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On a generalization of an approximation operator defined by A. Lupaş [1]

Ulrich Abel and Mircea Ivan

Dedicated to Professor Alexandru Lupaş on the ocassion of his 65th birthday

Abstract

In this paper we study local approximation properties of a family of positive linear operators introduced by Pethe and Jain. We obtain estimates for the rate of convergence and derive the complete asymptotic expansion for these operators. They generalize the Szász–Mirakjan operators and approximate functions satisfying a certain growth condition on the infinite interval $[0,\infty)$. On the other hand they contain as a special case an operator defined by A. Lupaş [17, Problem 4, p. 227] in 1995.

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1 Introduction

Let E be the class of all functions of exponential type on $[0,\infty)$ with the property $|f(t)| \le Ke^{At}$ $(t \ge 0)$ for some finite constants K, $A > 0$. In 1977,

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Pethe and Jain [15] introduced a generalization $M_{n,\alpha}$ $(n = 1, 2, \ldots)$ of the Szász–Mirakjan operators associating to each function $f \in E$ the series

$$(M_{n,\alpha} f)(x) = (1 + n\alpha)^{-x/\alpha} \sum_{\nu=0}^{\infty} \left(\alpha + \frac{1}{n}\right)^{-\nu} \frac{x^{(\nu,-\alpha)}}{\nu!} \, f\left(\frac{\nu}{n}\right) \qquad (x \geq 0),$$

where $x^{(\nu,-\alpha)} = x(x+\alpha)\ldots(x + (\nu-1)\alpha)$, $x^{(0,-\alpha)} = 1$ and $0 \leq n\alpha \leq 1$ (cf. [13, Example 4, p. 48]). More precisely, the parameter $\alpha = \alpha_n$ is coupled with n in such a way that $0 \leq \alpha_n \leq 1/n$. We will refer to this family of operators as Favard, Pethe and Jain operators, briefly FPJ operators.

The purpose of this paper is to study the local rate of convergence and to the asymptotic behaviour of this family of positive linear operators.

To this end, we write the FPJ operators in a slightly different form which is more convenient for our investigation. Putting $c = (\alpha n)^{-1}$ we get a sequence of reals satisfying $c = c_n \geq \beta$ $(n = 0, 1, \ldots)$, for a certain constant $\beta > 0$. Then, the operators $M_{n,\alpha}$ are equivalent to the operators $S_{n,c}$ $(n = 1, 2, \ldots)$ given by

$$(S_{n,c} f)(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}^{[c]}(x) \; f\left(\frac{\nu}{n}\right) \qquad (x \geq 0), \tag{1}$$

where

$$p_{n,\nu}^{[c]}(x) = \left(\frac{c}{1+c}\right)^{ncx} \binom{ncx + \nu - 1}{\nu} (1+c)^{-\nu} \, (\nu = 0, 1, \ldots). \tag{2}$$

Note that the operators $S_{n,c}$ are well-defined, for all sufficiently large n, since the infinite sum in (1) is convergent if $n > A/\log(1+c)$, provided that $|f(t)| \leq Ke^{At}$ $(t \geq 0)$, that is $f \in E$. In particular there holds $S_{n,c}e_0 = e_0$, where e_r denote the monomials given by $e_r(x) = x^r$ $(r = 0, 1, \ldots)$. Simple computations yield $S_{n,c}e_1 = e_1$ and $S_{n,c}e_2 = e_2 + ((1+c)/(nc))\, e_1$. Thus, the Bohman–Korovkin theorem implies

$$\lim_{n\to\infty} (S_{n,c} f)(x) = f(x),$$

for all bounded continuous functions $f \in E$. Later on we shall show that the approximation property is valid for all $f \in E$.

