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On a subclass of n-uniformly close to convex functions [1]

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Abstract

In this paper we define a subclass on n-uniformly close to convex functions and we obtain some properties regarding this class.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : \ f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : \ f \text{ is univalent in } U\}$.

We recall here the definition of the well - known class of starlike functions:

$$S^* = \left\{ f \in A : Re\frac{zf'(z)}{f(z)} > 0 \ , \ z \in U \right\},$$

Let D^n be the Sălăgean differential operator (see [5]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

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$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^n f(z) = D(D^{n-1} f(z))$$

Remark 1.1. *If* $f \in S$, $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$ *then*

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j .$$

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt \;, \quad a \in \mathbb{C} \;, \quad Re\ a \geq 0.$$

For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and in the case $a = 1, 2, 3, ...$ we obtain the Bernardi integral operator.

The purpose of this note is to define, using the Sălăgean differential operator, a subclass on n-uniformly close to convex functions and to obtain some properties regarding this class.

2 Preliminary results

Let $k \in [0, \infty)$, $n \in \mathbb{N}^*$. We define the class $(k, n) - S^*$ (see the definition of the class $(k, n) - ST$ in [1]) by $f \in S^*$ and

$$Re \left(\frac{D^n f(z)}{f(z)} \right) > k \left| \frac{D^n f(z)}{f(z)} - 1 \right| \;, \; z \in U \,.$$

Remark 2.1. *(for more details see [1]) We denote by* p_k , $k \in [0, \infty)$ *the function which maps the unit disk conformally onto the region* Ω_k*, such that* $1 \in \Omega_k$ *and*

$$\partial\Omega_k = \left\{ u + iv \;:\; u^2 = k^2(u-1)^2 + k^2 v^2 \right\} \,.$$

The domain Ω_k *is elliptic for* $k > 1$*, hyperbolic when* $0 < k < 1$*, parabolic for* $k = 1$*, and a right half-plane when* $k = 0$*. In this conditions, a function*

f is in the class $(k,n)-S^*$ if and only if $\dfrac{D^n f(z)}{f(z)} \prec p_k$ or $\dfrac{D^n f(z)}{f(z)}$ take all values in the domain Ω_k . Because the domain Ω_k is convex, as an immediate consequence of the well known Rogosinski result for subordinate functions, we obtain for $p \prec p_k$, $p(z) = 1 + p_1 z + p_2 z^2 + \dots$, $z \in U$,

$$|p_n| \le |P_1| := P_1(k) = \begin{cases} \dfrac{8(\arccos k)^2}{\pi^2(1-k^2)}\,, & 0 \le k < 1\,, \\ \dfrac{8}{\pi^2}\,, & k = 1\,, \\ \dfrac{\pi^2}{4\sqrt{\kappa}(k^2-1)K^2(\kappa)(1+\kappa)}\,, & k > 1\,. \end{cases}$$

for $n = 1, 2, \dots$, where $K(\kappa)$ is Legendre's complete elliptic integral of the first kind, κ is chosen such that $k = \cosh[\pi K'(\kappa)]/[4K(\kappa)]$ and $K'(\kappa)$ is complementary integral of $K(\kappa)$.

With the notations from Remark 2.1 we have:

Theorem 2.1. *[1] Let $k \in [0, \infty)$ and $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$ belongs to the class $(k,n)-S^*$. Then $|a_2| \le \dfrac{P_1}{2^n - 1}$ and*

$$|a_j| \le \frac{P_1}{j^n - 1} \prod_{s=2}^{j-1} \left(1 + \frac{P_1}{s^n - 1}\right),\ j = 3, 4, \dots,\ n \in \mathbb{N}^*\,.$$

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [2], [3], [4]).

Theorem 2.2. *Let q be convex in U and $j : U \to \mathbb{C}$ with $Re[j(z)] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ and satisfied $p(z) + j(z) \cdot zp'(z) \prec q(z)$, then $p(z) \prec q(z)$.*

3 Main results

Definition 3.1. *Let $f \in A$, $k \in [0, \infty)$ and $n \in \mathbb{N}^*$. We say that the function f is in the class $(k,n)-CC$ with respect to the function $g \in (k,n)-S^*$ if*

$$Re\left(\frac{D^n f(z)}{g(z)}\right) > k \cdot \left|\frac{D^n f(z)}{g(z)} - 1\right|,\ z \in U\,.$$

Remark 3.1. *Geometric interpretation:* $f \in (k,n) - CC$ *with respect to the function* $g \in (k,n) - S^*$ *if and only if* $\dfrac{D^n f(z)}{g(z)} \prec p_k$ *(see Remark 2.1) or* $\dfrac{D^n f(z)}{g(z)}$ *take all values in the domain* Ω_k *(see Remark 2.1).*

Remark 3.2. *From the geometric properties of the domains* Ω_k *we have that*
$(k_1, n) - CC \subset (k_2, n) - CC$, *where* $k_1 \geq k_2$.

