General Mathematics Vol. 15, No. 1 (2007), 81–92

Monosplines and inequalities for the remainder term of quadrature formulas [1]

Ana Maria Acu

Dedicated to Professor Alexandru Lupaş on the occasion of his 65th birthday

Abstract

In this paper we studied some quadrature formulas witch are obtained using connection between the monosplines and the quadrature formulas. For the remainder term we give some inequalities.

2000 Mathematics Subject Classification: 26D15, 65D30
Key words and phrases: quadrature rule, numerical integration, error bounds

1 Introduction

We denote

$$W_p^n[a,b] := \left\{ f \in C^{n-1}[a,b]\,,\ f^{(n-1)} \text{ absolutely continuous },\ \left\| f^{(n)} \right\|_p < \infty \right\}$$

with

$$\|f\|_p := \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}} \text{ for } 1 \le p < \infty$$
$$\|f\|_\infty := \text{supvrai}_{x \in [a,b]} |f(x)| \ .$$

Definition 1. *The function $s(x)$ is called a spline function of degree n with knots $\{x_i\}_{i=1}^{m-1}$ if $a := x_0 < x_1 < \cdots < x_{m-1} < x_m := b$ and*

i) for each $i = 0, \ldots, m-1$, $s(x)$ coincides on (x_i, x_{i+1}) with a polynomial of degree not greater then n;

ii) $s(x), s'(x), \ldots, s^{(n-1)}(x)$ are continuous functions on $[a,b]$.

[1] *Received 18 December, 2006*
Accepted for publication (in revised form) 13 January, 2007

Definition 2. *Functions of the form*

$$M_n(t)=\frac{t^n}{n!}+s_{n-1}(t)\,,$$

where $s_{n-1}(t)$ is a spline of degree $n-1$ are called monosplines.

Let

$$(1)\quad M_n(t)=\frac{(b-t)^n}{n!}-\sum_{k=0}^{n-1}A_{k,m}\frac{(b-t)^{n-k-1}}{(n-k-1)!}-\sum_{i=1}^{m-1}\sum_{k=0}^{n-1}A_{k,i}\frac{(x_i-t)_+^{n-k-1}}{(n-k-1)!}$$

be the monospline of degree n and let

$$(2)\qquad \int_a^b f(t)dt=\sum_{i=0}^{m}\sum_{k=0}^{n-1}A_{k,i}f^{(k)}(x_i)+\mathcal{R}[f]\,,$$

be the quadrature formula. Between the monospline (1) and the quadrature formula (2) there is a connection, namely, the coefficients $\{A_{k,i}\}_{k=0}^{n-1}\,{}_{i=1}^{m}$of the quadrature formula are the same with the coefficients of monospline (1), $A_{k,0}=(-1)^{n-k-1}M_n^{(n-k-1)}(a)\,,\ k=\overline{0,n-1}$ and the remainder term of quadrature formula have the representation :

$$\mathcal{R}[f]=\int_a^b M_n(t)f^{(n)}(t)dt\,,\ f\in W_1^n[a,b].$$

Definition 3. *Functions of the form*

$$M_n(t)=v(t)+s_{n-1}(t)\,,$$

where $s_{n-1}(t)$ is a spline of degree $n-1$ and v is the n^{th} integral of weight function $w:[a,b]\to\mathbb{R}$, are called generalized monosplines.

If we choose the weight functions $w(t)=(b-t)(t-a)$, then we obtain the generalized monosplines

$$\begin{aligned}(3)\,M_n(t) &= (a-b)\frac{(t-b)^{n+1}}{(n+1)!}-2\frac{(t-b)^{n+2}}{(n+2)!}+(-1)^{n-1}\sum_{k=0}^{n-1}A_{k,m}\frac{(b-t)^{n-k-1}}{(n-k-1)!}\\ &+ (-1)^{n-1}\sum_{i=1}^{m-1}\sum_{k=0}^{n-1}A_{k,i}\frac{(x_i-t)_+^{n-k-1}}{(n-k-1)!}\,.\end{aligned}$$

