General Mathematics Vol. 15, No. 1 (2007), 59–66

Bounds for the Čebyšev Functional of a Convex and a Bounded Function [1]

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Dedicated to Professor Ph.D. Alexandru Lupaş on the occasion of his 65th birthday

Abstract

Upper and lower bounds for the Čebyšev functional of a convex and a bounded function are given. Some applications for quadrature rules and probability density functionsare also provided.

2000 Mathematics Subject Classification: Primary 26D15 Secondary 26D10

Key words and phrases: Čebyšev Functional, Convex Functions, Integral Inequalities

1 Introduction

For two Lebesgue functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1}$$

[1] *Received 18 December, 2006*

Accepted for publication (in revised form) 9 January, 2007

In 1971, F.V. Atkinson [1] showed that if f, g are twice differentiable and convex on $[a, b]$ and

$$\int_a^b \left(t - \frac{a+b}{2}\right) g(t)\, dt = 0, \tag{2}$$

then $C(f, g)$ is nonnegative.

This result is, in fact, implied by that of A. Lupaş [3] who proved that for any two convex functions $f, g : [a, b] \to \mathbb{R}$ the lower bound for the Čebyšev functional is:

$$C(f, g) \geq \frac{12}{(b-a)^3} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt \cdot \int_a^b \left(t - \frac{a+b}{2}\right) g(t)\, dt, \tag{3}$$

with true equality holding when at least one of f or g is a linear function on $[a, b]$.

As pointed out in [4, p. 262], if the functions f, g are convex and one is symmetric, then $C(f, g) \geq 0$.

For other results for convex integrands, see [4, p. 256] and [4, p. 262] where further references are given.

In this note we provide some bounds for the Čebyšev functional in the case of a convex function g and a bounded function f. Some applications are given as well.

2 The Results

For an integrable function $f : [a, b] \to \mathbb{R}$, define the $(\gamma - 2)$ −moment by

$$M_{2,\gamma}(f) := \int_a^b (t - \gamma)^2 f(t)\, dt.$$

For a convex function $g : [a, b] \to \mathbb{R}$ for which the derivatives $g'_-(b)$ and $g'_+(a)$ are finite, define

$$\Gamma(f, g) := \frac{g'_-(b)\, M_{2,b}(f) - g'_+(a)\, M_{2,a}(f)}{2(b-a)^2},$$

where f is integrable on $[a,b]$.

The following result holds:

Theorem 2.1. *If* $f:[a,b]\to\mathbb{R}$ *is a Lebesgue measurable function such that there exist constants* $m,M\in\mathbb{R}$ *with*

$$m\le f(t)\le M \quad \text{for a.e.} \quad t\in[a,b], \tag{4}$$

and $g:[a,b]\to\mathbb{R}$ *is a convex function on* $[a,b]$ *with the lateral derivatives* $g'_{+}(a)$ *and* $g'_{-}(b)$ *finite, then,*

$$\frac{1}{6}m(b-a)\left[g'_{-}(b)-g'_{+}(a)\right]-\Gamma(f,g)\le \tag{5}$$

$$\le C(f,g)\le\frac{1}{6}M(b-a)\left[g'_{-}(b)-g'_{+}(a)\right]-\Gamma(f,g).$$

Proof. We use Sonin's identity [4, p. 246]:

$$C(f,g)=\frac{1}{b-a}\int_a^b(f(t)-\gamma)\left(g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right)dt, \tag{6}$$

for any $\gamma\in\mathbb{R}$, and the following inequality for convex functions obtained by S.S. Dragomir in [2]:

$$\frac{1}{b-a}\int_a^b g(s)\,ds-g(t)\le\frac{1}{2(b-a)}\left[(b-t)^2g'_{-}(b)-(t-a)^2g'_{+}(a)\right] \tag{7}$$

for any $t\in[a,b]$. The constant $\frac{1}{2}$ is sharp.

