,'\|_{\infty},$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 2005, B. G. Pachpatte [8] established new inequality of the type(1.1) involving two functions and their derivatives as given in the following theorem:

Theorem 2. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions on $[a,b]$ and differentiable on (a,b), whose derivatives $f', g' : (a,b) \to \mathbb{R}$ are bounded on (a,b), i.e., $\|f\,'\|_{\infty} := \sup\limits_{t\in(a,b)} |f\,'(t)| < \infty$, $\|g\,'\|_{\infty} := \sup\limits_{t\in(a,b)} |g\,'(t)| < \infty$, then*

$$(1.2) \qquad \left| f(x)\,g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)\,dy + f(x)\int_a^b g(y)\,dy \right] \right| \leq$$

$$\leq \frac{1}{2}\left(|g(x)|\,\|f\,'\|_{\infty} + |f(x)|\,\|g\,'\|_{\infty} \right) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a),$$

for all $x \in [a,b]$.

In [3], S. S. Dragomir and S. Wang established another Ostrowski like inequality for $\|.\|_1$ −norm as given in the following theorem:

Theorem 3. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on (a,b), whose derivative $f':[a,b]\longrightarrow\mathbb{R}$ belongs to $\mathbf{L}_1(a,b)$. Then, we have the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_1, \tag{1.3}$$

for all $x\in[a,b]$.

In the last few years, the study of such inequalities has been the focus of many mathematicians and a number of research papers have appeared which deal with various generalizations, extensions and variants, see[3] and references given therein. Inspired and motivated by the research work going on related to inequalities (1.1-1.3), we establish here new Ostrowski like inequalities for the product of two continuous functions whose derivatives are in $\mathbf{L}_1(a,b)$. The results are presented in an elementary way and provide new estimates on these types of inequalities.

2 Main Results

Our main result is given in the following theorem:

Theorem 4. *Let $f,\ g:[a,b]\to\mathbb{R}$ be continuous mappings on $[a,b]$ and differentiable on (a,b), whose derivatives $f',g':(a,b)\to\mathbb{R}$ belong to $\mathbf{L}_1(a,b)$ i.e., $\|f'\|_1=\left(\int_a^b|f(t)|\,dt\right)$, $\|g'\|_1=\left(\int_a^b|g(t)|\,dt\right)$, then*

$$\left|f(x)\,g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)\,dy+f(x)\int_a^b g(y)\,dy\right]\right|\le \tag{2.1}$$

$$\le\frac{1}{2}\left[|g(x)|\,\|f\,'\|_1+|f(x)|\,\|g\,'\|_1\right]\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]$$

for all $x \in [a, b]$.

Proof. *For any* x, $y \in [a, b]$, *we have the following identities*

$$f(x) - f(y) = \int_y^x f'(t)dt \tag{2.2}$$

and

$$g(x) - g(y) = \int_y^x g'(t)dt. \tag{2.3}$$

Multiplying both sides of (2.2) and (2.3) by $g(x)$ *and* $f(x)$ *respectively and adding we get*

$$2f(x)g(x) - [g(x)f(y) + f(x)g(y)] = g(x)\int_y^x f'(t)dt + f(x)\int_y^x g'(t)dt. \tag{2.4}$$

Integrating both sides of (2.4) with respect to y *over* $[a, b]$ *and rewriting, we have:*

$$f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)dy + f(x)\int_a^b g(y)dy\right] = \tag{2.5}$$

$$= \frac{1}{2(b-a)}\int_a^b \left[g(x)\int_y^x f'(t)dt + f(y)\int_y^x g'(t)dt\right] dy.$$