Between the monospline (3) and the quadrature formula

$$\int_a^b w(t)f(t)dt = \sum_{i=0}^{m}\sum_{k=0}^{n-1} A_{k,i} f^{(k)}(x_i) + \mathcal{R}[f]\,, \tag{4}$$

there is a connection, namely, the coefficients $\{A_{k,i}\}_{k=0}^{n-1}\,{}_{i=1}^{m}$ of the quadrature formula are the same with the coefficients of monospline (3), $A_{k,0} = (-1)^{k+1} M_n^{(n-k-1)}(a)$, $k = \overline{0, n-1}$ and the remainder term of quadrature formula has the representation :

$$\mathcal{R}[f] = (-1)^n \int_a^b M_n(t) f^{(n)}(t)dt\,,\ f \in W_1^n[a,b].$$

In the paper [1], [2], [3], [4] and [5] are considered generalization of the trapezoid, mid-point and Simpson′ s quadrature rule. For example, in [3] is studied a generalization of the mid-point quadrature rule:

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1} \left[1+(-1)^k\right] \frac{(b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) + (-1)^n \int_a^b K_n(t) f^{(n)}(t)dt\,,$$

where

$$K_n(t) = \begin{cases} \dfrac{(t-a)^n}{n!}\,, & t \in \left[a, \dfrac{a+b}{2}\right] \\ \dfrac{(t-b)^n}{n!}\,, & t \in \left(\dfrac{a+b}{2},\ b\right] \end{cases}$$

We observe that for $n = 1$ we get the mid-point rule

$$\int_a^b f(t)dt = (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b K_1(t) f'(t)dt\,.$$

We will study the case of the quadrature formula with the weight function $w(t) = (b-t)(t-a)$.

2 Main results

Lemma 1. *If* $f \in W_1^n[a,b]$, *then*

$$\int_a^b w(t)f(t)dt = \sum_{k=0}^{n-1} \left[(-1)^k + 1\right] \cdot \frac{(b-a)^{k+3}}{2^{k+2}(k+1)!(k+3)} f^{(k)}\left(\frac{a+b}{2}\right) + \mathcal{R}[f]\,, \tag{5}$$

where $w(t)=(b-t)(t-a)$

$$\mathcal{R}[f]=(-1)^n\int_a^b M_n(t)f^{(n)}(t)dt \tag{6}$$

and

$$M_n(t)=\begin{cases}(b-a)\dfrac{(t-a)^{n+1}}{(n+1)!}-2\dfrac{(t-a)^{n+2}}{(n+2)!}\,,\ t\in\left[a,\ \dfrac{a+b}{2}\right)\\ (a-b)\dfrac{(t-b)^{n+1}}{(n+1)!}-2\dfrac{(t-b)^{n+2}}{(n+2)!}\,,\ t\in\left[\dfrac{a+b}{2},b\right]\end{cases}. \tag{7}$$

Proof. We denoting

$$P_n(t)=(b-a)\frac{(t-a)^{n+1}}{(n+1)!}-2\frac{(t-a)^{n+2}}{(n+2)!}\ \text{ and}$$
$$Q_n(t)=(a-b)\frac{(t-b)^{n+1}}{(n+1)!}-2\frac{(t-b)^{n+2}}{(n+2)!}$$

we observe that successive integration by parts yields the relation

$$(-1)^n\int_a^b M_n(t)f^{(n)}(t)dt=(-1)^n\int_a^{\frac{a+b}{2}}P_n(t)f^{(n)}(t)dt+(-1)^n\int_{\frac{a+b}{2}}^b Q_n(t)f^{(n)}(t)dt$$
$$=(-1)^n\sum_{k=0}^{n-1}(-1)^{n-1-k}\,P_n^{(n-1-k)}(t)f^{(k)}(t)\Big|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}}P_n^{(n)}(t)f(t)dt$$
$$+(-1)^n\sum_{k=0}^{n-1}(-1)^{n-1-k}\,Q_n^{(n-1-k)}(t)f^{(k)}(t)\Big|_{\frac{a+b}{2}}^b+\int_{\frac{a+b}{2}}^b Q_n^{(n)}(t)f(t)dt$$
$$=\sum_{k=0}^{n-1}(-1)^{k+1}\left[(b-a)\frac{(t-a)^{k+2}}{(k+2)!}-2\frac{(t-a)^{k+3}}{(k+3)!}\right]f^{(k)}(t)\Bigg|_a^{\frac{a+b}{2}}$$
$$+\sum_{k=0}^{n-1}(-1)^{k+1}\left[(a-b)\frac{(t-b)^{k+2}}{(k+2)!}-2\frac{(t-b)^{k+3}}{(k+3)!}\right]f^{(k)}(t)\Bigg|_{\frac{a+b}{2}}^b+\int_a^b(b-t)(t-a)f(t)dt$$
$$=-\sum_{k=0}^{n-1}\left[(-1)^k+1\right]\cdot\frac{(b-a)^{k+3}}{2^{k+2}(k+1)!(k+3)}f^{(k)}\left(\frac{a+b}{2}\right)+\int_a^b w(t)f(t)dt\,,$$