By Sonin's identity for $\gamma=M$, we have,

$$C(f,g)=\frac{1}{b-a}\int_a^b(M-f(t))\left(\frac{1}{b-a}\int_a^b g(s)\,ds-g(t)\right)dt. \tag{8}$$

From (7) we get,

$$\left(\frac{1}{b-a}\int_a^b g(s)\,ds-g(t)\right)(M-f(t))\le \tag{9}$$

$$\leq \frac{1}{2(b-a)}\left[g'_{-}(b)(b-t)^2(M-f(t)) - g'_{+}(a)(t-a)^2(M-f(t))\right]$$

for a.e. $t \in [a,b]$.

Integrating (9) over t on $[a,b]$ and using the representation (8), we get

$$C(f,g) \leq \frac{1}{2(b-a)^2}\left[M\int_a^b \left[g'_{-}(b)(b-t)^2 - g'_{+}(a)(t-a)^2\right]dt - \right. \tag{10}$$

$$\left. -g'_{-}(b)\int_a^b (b-t)^2 f(t)\,dt + g'_{+}(a)\int_a^b (t-a)^2 f(t)\,dt\right].$$

Since $\int_a^b \left[g'_{-}(b)(b-t)^2 - g'_{+}(a)(t-a)^2\right]dt = \frac{(b-a)^3}{3}\left[g'_{-}(b) - g'_{+}(a)\right]$ then (10) provides the second part of (5).

Again, by Sonin's identity,

$$C(f,g) = \frac{1}{b-a}\int_a^b (m-f(t))\left(\frac{1}{b-a}\int_a^b g(s)\,ds - g(t)\right)dt.$$

Utilising (7) and the fact that $m - f(t) \leq 0$ for a.e. $t \in [a,b]$, we obtain,

$$C(f,g) \geq$$

$$\geq \frac{1}{2(b-a)^2}\int_a^b \left[(b-t)^2 g'_{-}(b)(m-f(t)) - (t-a)^2 g'_{+}(a)(m-f(t))\right]dt =$$

$$= \frac{1}{2(b-a)^2}\left[m\int_a^b \left[(b-t)^2 g'_{-}(b) - (t-a)^2 g'_{+}(a)\right]dt - 2(b-a)\Gamma(f,g)\right],$$

giving the first part of (5).

The following particular result holds.

Corollary 2.1. *Let $f : [a,b] \to \mathbb{R}$ be an essentially bounded function on $[a,b]$, i.e., $f \in L_\infty[a,b]$ and $\|f\|_\infty := ess\sup_{t\in[a,b]}|f(t)|$, its norm. If $g : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$ with the lateral derivatives $g'_{+}(a)$ and $g'_{-}(b)$ finite, then we have the inequality:*

$$|C(f,g) + \Gamma(f,g)| \leq \frac{1}{6}\|f\|_\infty (b-a)\left[g'_{-}(b) - g'_{+}(a)\right]. \tag{11}$$

3 Applications for the Trapezoid Rule

The following result is a perturbed version of the trapezoid rule.

Proposition 3.1. *Let $h : [a,b] \to \mathbb{R}$ be a differentiable function with the property that the derivative $h' : (a,b) \to \mathbb{R}$ is convex on (a,b). If $h''_+(a)$, $h''_-(b)$ are finite, then,*

$$(12) \quad \left| \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt - \frac{(b-a)^2}{12}\cdot\frac{h''_+(a)+h''_-(b)}{2} \right| \leq$$

$$\leq \frac{1}{12}(b-a)^2 \cdot \left[h''_-(b) - h''_+(a)\right].$$

Proof. Consider the functions $f, g : [a,b] \to \mathbb{R}$ defined by $f(t) = t - \frac{a+b}{2}$, $g(t) = h'(t)$. For these functions, a simple calculation shows that, $\Gamma(f,g) = -\frac{(b-a)^2}{12}\cdot\frac{h''_+(a)+h''_-(b)}{2}$, since,

$$\int_a^b (t-b)^2\left(t - \frac{a+b}{2}\right)dt = -\frac{(b-a)^4}{12}$$

and $\int_a^b (t-a)^2\left(t-\frac{a+b}{2}\right)dt = \frac{(b-a)^4}{12}$.

Clearly, also, $\|f\|_\infty = \frac{1}{2}(b-a)$. Utilising the elementary identity,

$$\frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)h'(t)\,dt = \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt$$

and the fact that, for f, g as defined previously,

$$C(f,g) = \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)h'(t)\,dt,$$

that a direct application of Corollary 2.1 reveals the desired inequality (12).