Theorem 3.1. *If* $F(z) \in (k,n) - S^*$, *with* $k \in [0, \infty)$ *and* $n \in \mathbb{N}^*$, *then* $f(z) = L_a F(z) \in (k,n) - S^*$, *where* L_a *is the integral operator defined by (1).*

Proof. By differentiating (1) we obtain

$$(1 + a)F(z) = af(z) + zf'(z)\,. \tag{2}$$

By means of the application of the linear operator D^n we have

$$(1 + a)D^n F(z) = aD^n f(z) + D^{n+1} f(z)\,. \tag{3}$$

From (2) and (3) we obtain

$$\frac{(1+a)D^n F(z)}{(1+a)F(z)} = \frac{aD^n f(z) + D^{n+1} f(z)}{af(z) + zf'(z)} = \frac{f(z)\left[a\dfrac{D^n f(z)}{f(z)} + \dfrac{D^{n+1} f(z)}{f(z)}\right]}{f(z)\left[a + \dfrac{zf'(z)}{f(z)}\right]}$$

or

$$\frac{D^n F(z)}{F(z)} = \frac{a\dfrac{D^n f(z)}{f(z)} + \dfrac{D^{n+1} f(z)}{f(z)}}{a + \dfrac{zf'(z)}{f(z)}}\,. \tag{4}$$

With notation $p(z) = \dfrac{D^n f(z)}{f(z)}$, where $p(0) = 1$, we obtain

$$zp'(z) = z\frac{(D^n f(z))' f(z) - (D^n f(z))\, f'(z)}{f^2(z)} =$$

$$= \frac{z\left(D^n f(z)\right)'}{f(z)} - \frac{D^n f(z)}{f(z)} \cdot \frac{zf'(z)}{f(z)} = \frac{D^{n+1} f(z)}{f(z)} - p(z) \cdot \frac{zf'(z)}{f(z)}$$

or

$$\frac{D^{n+1} f(z)}{f(z)} = zp'(z) + p(z) \cdot \frac{zf'(z)}{f(z)} \,. \tag{5}$$

From (4) and (5) we have

$$\frac{D^n F(z)}{F(z)} = \frac{p(z)\left[a + \dfrac{zf'(z)}{f(z)}\right] + zp'(z)}{a + \dfrac{zf'(z)}{f(z)}}$$

or

$$\frac{D^n F(z)}{F(z)} = p(z) + \frac{1}{a + \dfrac{zf'(z)}{f(z)}} \cdot zp'(z) \tag{6}$$

From hypothesis we have $\dfrac{D^n F(z)}{F(z)} \prec p_k(z)$, where p_k maps the unit disk conformally onto the convex domain Ω_k (see Remark 2.1).

Using (6) we obtain $p(z) + \dfrac{1}{a + \dfrac{zf'(z)}{f(z)}} \cdot zp'(z) \prec p_k(z)$.

Using the hypothesis, from Theorem 2.2, we have $p(z) \prec p_k(z)$ or $\dfrac{D^n f(z)}{f(z)}$ take all values in the domain Ω_k. This means that $f(z) \in (k,n) - S^*$.

Theorem 3.2. *If* $F(z) \in (k,n) - CC$, $k \in [0,\infty)$, $n \in \mathbb{N}^*$, *with respect to the function* $G(z) \in (k,n) - S^*$, *and* $f(z) = L_a F(z)$, $g(z) = L_a G(z)$, *where* L_a *is the integral operator defined by (1), then* $f(z) \in (k,n) - CC$, $k \in [0,\infty)$, $n \in \mathbb{N}^*$, *with respect to the function* $g(z) \in (k,n) - S^*$.