namely

$$\int_a^b w(t)f(t)dt = \sum_{k=0}^{n-1}\left[(-1)^k+1\right]\cdot\frac{(b-a)^{k+3}}{2^{k+2}(k+1)!(k+3)}f^{(k)}\left(\frac{a+b}{2}\right) + (-1)^n\int_a^b M_n(t)f^{(n)}(t)dt\,.$$

Remark 1. *To observe that the quadrature formula (5) it is the open type.*

Remark 2. *If for the generalized monospline (3) we consider the particular case* $m = 2,\ x_0 = a,\ x_1 = \dfrac{a+b}{2},\ x_2 = b,\ A_{k,2} = 0,\ A_{k,1} = \left[(-1)^k + 1\right] \cdot \dfrac{(b-a)^{k+3}}{2^{k+2}(k+1)!(k+3)},\ k = \overline{0, n-1},$ *we will have*

$$M_n(t) = (a-b)\frac{(t-b)^{n+1}}{(n+1)!} - 2\frac{(t-b)^{n+2}}{(n+2)!} +$$

$$\frac{2(-1)^{n-1}}{(n+2)!}\sum_{k=0}^{n-1}\left[(-1)^k + 1\right](k+2)\binom{n+2}{k+3}\left(\frac{b-a}{2}\right)^{k+3}\left(\frac{a+b}{2} - t\right)_+^{n-k-1}.$$

If $t \in [a, \dfrac{a+b}{2})$, *then*

$$M_n(t) = (a-b)\frac{(t-b)^{n+1}}{(n+1)!} - 2\frac{(t-b)^{n+2}}{(n+2)!} + \frac{2(-1)^{n-1}}{(n+2)!}$$

$$\cdot\left[(n+2)\frac{b-a}{2}(b-t)^{n+1} + (n+2)\frac{b-a}{2}(a-t)^{n+1} - (b-t)^{n+2} + (a-t)^{n+2}\right]$$

$$= (b-a)\frac{(t-a)^{n+1}}{(n+1)!} - 2\frac{(t-a)^{n+2}}{(n+2)!}.$$

If $t \in \left[\dfrac{a+b}{2}, b\right]$, *then*

$$M_n(t) = (a-b)\frac{(t-b)^{n+1}}{(n+1)!} - 2\frac{(t-b)^{n+2}}{(n+2)!}.$$

Theorem 1. *The generalized monospline of degree* n, $M_n(t), n > 1$, *defined in (7), verifies*

(8) $$\int_a^b M_n(t)dt = 0, \quad \text{if } n \text{ is odd},$$

(9) $$\int_a^b |M_n(t)|\,dt = \frac{(b-a)^{n+3}}{2^{n+1}(n+1)!(n+3)},$$

(10) $$\max_{t\in[a,b]} |M_n(t)| = \frac{(b-a)^{n+2}}{2^{n+1}n!(n+2)}.$$

Proof. We have

$$\int_a^b M_n(t)dt = \int_a^{\frac{a+b}{2}} P_n(t)dt + \int_{\frac{a+b}{2}}^b Q_n(t)dt =$$

$$= \left[(b-a)\frac{(t-a)^{n+2}}{(n+2)!} - 2\frac{(t-a)^{n+3}}{(n+3)!}\right]\Bigg|_a^{\frac{a+b}{2}} + \left[(a-b)\frac{(t-b)^{n+2}}{(n+2)!} - 2\frac{(t-b)^{n+3}}{(n+3)!}\right]\Bigg|_{\frac{a+b}{2}}^b$$