Using (2.5), we have by Hölder's integral inequality and mean value theorem, *that*

$$\left| f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)dy + f(x)\int_a^b g(y)dy \right] \right| =$$

$$= \frac{1}{2(b-a)}\left| \left[g(x)\int_a^b f'(y)(x-y)dy + f(x)\int_a^b g'(y)(x-y)dy \right] \right| =$$

$$= \frac{1}{2(b-a)}\left| g(x)(x-a)\int_a^b f'(y)dy + f(x)(b-x)\int_a^b g'(y)dy \right| =$$

$$\leq \frac{1}{2(b-a)}\left[|g(x)| \left\| f' \right\|_1 (x-a) + |f(x)| \left\| g' \right\|_1 (b-x) \right] \leq$$

$$\leq \frac{1}{2(b-a)}\max(x-a, b-x)\left[|g(x)| \left\| f' \right\|_1 + |f(x)| \left\| g' \right\|_1 \right] \leq$$

$$\leq \frac{1}{2}\left[|g(x)| \left\| f' \right\|_1 + |f(x)| \left\| g' \right\|_1 \right]\left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right],$$

for all $x \in [a,b]$.
This completes the proof.

Remark 1. *We note that, by taking $g(x) = 1$ and hence $g'(x) = 0$ in theorem 4, we recapture the inequality in (1.3).*

2. Integrating both sides of (2.5) with respect to x over $[a,b]$, rewriting the resulting identity and using the Hölder's integral inequality, we obtain the following Grüss type inequality:

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx \right)\left(\frac{1}{b-a}\int_a^b g(x)dx \right) \right| \leq \tag{2.6}$$

$$\leq \frac{1}{2}\left[|g(x)| \left\| f' \right\|_1 + |f(x)| \left\| g' \right\|_1 \right]$$

3. For other inequalities of the type (2.6), see the book [4], where many other references are given.

A slight variant of theorem 4 is embodied in the following theorem.

Theorem 5. *Let f, g, f', g' be as in theorem 4, then*

$$f(x)\,g(x) - \frac{1}{b-a}\left[g(x)\int_a^b f(y)\,dy + f(x)\int_a^b g(y)\,dy\right] + \tag{2.7}$$

$$+\frac{1}{b-a}\int_a^b f(y)g(y)dy \leq \|f\,'\|_{1,[y,x]}\,\|g\,'\|_{1,[y,x]}\,.$$

for all $x, y \in [a,b]$.

Proof. *From the hypothesis, the identities (2.2) and (2.3) hold. Multiplying the left and right sides of (2.2) and (2.3) we get*

$$f(x)g(x) - [g(x)f(y) + f(x)g(y)] + f(y)g(y) = \int_y^x f'(t)dt \int_y^x g'(t)dt. \tag{2.8}$$

Integrating both sides of (2.8) with respect to y over $[a,b]$and rewriting we have

$$f(x)\,g(x) - \frac{1}{b-a}\left[g(x)\int_a^b f(y)\,dy + f(x)\int_a^b g(y)\,dy\right] + \tag{2.9}$$

$$+\frac{1}{b-a}\int_a^b f(y)g(y)dy = \frac{1}{b-a}\int_a^b\left(\int_y^x f'(t)dt\int_y^x g'(t)dt\right)dy.$$

From (2.9) using the properties of modulus, we obtain:

$$\left|f(x)\,g(x) - \frac{1}{b-a}\left[g(x)\int_a^b f(y)\,dy + f(x)\int_a^b g(y)\,dy\right] + \frac{1}{b-a}\int_a^b f(y)g(y)dy\right| \leq$$

$$\leq \|f\,'\|_{1,[y,x]}\,\|g\,'\|_{1,[y,x]}\,.$$

Remark 2. *Integrating both sides of (2.9) with respect to x over $[a,b]$, rewriting the resulting identity, and using the Hölder's integral inequality we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right|\leq \tag{2.10}$$

$$\leq\frac{1}{2\,(b-a)}\int_a^b\left[|g(x)|\,\|f\,'\|_{1,[y,x]}+|f(x)|\,\|g'\|_{1,[y,x]}\right]\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]$$

for all $x\in[a,b]$.

2. We note that the norms $\|f\,'\|_{1,[y,x]}$ and $\|g'\|_{1,[y,x]}$ are valid for all x, $y\in[a,b]$,therefore we can recapture the norms over $[a,b]$.

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