Proof. Using (1) and the linear operator D^n we obtain

$$(1+a)D^n F(z) = aD^n f(z) + D^{n+1} f(z)$$

and

$$(1+a)G(z) = ag(z) + zg'(z) \,.$$

From the above we have

$$\frac{(1+a)D^nF(z)}{(1+a)G(z)}=\frac{aD^nf(z)+D^{n+1}f(z)}{ag(z)+zg'(z)}$$

or

$$\frac{D^nF(z)}{G(z)}=\frac{a\dfrac{D^nf(z)}{g(z)}+\dfrac{D^{n+1}f(z)}{g(z)}}{a+\dfrac{zg'(z)}{g(z)}}$$

If we denote $p(z)=\dfrac{D^nf(z)}{g(z)}$, with $p(0)=1$, we have

$$\frac{D^nF(z)}{G(z)}=\frac{ap(z)+\dfrac{D^{n+1}f(z)}{g(z)}}{a+\dfrac{zg'(z)}{g(z)}} \tag{7}$$

With simple calculations we obtain

$$zp'(z)=\frac{z\left(D^nf(z)\right)'}{g(z)}-\frac{D^nf(z)}{g(z)}\cdot\frac{zg'(z)}{g(z)}=\frac{D^{n+1}f(z)}{g(z)}-p(z)\cdot\frac{zg'(z)}{g(z)}$$

and thus

$$\frac{D^{n+1}f(z)}{g(z)}=zp'(z)+p(z)\cdot\frac{zg'(z)}{g(z)} \tag{8}$$

From (7) and (8) we obtain

$$\frac{D^nF(z)}{G(z)}=p(z)+\frac{1}{a+\dfrac{zg'(z)}{g(z)}}\cdot zp'(z)=p(z)+j(z)\cdot zp'(z)\,, \tag{9}$$

where from the hypothesis and the Theorem 3.1 we have $Re\,j(z)>0$ $z\in U$.

From $F(z)\in(k,n)-CC$ with respect to the function $G(z)\in(k,n)-S^*$, using Remark 3.1, we obtain $p(z)+j(z)\cdot zp'(z)\prec p_k(z)$, where p_k maps the unit disk conformally onto the convex domain Ω_k (see Remark 2.1).

From Theorem 2.2, we have $p(z)\prec p_k(z)$ or $\dfrac{D^nf(z)}{g(z)}$ take all values in the domain Ω_k. This means that $f(z)\in(k,n)-CC$ with respect to the function $g(z)\in(k,n)-S^*$.

Theorem 3.3. *If* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *belong to the class* $(k, n) - CC$*, with respect to the function* $g(z) \in (k, n) - S^*$*,* $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$*, where* $k \in [0, \infty)$*,* $n \in \mathbb{N}^*$*, then*

$$|a_2| \le \frac{P_1}{2^n - 1}\,;\ |a_3| \le \frac{P_1\,(P_1 - 1 + 2^n)}{(2^n - 1)\,(3^n - 1)}\,;$$

$$|a_j| \le \frac{P_1}{j^n - 1} \cdot \prod_{t=2}^{j-1} \frac{P_1 - 1 + t^n}{t^n - 1}\,,\ j \ge 4\,,$$

where P_1 *is given in Remark 2.1.*

Proof. We have $f(z) \in (k, n) - CC$ if and only if $h(z) = \dfrac{D^n f(z)}{g(z)} \prec p_k(z)$, where $p_k\,(U) = \Omega_k$ (see Remark 3.1). Let $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$, $z \in U$. Taking account the Rogosinski subordination theorem, we have $|c_j| \le P_1\,,\ j \ge 1\,.$

Using the hypothesis and the Remark 1.1 we have

$$\frac{z + \sum_{j=2}^{\infty} j^n a_j z^j}{z + \sum_{j=2}^{\infty} b_j z^j} = 1 + c_1 z + c_2 z^2 + \cdots\,.$$

From the equality of the powers coefficients we obtain

$$2^n a_2 = c_1 + b_2\ ;\ 3^n a_3 = c_2 + b_3 + c_1 b_2$$

and

$$j^n a_j = c_{j-1} + c_1 b_{j-1} + c_2 b_{j-2} + c_3 b_{j-3} + \cdots c_{j-2} b_2 + b_j\,,\ j \ge 4\,. \tag{10}$$

Using $|c_j| \le P_1\,,\ j \ge 1\,,\ 2^n a_2 = c_1 + b_2$ and Theorem 2.1 we have

$$2^n\,|a_2| \le P_1 + \frac{P_1}{2^n - 1} = \frac{2^n}{2^n - 1} \cdot P_1$$

and thus $|a_2| \leq \dfrac{P_1}{2^n-1}$.

In a similarly way we obtain $|a_3| \leq \dfrac{P_1\left(P_1-1+2^n\right)}{\left(2^n-1\right)\left(3^n-1\right)}$.