$$= [1+(-1)^n]\,\frac{(b-a)^{n+3}}{2^{n+2}(n+1)!(n+3)}\,.$$

If n is odd, then $\int_a^b M_n(t)dt = 0\,.$

$$\int_a^b |M_n(t)|\,dt = \int_a^{\frac{a+b}{2}} |P_n(t)|\,dt + \int_{\frac{a+b}{2}}^b |Q_n(t)|\,dt =$$

$$\int_a^{\frac{a+b}{2}} \left[(b-a)\frac{(t-a)^{n+1}}{(n+1)!} - 2\frac{(t-a)^{n+2}}{(n+2)!}\right] dt + \int_{\frac{a+b}{2}}^b \left[(b-a)\frac{(b-t)^{n+1}}{(n+1)!} - 2\frac{(b-t)^{n+2}}{(n+2)!}\right] dt$$

$$= \frac{(b-a)^{n+3}}{2^{n+1}(n+1)!(n+3)}\,.$$

$$\max_{t\in[a,b]} |M_n(t)| = \max\left\{\max_{t\in\left[a,\frac{a+b}{2}\right]} |P_n(t)|\,,\ \max_{t\in\left[\frac{a+b}{2},b\right]} |Q_n(t)|\right\}$$

$$= \max\left\{P_n\left(\frac{a+b}{2}\right), \left|Q_n\left(\frac{a+b}{2}\right)\right|\right\} = \frac{(b-a)^{n+2}}{2^{n+1}n!(n+2)}\,.$$

Theorem 2. *If $f \in W_1^n[a,b]$, $n > 1$ and there exist numbers γ_n, Γ_n such that $\gamma_n \le f^{(n)}(t) \le \Gamma_n$, $t \in [a,b]$, then*

$$|\mathcal{R}[f]| \le \frac{\Gamma_n - \gamma_n}{2^{n+2}} \cdot \frac{(b-a)^{n+3}}{(n+1)!(n+3)}\,,\ \text{if } n \text{ is odd} \tag{11}$$

and

$$|\mathcal{R}[f]| \le \frac{(b-a)^{n+3}}{2^{n+1}(n+1)!(n+3)}\left\|f^{(n)}\right\|_\infty\,,\ \text{if } n \text{ is even.} \tag{12}$$

Proof. Let n be odd. Using relations (6) and (8) we can written

$$\mathcal{R}[f] = (-1)^n \int_a^b M_n(t) f^{(n)}(t) dt = (-1)^n \int_a^b M_n(t) \left[f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right] dt$$

such that we have

$$|\mathcal{R}[f]| \le \max_{t \in [a,b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \int_a^b |M_n(t)| \, dt \,. \tag{13}$$

We also have

$$\max_{t \in [a,b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \le \frac{\Gamma_n - \gamma_n}{2} \,. \tag{14}$$

From (9), (13) and (14) we have

$$|\mathcal{R}[f]| \le \frac{\Gamma_n - \gamma_n}{2^{n+2}} \cdot \frac{(b-a)^{n+3}}{(n+1)!(n+3)} \,.$$

Let n be even. Then we have

$$|\mathcal{R}[f]| \le \left\| f^{(n)} \right\|_\infty \cdot \int_a^b |M_n(t)| \, dt = \frac{(b-a)^{n+3}}{2^{n+1}(n+1)!(n+3)} \left\| f^{(n)} \right\|_\infty \,.$$

Theorem 3. *Let $f \in W_1^n[a,b], n > 1$ and let n be odd. If there exist a real number γ_n such that $\gamma_n \le f^{(n)}(t)$, then*

$$|\mathcal{R}[f]| \le (T_n - \gamma_n) \cdot \frac{(b-a)^{n+3}}{2^{n+1} n! (n+2)} \tag{15}$$

where

$$T_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \,.$$

If there exist a real number Γ_n such that $f^{(n)}(t) \le \Gamma_n$, then

$$|\mathcal{R}[f]| \le (\Gamma_n - T_n) \cdot \frac{(b-a)^{n+3}}{2^{n+1} n! (n+2)} \,. \tag{16}$$

Proof. Using relation (8) we can written

$$|\mathcal{R}[f]| = \left| \int_a^b \left(f^{(n)}(t) - \gamma_n \right) M_n(t) dt \right| .$$

From (10) we have

$$|\mathcal{R}[f]| \leq \max_{t\in[a,b]} |M_n(t)| \cdot \int_a^b \left(f^{(n)}(t) - \gamma_n \right) dt$$

$$= \frac{(b-a)^{n+2}}{2^{n+1} n!(n+2)} \left[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a) \right] = \frac{(b-a)^{n+3}}{2^{n+1} n!(n+2)} (T_n - \gamma_n) .$$

In a similar way we can prove that (16) holds.