Since, for fixed n, ν and x,

$$\lim_{c \to +\infty} p_{n,\nu}^{[c]}(x) = e^{-nx} \frac{(nx)^{\nu}}{\nu!},$$

we obtain in the limiting case $c \to +\infty$ the classical Szász–Mirakjan operators

$$(S_n f)(x) \equiv (S_{n,\infty} f)(x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^{\nu}}{\nu} f\left(\frac{\nu}{n}\right) (x \geq 0)$$

In the special case $c = 1$ we obtain the operator $L_n \equiv S_{n,1}$, given by

$$(L_n f)(x) = 2^{-nx} \sum_{\nu=0}^{\infty} \binom{nx + \nu - 1}{\nu} 2^{-\nu} f\left(\frac{\nu}{n}\right) (x \geq 0), \tag{3}$$

introduced independently by A. Lupaş [17, Problem 4, p. 227] in 1995. Lupaş established the Korovkin condition guaranteeing the approximation property. He remarked that the operators L_n have a form very similar to the Szász–Mirakjan operators and invited to find further properties. In 1999, O. Agratini [11] investigated the operators of Lupaş (for an announcement of his results see [10]). He derived an asymptotic formula and some quantitative estimates for the rate of convergence. Moreover, he defined the Kantorovich type version and a Durrmeyer variant of the operators L_n.

In this paper we give estimates for the rate of convergence. Furthermore, we derive the complete asymptotic expansion for the sequence of operators $S_{n,c}$ in the form

$$(S_{n,c} f)(x) \sim f(x) + \sum_{k=1}^{\infty} a_k(f, c; x)\, n^{-k} \qquad (n \to \infty), \tag{4}$$

provided that f admits derivatives of sufficiently high order at $x > 0$. Formula (4) means that, for all $q = 1, 2, \ldots$, there holds

$$(S_{n,c} f)(x) = f(x) + \sum_{k=1}^{q} a_k(f, c; x)\, n^{-k} + o(n^{-q}) \qquad (n \to \infty).$$

The coefficients $a_k(f,c;x)$, which are dependent on the sequence c, will be given in an explicit form. It turns out that Stirling numbers of first and second kind play an important role. As a special case we obtain the complete asymptotic expansion for the sequence of Szász–Mirakjan operators S_n.

We remark that in [1, 3, 5, 6] the first author gave analogous results for the Meyer–König and Zeller operators, the Bernstein–Kantorovich operators, the Bernstein–Durrmeyer operators, and the operators of K. Balázs and Szabados, respectively. See also [9]. Asymptotic expansions of multivariate operators can be found in [4, 8].

2 Rate of convergence

We obtain the following estimate of the rate of convergence when the Lupaş operators are applied to bounded function $f \in E$. All results are consequences of general results (see, e.g., [12, p. 268, Theorem 5.1.2]).

Theorem 2.1 *Let $f \in E$ be bounded. Then, for all $x \in [0,\infty)$, $n \in \mathbb{N}$, and $\delta > 0$, there holds*

$$\left|\left(S_{n,c}f\right)(x) - f(x)\right| \le \left(1 + \delta^{-1}\sqrt{\frac{1+c}{cn}x}\right)\omega\left(f;\delta\right).$$

Moreover, if f is differentiable on $[0,\infty)$ with f' bounded on $[0,\infty)$, we also have

$$\left|S_{n,c}(f;x) - f(x)\right| \le \sqrt{\frac{1+c}{cn}x}\left(1 + \delta^{-1}\sqrt{\frac{1+c}{cn}x}\right)\omega\left(f';\delta\right).$$

Theorem 2.1 applied to $\delta = \sqrt{(1+c)x/\left(cn\right)}$ implies

Corollary 2.1 *Let $f \in E$ be bounded. Then, for all $x \in [0,\infty)$, and $n \in \mathbb{N}$, there holds*

$$\left|\left(S_{n,c}f\right)(x) - f(x)\right| \le 2\omega\left(f;\sqrt{\frac{1+c}{cn}x}\right).$$

Moreover, if f is differentiable on $[0,\infty)$ with f' bounded on $[0,\infty)$, we also have

$$\left|\left(S_{n,c}f\right)(x)-f(x)\right|\leq 2\sqrt{\frac{1+c}{cn}}x\omega\left(f';\sqrt{\frac{1+c}{cn}x}\right).$$

3 Asymptotic expansion

For $q\in\mathbb{N}$ and $x\in(0,\infty)$, let $K[q;x]$ be the class of all functions $f\in E$ of polynomial growth which are q times differentiable at x. The following theorem presents as our main result the complete asymptotic expansion for the FPJ operators $S_{n,c}$.