Using $|c_j| \leq P_1$, $j \geq 1$ and Theorem 2.1 we obtain from (10) the estimations

$$j^n\left|a_j\right| \leq P_1\left\{1+\frac{P_1}{2^n-1}+\sum_{l=3}^{j-1}\left[\frac{P_1}{l^n-1}\cdot\prod_{s=2}^{l-1}\left(1+\frac{P_1}{s^n-1}\right)\right]\right\}+\frac{P_1}{j^n-1}\cdot$$
$$\cdot\prod_{t=2}^{j-1}\left(1+\frac{P_1}{t^n-1}\right).$$

By mathematical induction for $j \geq 4$ we have

$$1+\frac{P_1}{2^n-1}+\sum_{l=3}^{j-1}\left[\frac{P_1}{l^n-1}\cdot\prod_{s=2}^{l-1}\left(1+\frac{P_1}{s^n-1}\right)\right]=\prod_{t=2}^{j-1}\frac{P_1-1+t^n}{t^n-1}$$

and thus we obtain

$$j^n\left|a_j\right| \leq P_1\cdot\prod_{t=2}^{j-1}\frac{P_1-1+t^n}{t^n-1}+\frac{P_1}{j^n-1}\cdot\prod_{t=2}^{j-1}\left(1+\frac{P_1}{t^n-1}\right)$$

or

$$j^n\left|a_j\right| \leq j^n\frac{P_1}{j^n-1}\cdot\prod_{t=2}^{j-1}\frac{P_1-1+t^n}{t^n-1}\,,\ j \geq 4\,.$$

Thus

$$\left|a_j\right| \leq \frac{P_1}{j^n-1}\cdot\prod_{t=2}^{j-1}\frac{P_1-1+t^n}{t^n-1}\,,\ j \geq 4\,,$$

Theorem 3.4. *Let* $a \in \mathbb{C}$, $Re\, a \geq 0$, $n \in \mathbb{N}^*$ *and* $k \in [0,\infty)$. *If* $F(z) \in (k,n)-CC$, $F(z)=z+\sum_{j=2}^{\infty}a_jz^j$, *and* $f(z)=L_aF(z)$, $f(z)=z+\sum_{j=2}^{\infty}b_jz^j$,*where* L_a *is the integral operator defined by (1), then*

$$|b_2| \leq \left|\frac{a+1}{a+2}\right|\frac{P_1}{2^n-1}\,;\ |b_3| \leq \left|\frac{a+1}{a+3}\right|\frac{P_1\left(P_1-1+2^n\right)}{\left(2^n-1\right)\left(3^n-1\right)}\,;$$

$$|b_j| \le \left|\frac{a+1}{a+j}\right| \frac{P_1}{j^n - 1} \cdot \prod_{t=2}^{j-1} \frac{P_1 - 1 + t^n}{t^n - 1}\,,\ j \ge 4\,,$$

where P_1 is given in Remark 2.1.

Proof. From $f(z) = L_a F(z)$ we have

$$(1+a)F(z) = af(z) + zf'(z)\,.$$

Using the above series expansions we obtain

$$(1+a)z + \sum_{j=2}^{\infty}(1+a)a_j z^j = az + \sum_{j=2}^{\infty} ab_j z^j + z + \sum_{j=2}^{\infty} jb_j z^j$$

and thus $b_j(a+j) = (1+a)a_j\ j \ge 2$.

From the above we have $b_j \le \left|\frac{a+1}{a+j}\right| \cdot |a_j|\ ,\ j \ge 2$. Using the estimations from Theorem 3.3 we obtain the needed results.

For $a = 1$, when the integral operator L_a become the Libera integral operator, we obtain from the above theorem:

Corollary 3.1. *Let $n \in \mathbb{N}^*$ and $k \in [0,\infty)$. If $F(z) \in (k,n) - CC$, $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $f(z) = L(F(z))$, $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where L is the Libera integral operator defined by $L(F(z)) = \frac{2}{z}\int_0^z F(t)dt$, then*

$$|b_2| \le \frac{2}{3}\frac{P_1}{2^n - 1}\,;\ |b_3| \le \frac{1}{2}\frac{P_1\,(P_1 - 1 + 2^n)}{(2^n - 1)\,(3^n - 1)}\,;$$

$$|b_j| \le \frac{2}{j+1}\frac{P_1}{j^n - 1} \cdot \prod_{t=2}^{j-1} \frac{P_1 - 1 + t^n}{t^n - 1}\,,\ j \ge 4\,,$$

where P_1 is given in Remark 2.1.

Remark 3.3. *Similarly results with the results from the Corollary 3.1 are easy to obtain from Theorem 3.4 by taking $a = 0$, respectively $a = 1, 2, 3, \cdots$.*

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