Now, we will study the case of the quadrature formula of close type, with the weight function $w(t) = (b-t)(t-a)$.

Lemma 2. *If $f \in W_1^n[a,b]$, then*

$$\begin{aligned} \int_a^b w(t) f(t) dt &= \sum_{k=0}^{n-1} \left(\frac{b-a}{4} \right)^{k+3} \frac{1}{k!} \frac{3k+10}{(k+2)(k+3)} f^{(k)}(a) \\ &+ \sum_{k=0}^{n-1} \left[(-1)^k + 1 \right] \left(\frac{b-a}{4} \right)^{k+3} \frac{1}{(k+1)!} \frac{3k+11}{k+3} f^{(k)} \left(\frac{a+b}{2} \right) \\ &+ \sum_{k=0}^{n-1} (-1)^k \left(\frac{b-a}{4} \right)^{k+3} \frac{1}{k!} \frac{3k+10}{(k+2)(k+3)} f^{(k)}(b) + \mathcal{R}[f] , \end{aligned} \tag{17}$$

where $w(t) = (b-t)(t-a)$

$$\mathcal{R}[f] = (-1)^n \int_a^b M_n(t) f^{(n)}(t) dt \tag{18}$$

and

(19)

$$M_n(t) = \begin{cases} \frac{3}{n!} \left(\frac{b-a}{4} \right)^2 \left(t - \frac{3a+b}{4} \right)^n + \frac{b-a}{2(n+1)!} \left(t - \frac{3a+b}{4} \right)^{n+1} - \frac{2}{(n+2)!} \left(t - \frac{3a+b}{4} \right)^{n+2}, \\ \qquad\qquad t \in \left[a, \frac{a+b}{2}\right) \\ \frac{3}{n!} \left(\frac{b-a}{4} \right)^2 \left(t - \frac{a+3b}{4} \right)^n + \frac{a-b}{2(n+1)!} \left(t - \frac{a+3b}{4} \right)^{n+1} - \frac{2}{(n+2)!} \left(t - \frac{a+3b}{4} \right)^{n+2}, \\ \qquad\qquad t \in \left[\frac{a+b}{2}, b \right] \end{cases}$$

Proof. We denoting

$$P_n(t) = \frac{3}{n!}\left(\frac{b-a}{4}\right)^2\left(t-\frac{3a+b}{4}\right)^n+\frac{b-a}{2(n+1)!}\left(t-\frac{3a+b}{4}\right)^{n+1}-\frac{2}{(n+2)!}\left(t-\frac{3a+b}{4}\right)^{n+2},$$
$$Q_n(t) = \frac{3}{n!}\left(\frac{b-a}{4}\right)^2\left(t-\frac{a+3b}{4}\right)^n+\frac{a-b}{2(n+1)!}\left(t-\frac{a+3b}{4}\right)^{n+1}-\frac{2}{(n+2)!}\left(t-\frac{a+3b}{4}\right)^{n+2},$$