Theorem 3.1 *Let $q\in\mathbb{N}$ and $x\in(0,\infty)$. For each function $f\in K[2q;x]$, the FPJ operators possess the asymptotic expansion*

$$\left(S_{n,c}f\right)(x)=f(x)+\sum_{k=1}^{q}a_k(f,c;x)n^{-k}+o\left(n^{-q}\right)\qquad(n\to\infty)$$

with the coefficients

$$(5)\quad a_k(f,c;x)=\sum_{s=k}^{2k}\frac{f^{(s)}(x)}{s!}x^{s-k}\sum_{j=0}^{k}(-c)^{j-k}T(s,k,j)\qquad(k=1,2,\ldots),$$

where the numbers $T(s,k,j)$ are defined by

$$(6)\qquad T(s,k,j)=\sum_{r=k}^{s}(-1)^{s-r}\binom{s}{r}S_{r-j}^{r-k}\sigma_r^{r-j}\qquad(0\leq j\leq k\leq s).$$

The quantities S_j^i and σ_j^i in Eq. (6) denote the Stirling numbers of the first resp. second kind. The Stirling numbers are defined by

$$(7)\qquad x^{\underline{j}}=\sum_{i=0}^{j}S_j^ix^i,\qquad x^j=\sum_{i=0}^{j}\sigma_j^ix^{\underline{i}}\qquad(j=0,1,\ldots),$$

where $x^{\underline{i}}\equiv x^{(i,1)}=x(x-1)\ldots(x-i+1)$, $x^{\underline{0}}=1$ is the falling factorial.

Remark 3.1 *If $f \in \bigcap_{q=1}^{\infty} K[q;x]$, the FPJ operators possess the complete asymptotic expansion*

$$(S_{n,c}f)(x) = f(x) + \sum_{k=1}^{\infty} a_k(f,c;x) n^{-k} \qquad (n \to \infty),$$

where the coefficients $a_k(f,c;x)$ are as defined in (5).

Remark 3.2 *For the convenience of the reader, we list the explicit expressions for the initial coefficients $a_k(f,c;x)$:*

$$a_1(f,c;x) = \frac{(1+c)\,xf''(x)}{2c}$$

$$a_2(f,c;x) = (1+c)\,x\frac{4(c+2)\,f^{(3)}(x) + 3(c+1)\,xf^{(4)}(x)}{24c^2}$$

$$a_3(f,c;x) = \frac{1}{48c^3}\left[2(1+c)\,x\left(c^2+6c+6\right)f^{(4)}(x)\right.$$
$$\left. +4(c+2)\,(x(1+c))^2 f^{(5)}(x) + ((1+c)\,x)^3 f^{(6)}(x)\right].$$

An immediate consequence of Theorem 3.1 is the following Voronovskaja-type formula.

Corollary 3.1 *Let $x \in (0,\infty)$. For each function $f \in K[2;x]$, the operators $S_{n,c}$ satisfy*

$$\lim_{n\to\infty} n\left((S_{n,c}f)(x) - f(x)\right) = \frac{1+c}{2c} xf''(x).$$

Remark 3.3 *The second central moment of the operators $S_{n,c}$ is given by*

$$\left(S_{n,c}\psi_x^2\right)(x) = \frac{(1+c)\,x}{cn},$$

where, for each real x, we put $\psi_x(t) = t - x$.

In the limiting case $c \to +\infty$ we obtain the following result for the Szász–Mirakjan operators S_n.

Corollary 3.2 [7, Corollary 3] *Let $q \in \mathbb{N}$ and $x \in (0, \infty)$. For each function $f \in K[2q; x]$, the Szász–Mirakjan operators possess the asymptotic expansion*

$$(S_n f)(x) = f(x) + \sum_{k=1}^{q} b_k(f; x) n^{-k} + o\left(n^{-q}\right) \qquad (n \to \infty)$$

with the coefficients

$$b_k(f; x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} x^{s-k} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} \sigma_r^{r-k} \qquad (k = 1, 2, \ldots).$$

In the case $q = 2$, i.e., for $f \in K[2; x]$, we have the well-known Voronovskaja-type formula

$$\lim_{n \to \infty} n\left(\left(S_n f\right)(x) - f(x)\right) = \frac{1}{2} x f''(x).$$

We give the series explicitly, for $q = 3$:

$$(S_n f)(x) = f(x) + \frac{x f''(x)}{2n} + \frac{4x f^{(3)}(x) + 3x^2 f^{(4)}(x)}{24n^2} +$$

$$+ \frac{1}{48n^3}\left(2x f^{(4)}(x) + 4x^2 f^{(5)}(x) + x^3 f^{(6)}(x)\right) + o\left(n^{-3}\right)$$

as $n \to \infty$.

In the special case $c = 1$ we obtain the following result on the operators L_n of Lupaş.

Corollary 3.3 *Let $q \in \mathbb{N}$ and $x \in (0, \infty)$. For each function $f \in K[2q; x]$, the operators L_n of Lupaş possess the asymptotic expansion*

$$(L_n f)(x) = f(x) + \sum_{k=1}^{q} c_k(f; x) n^{-k} + o\left(n^{-q}\right) \qquad (n \to \infty)$$

with the coefficients

$$c_k(f;x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} x^{s-k} \sum_{j=0}^{k} (-1)^{j-k} T(s,k,j) \qquad (k = 1, 2, \dots),$$

where the numbers $T(s,k,j)$ are defined by (6)*. In the case $q = 2$, i.e., for $f \in K[2;x]$, we have the Voronovskaja-type formula*

$$\lim_{n\to\infty} n\left((L_n f)(x) - f(x)\right) = x f''(x).$$

We give the series explicitly, for $q = 3$:

$$(S_n f)(x) = f(x) + \frac{x f''(x)}{2n} + \frac{2x f^{(3)}(x) + x^2 f^{(4)}(x)}{2n^2} +$$

$$+ \frac{1}{12n^3}\left(13x f^{(4)}(x) + 12x^2 f^{(5)}(x) + 2x^3 f^{(6)}(x)\right) + o\left(n^{-3}\right)$$

as $n \to \infty$.

4 Auxiliary results and proofs

In order to prove our main result we shall need some auxiliary results. Throughout the paper let e_r denote the monomials $e_r(t) = t^r$ $(r = 0, 1, \dots)$ and, for each real x, put (as above) $\psi_x(t) = t - x$. The proof of Theorem 3.1 is based on the following lemmas.

Lemma 4.1 *The moments of the Baskakov–Kantorovich operators possess the representation*

$$(S_{n,c} e_r)(x) = \sum_{k=0}^{r} n^{-k} x^{r-k} \sum_{j=0}^{k} S_{r-j}^{r-k} \sigma_r^{r-j} (-c)^{j-k} \qquad (r = 0, 1, \dots).$$

Proof. Taking advantage of the second identity of (7), we obtain

$$(S_{n,c} e_r)(x) = n^{-r} \left(\frac{c}{1+c}\right)^{ncx} \sum_{\nu=0}^{\infty} \binom{ncx+\nu-1}{\nu} (1+c)^{-\nu} \sum_{j=0}^{r} \sigma_r^j \nu^{\underline{j}} =$$

$$= n^{-r} \left(\frac{c}{1+c} \right)^{ncx} \sum_{j=0}^{r} \sigma_r^j \left(ncx + j - 1 \right)^{\underline{j}} \sum_{\nu=0}^{\infty} \binom{ncx + j - 1}{\nu} \left(1 + c \right)^{-\nu - j} =$$

$$= n^{-r} \sum_{j=0}^{r} \sigma_r^j \left(-ncx \right)^{\underline{j}} \left(-c \right)^{-j} .$$

Using the first identity of (7) we have

$$\left(S_{n,c} e_r \right) (x) =$$

$$= n^{-r} \sum_{j=0}^{r} \sigma_r^j \left(-c \right)^{-j} \sum_{k=0}^{j} S_j^k \left(-ncx \right)^k = \sum_{k=0}^{r} n^{k-r} x^k \sum_{j=k}^{r} S_j^k \sigma_r^j \left(-c \right)^{k-j} =$$

$$= \sum_{k=0}^{r} n^{-k} x^{r-k} \sum_{j=0}^{k} S_{r-j}^{r-k} \sigma_r^{r-j} \left(-c \right)^{j-k} .$$

This completes the proof of Lemma 4.1.