we observe that successive integration by parts yields the relation

$$(-1)^n\int_a^b M_n(t)f^{(n)}(t)dt=(-1)^n\int_a^{\frac{a+b}{2}} P_n(t)f^{(n)}(t)dt+(-1)^n\int_{\frac{a+b}{2}}^b Q_n(t)f^{(n)}(t)dt$$
$$= (-1)^n\sum_{k=0}^{n-1}(-1)^{n-1-k}\, P_n^{(n-1-k)}(t)f^{(k)}(t)\Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} P_n^{(n)}(t)f(t)dt$$
$$+(-1)^n\sum_{k=0}^{n-1}(-1)^{n-1-k}\, Q_n^{(n-1-k)}(t)f^{(k)}(t)\Big|_{\frac{a+b}{2}}^b + \int_{\frac{a+b}{2}}^b Q_n^{(n)}(t)f(t)dt$$
$$= \sum_{k=0}^{n-1}(-1)^{k+1}\left[\frac{3}{(k+1)!}\left(\frac{b-a}{4}\right)^2\left(t-\frac{3a+b}{4}\right)^{k+1}\right.$$
$$\left.+\frac{b-a}{2(k+2)!}\left(t-\frac{3a+b}{4}\right)^{k+2}-\frac{2}{(k+3)!}\left(t-\frac{3a+b}{4}\right)^{k+3}\right]f^{(k)}(t)\Bigg|_a^{\frac{a+b}{2}}$$
$$+\sum_{k=0}^{n-1}(-1)^{k+1}\left[\frac{3}{(k+1)!}\left(\frac{b-a}{4}\right)^2\left(t-\frac{a+3b}{4}\right)^{k+1}+\frac{a-b}{2(k+2)!}\left(t-\frac{a+3b}{4}\right)^{k+2}\right.$$
$$\left.-\frac{2}{(k+3)!}\left(t-\frac{a+3b}{4}\right)^{k+3}\right]f^{(k)}(t)\Bigg|_{\frac{a+b}{2}}^b+\int_a^b(b-t)(t-a)f(t)dt =$$
$$-\sum_{k=0}^{n-1}\left(\frac{b-a}{4}\right)^{k+3}\frac{1}{k!}\frac{3k+10}{(k+2)(k+3)}f^{(k)}(a)$$
$$-\sum_{k=0}^{n-1}\left[(-1)^k+1\right]\left(\frac{b-a}{4}\right)^{k+3}\frac{1}{(k+1)!}\frac{3k+11}{k+3}f^{(k)}\left(\frac{a+b}{2}\right)$$
$$-\sum_{k=0}^{n-1}(-1)^k\left(\frac{b-a}{4}\right)^{k+3}\frac{1}{k!}\frac{3k+10}{(k+2)(k+3)}f^{(k)}(b)+\int_a^b w(t)f(t)dt\,.$$

Theorem 4. *The generalized monospline of degree n , $M_n(t), n>1$, defined* *in (19), verifies*

$$\int_a^b M_n(t)dt = 0\,, \text{ if } n \text{ is odd}, \tag{20}$$

$$\int_a^b |M_n(t)|\,dt = \left(\frac{b-a}{4}\right)^{n+3} \frac{4}{(n+3)!}(3n^2+15n+16)\,, \tag{21}$$

$$\max_{t\in[a,b]} |M_n(t)| = \left(\frac{b-a}{4}\right)^{n+2} \frac{1}{(n+2)!}(3n^2+11n+8)\,. \tag{22}$$

Proof. We have

$$\int_a^b M_n(t)dt = \int_a^{\frac{a+b}{2}} P_n(t)dt + \int_{\frac{a+b}{2}}^b Q_n(t)dt =$$

$$= [1+(-1)^n]\left(\frac{b-a}{4}\right)^{n+3} \frac{2}{(n+3)!}(3n^2+15n+16)\,.$$

If n is odd, then $\int_a^b M_n(t)dt = 0\,.$

$$\int_a^b |M_n(t)|\,dt = \int_a^{\frac{a+b}{2}} |P_n(t)|\,dt + \int_{\frac{a+b}{2}}^b |Q_n(t)|\,dt$$

$$= \left(\frac{b-a}{4}\right)^{n+3} \frac{4}{(n+3)!}(3n^2+15n+16)\,.$$

$$\max_{t\in[a,b]} |M_n(t)| = \max\left\{ \max_{t\in\left[a,\frac{a+b}{2}\right]} |P_n(t)|\,,\ \max_{t\in\left[\frac{a+b}{2},b\right]} |Q_n(t)| \right\}$$

$$= \max\left\{ P_n\left(\frac{a+b}{2}\right), \left|Q_n\left(\frac{a+b}{2}\right)\right| \right\} = \left(\frac{b-a}{4}\right)^{n+2} \frac{1}{(n+2)!}(3n^2+11n+8)\,.$$

Theorem 5. *If $f \in W_1^n[a,b]$, $n > 1$ and there exist numbers γ_n, Γ_n such that $\gamma_n \le f^{(n)}(t) \le \Gamma_n$, $t \in [a,b]$, then*

$$|\mathcal{R}[f]| \le \frac{\Gamma_n - \gamma_n}{2}\left(\frac{b-a}{4}\right)^{n+3} \frac{4}{(n+3)!}(3n^2+15n+16)\,, \text{ if } n \text{ is odd} \tag{23}$$

and

$$|\mathcal{R}[f]| \le \left(\frac{b-a}{4}\right)^{n+3} \frac{4}{(n+3)!}(3n^2+5n+16)\left\|f^{(n)}\right\|_\infty\,, \text{ if } n \text{ is even.} \tag{24}$$

Proof. If n is odd, then we can written

$$|\mathcal{R}[f]| \leq \frac{\Gamma_n - \gamma_n}{2} \int_a^b |M_n(t)|\, dt = \frac{\Gamma_n - \gamma_n}{2} \left(\frac{b-a}{4}\right)^{n+3} \frac{4}{(n+3)!}(3n^2+15n+16)\,.$$

Let n be even. Then we have

$$|\mathcal{R}[f]| \leq \left\|f^{(n)}\right\|_\infty \cdot \int_a^b |M_n(t)|\, dt = \left(\frac{b-a}{4}\right)^{n+3} \frac{4}{(n+3)!}(3n^2+15n+16) \left\|f^{(n)}\right\|_\infty \,.$$

Theorem 6. *Let $f \in W_1^n[a,b], n > 1$ and let n be odd. If there exist a real number γ_n such that $\gamma_n \leq f^{(n)}(t)$, then*

$$|\mathcal{R}[f]| \leq (T_n - \gamma_n) \cdot \left(\frac{b-a}{4}\right)^{n+3} \cdot \frac{4}{(n+2)!} \cdot (3n^2 + 11n + 8) \tag{25}$$

where

$$T_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\,.$$

If there exist a real number Γ_n such that $f^{(n)}(t) \leq \Gamma_n$, then

$$|\mathcal{R}[f]| \leq (\Gamma_n - T_n) \cdot \left(\frac{b-a}{4}\right)^{n+3} \cdot \frac{4}{(n+2)!} \cdot (3n^2 + 11n + 8)\,. \tag{26}$$

Proof. We have

$$|\mathcal{R}[f]| \leq \max_{t \in [a,b]} |M_n(t)| \cdot \int_a^b \left(f^{(n)}(t) - \gamma_n\right) dt$$

$$= \left(\frac{b-a}{4}\right)^{n+2} \frac{1}{(n+2)!}(3n^2 + 11n + 8) \left[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a)\right]$$

$$= (T_n - \gamma_n) \cdot \left(\frac{b-a}{4}\right)^{n+3} \cdot \frac{4}{(n+2)!} \cdot (3n^2 + 11n + 8)\,.$$

In a similar way we can prove that (26) holds.

References

[1] G.A.Anastassiou, *Ostrowski Type Inequalities*, Proc. Amer. Math.Soc.,Vol 123,No 12,(1995),3775-3781.

[2] P. Cerone and S.S.Dragomir, *Midpoint-type Rules from an Inequalities Point of View*, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, (2000),135-200.

[3] P. Cerone and S.S.Dragomir, *Trapezoidal-type Rules from an Inequalities Point of View*, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, (2000),65-134.

[4] Lj. Dedić, M. Matić and J.Pečarić, *On Euler trapezoid formulae*, Appl. Math. Comput.,123(2001),37-62.

[5] C.E.M. Pearce, J.Pečarić, N. Ujević and S. Varošanec, *Generalizations of some inequalities of Ostrowski-Grüss type*, Math. Inequal. Appl., 3(1), (2000), 25-34.

[6] Nenad Ujević, *Error Inequalities for a Generalized Quadrature Rule* , General Mathematics, Vol. 13, No. 4(2005), 51-64.

University "Lucian Blaga" of Sibiu
Department of Mathematics
Str. Dr. I. Rațiu, No. 5-7
550012 - Sibiu, Romania
e-mail: acuana77@yahoo.com