Lemma 4.2 *For $s = 0, 1, \ldots,$ the central moments of the FPJ operators possess the representation*

$$\left(S_{n,c} \psi_x^s \right) (x) = \sum_{k=\lfloor (s+1)/2 \rfloor}^{s} n^{-k} x^{s-k} \sum_{j=0}^{k} \left(-c \right)^{j-k} T \left(s, k, j \right) ,$$

where the numbers $T(s,k,j)$ are as defined in Eq. (6).

Proof. Application of the binomial formula yields for the central moments

$$\left(S_{n,c} \psi_x^s \right) (x) = \sum_{r=0}^{s} \binom{s}{r} \left(-x \right)^{s-r} \left(S_{n,c} e_r \right) (x) =$$

$$= \sum_{k=0}^{s} n^{-k} x^{s-k} \sum_{j=0}^{k} \left(-c \right)^{j-k} \sum_{r=k}^{s} \left(-1 \right)^{s-r} \binom{s}{r} S_{r-j}^{r-k} \sigma_r^{r-j} =$$

$$= \sum_{k=0}^{s} n^{-k} x^{s-k} \sum_{j=0}^{k} \left(-c \right)^{j-k} T \left(s, k, j \right) .$$

It remains to prove that $\left(S_{n,c}\psi_x^s\right)(x) = O\left(n^{-\lfloor (s+1)/2\rfloor}\right)$. Since this is clear in the case $s=0$, let now $s>0$. It is sufficient to show that, for $0\le j\le k$, there holds $Z(s,k,j)=0$ if $2k<s$. Obviously $Z(s,0,0)=0$ $(s=1,2,\ldots)$. Therefore, we have to consider only the case $k\ge 1$.

We first recall some known facts about Stirling numbers which will be useful in the sequel. The Stirling numbers of first resp. second kind possess the representation

$$(8)\qquad S_{r-j}^{r-k} = \sum_{\mu=0}^{k-j} C_{k-j,k-j-\mu}\binom{r-j}{k-j+\mu}, \qquad \sigma_r^{r-j} = \sum_{\nu=0}^{j} \overline{C}_{j,j-\nu}\binom{r}{j+\nu}$$

$$(0\le j\le k\le r).$$

(see [16, p.151, Eq. (5), resp. p. 171, Eq. (7)]). The coefficients $C_{k,i}$ resp. $\overline{C}_{k,i}$ are independent on r and satisfy certain partial difference equations ([16, p. 150]). Some closed expressions for $C_{k,i}$ and $\overline{C}_{k,i}$ can be found in [1, p. 113]. Taking advantage of representation (8) we obtain, for $0\le j\le k$,

$$S_{r-j}^{r-k}\sigma_r^{r-j} = \sum_{\mu=0}^{k-j}\sum_{\nu=0}^{j} \frac{C_{k-j,k-j-\mu}}{(k-j+\mu)!}\frac{\overline{C}_{j,j-\nu}}{(j+\nu)!} r^{\underline{j+\nu}}(r-j)^{\underline{k-j+\mu}} =$$

$$= \sum_{\mu=0}^{k-j}\sum_{\nu=0}^{j} r^{\underline{k}} P(k,j,\mu,\nu;r),$$

where

$$P(k,j,\mu,\nu;r) = \frac{C_{k-j,k-j-\mu}}{(k-j+\mu)!}\frac{\overline{C}_{j,j-\nu}}{(j+\nu)!}(r-j)^{\underline{\nu}}(r-k)^{\underline{\mu}}$$

is a polynomial in the variable r of degree $\le \mu+\nu$. Noting that the term with $r=k-1$ in Eq. (6) vanishes, we conclude that

$$Z(s,k,j) = \sum_{\mu=0}^{k-j}\sum_{\nu=0}^{j}\sum_{r=k}^{s}(-1)^{s-r}\binom{s}{r} r^{\underline{k}} P(k,j,\mu,\nu;r)\cdot$$

$$\cdot s^{\underline{k}} \sum_{\mu=0}^{k-j}\sum_{\nu=0}^{j}\sum_{r=0}^{s-k}(-1)^{s-k-r}\binom{s-k}{r} P(k,j,\mu,\nu;r+k).$$

Since $P(k,j,\mu,\nu;r+k)$ is a polynomial in the variable r of degree $\leq \mu+\nu \leq k$, the inner sum vanishes if $k < s-k$, that is, if $2k < s$. This completes the proof of Lemma 4.2.

In order to extend our main result from bounded functions to functions of polynomial growth, we need the following localization result.

Lemma 4.3 *Suppose that $g \in E$ is of polynomial growth, and let $x \in (0,\infty)$ be fixed. Moreover, assume that $g(t) = 0$ in a certain neighborhood $U_\delta(x)$ of the point x. Then*

$$(S_{n,c}g)(x) = O\left(n^{-m}\right) \quad \textit{for each} \quad m \in \mathbb{N}.$$

Proof. Assume that $|g(t)| \leq Mt^k$ for $t \in (0,\infty)$. Thus we obtain:

$$|(S_{n,c}g)(x)| \leq \sum_{\nu\geq 0 \left|\frac{\nu}{n}-x\right|\geq\delta} p_{n,\nu}^{[c]}(x)\left|g\left(\frac{\nu}{n}\right)\right| \leq$$

$$\leq M\delta^{-2m}\sum_{\nu=0}^{\infty} p_{n,\nu}^{[c]}(x)\left(\frac{\nu}{n}\right)^k\left(\frac{\nu}{n}-x\right)^{2m} =$$

$$= M\delta^{-2m}\sum_{\nu=0}^{\infty} p_{n,\nu}^{[c]}(x)\sum_{j=0}^{k}\binom{k}{j}(-x)^{k-j}\left(\frac{\nu}{n}-x\right)^{2m+j} =$$

$$= M\delta^{-2m}\sum_{j=0}^{k}\binom{k}{j}(-x)^{k-j}\left(S_{n,c}\psi_x^{2m+j}\right)(x) =$$

$$= O\left(n^{-m}\right) \qquad (n\to\infty)$$

since $\left(S_{n,c}\psi_x^m\right)(x) = O(n^{-\lfloor (m+1)/2\rfloor})$ as $n\to\infty$, by Lemma 4.2.

In order to derive our main result, the complete asymptotic expansion of the operators $S_{n,c}$, we use a general approximation theorem for positive linear operators due to Sikkema [18, Theorem 3] (cf. [19, Theorems 1 and 2]).

Lemma 4.4 *Let I be an interval. For $q \in \mathbb{N}$ and fixed $x \in I$, let $A_n : L_\infty(I) \to C(I)$ be a sequence of positive linear operators with the property*

$$A_n(\psi_x^s; x) = O(n^{-\lfloor (s+1)/2 \rfloor}) \ (n \to \infty) \ (s = 0, 1, \ldots, 2q+2). \tag{9}$$

Then, we have for each $f \in L_\infty(I)$ which is $2q$ times differentiable at x the asymptotic relation

$$A_n(f; x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} A_n(\psi_x^s; x) + o(n^{-q}) \qquad (n \to \infty). \tag{10}$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $o(n^{-q})$ in (10) can be replaced by $O(n^{-(q+1)})$.

Proof. By Remark 3.3, assumption (9) in Lemma 4.4 is valid for the operators $S_{n,c}$. Therefore, we can apply Lemma 4.4 and the assertion of Theorem 3.1 follows for bounded functions $f \in E$ after some calculations by Lemma 4.2. By the localization theorem (Lemma 4.3), the asymptotic expansion holds even for functions of polynomial growth.

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Fachhochschule Giessen-Friedberg,
University of Applied Sciences, Fachbereich MND,
Wilhelm-Leuschner-Strasse 13, 61169 Friedberg, Germany
E-mail address: *Ulrich.Abel@mnd.fh-friedberg.de*

Department of Mathematics,
Technical University of Cluj-Napoca,
Str. C. Daicoviciu 15, 400020 Cluj-Napoca, Romania
E-mail address: *Mircea.Ivan@math.utcluj.ro*