Remark 3.1. *Similar results may be stated if one considers quadrature rules for which the remainder* $R(f)$ *can be expressed in Peano kernel form, i.e.,* $R(f) = \int_a^b K(t) f^{(n)}(t)\,dt$, *where* $K(t)$ *is a kernel for which the supremum norm can be easily computed and the* $n-th$ *derivative of the function,* f, *is assumed to be convex on* (a,b). *The exploration of these bounds is left to the interested reader.*

4 Applications for Probability Density Functions

Let $f : [a,b] \to [0,\infty)$ be a *density function*, this means that f is integrable on $[a,b]$ and $\int_a^b f(t)\,dt = 1$ and let $F(x) := \int_a^x f(t)\,dt$, $x \in [a,b]$ be its *distribution function.* We also denote the *expectation* of f by $E(f)$, where $E(f) := \int_a^b t f(t)\,dt$, provided the integral exists and is finite, and the *mean deviation* $M_D(f)$, by, $M_D(f) := \int_a^b |t - E(f)| f(t)\,dt$.

Theorem 4.1. *Let* $f : [a,b] \to [0,\infty)$ *be a density function with the property that there exists* $m, M \geq 0$ *such that* $m \leq f(t) \leq M$ *for a.e.* $t \in [a,b]$ *then,*

$$\frac{1}{3} m (b-a)^2 \leq M_D(f) + \frac{1}{b-a} M_{2,\frac{a+b}{2}}(f) - \frac{\left(E(f) - \frac{a+b}{2}\right)^2}{b-a} \leq \tag{13}$$

$$\leq \frac{1}{3} M (b-a)^2 .$$

Proof. We apply Theorem 2.1 for $g : [a,b] \to \mathbb{R}$, $g(t) = |t - E(f)|$. Since $g'_-(b) = 1$, $g'_+(a) = -1$, then,

$$\Gamma(f,g) = \frac{1}{(b-a)^2} \int_a^b \left[\frac{(t-a)^2 + (t-b)^2}{2} \right] f(t)\,dt =$$

$$= \frac{1}{(b-a)^2} \int_a^b \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] f(t)\, dt =$$

$$= \frac{1}{(b-a)^2} M_{2,\frac{a+b}{2}}(f) + \frac{1}{4}.$$

On the other hand,

$$C(f,g) =$$

$$= \frac{1}{b-a} \int_a^b |t - E(f)|\, f(t)\, dt - \frac{1}{b-a} \int_a^b |t - E(f)|\, dt \cdot \frac{1}{b-a} \int_a^b f(t)\, dt =$$

$$= \frac{1}{b-a} M_D(f) - \frac{1}{(b-a)^2} \left[\frac{(b - E(f))^2 + (E(f) - a)^2}{2} \right] =$$

$$= \frac{1}{b-a} M_D(f) - \frac{1}{(b-a)^2} \left[\left(E(f) - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] =$$

$$= \frac{1}{b-a} M_D(f) - \frac{\left(E(f) - \frac{a+b}{2} \right)^2}{(b-a)^2} - \frac{1}{4}.$$

Making use of the inequality (5) we deduce the desired result (13).

If one is interested in providing bounds for the *absolute moment* around the midpoint $\frac{a+b}{2}$,$M_{\frac{a+b}{2}}(f) := \int_a^b \left| t - \frac{a+b}{2} \right| f(t)\, dt$, then on applying Theorem 2.1 for $g(t) = \left| t - \frac{a+b}{2} \right|$, we have the following.

Theorem 4.2. *Let $f : [a,b] \to [0, \infty)$ be as in Theorem 4.1, then*

$$\frac{1}{3} m (b-a)^2 \leq M_{\frac{a+b}{2}}(f) + \frac{1}{b-a} M_{2,\frac{a+b}{2}}(f) \leq \frac{1}{3} M (b-a)^2. \tag{14}$$

Remark 4.1. *Similar results may be stated if one considers higher moments*

$$M_{p,\gamma}(f) := \int_a^b |t - \gamma|^p f(t)\, dt, \qquad p \geq 1,$$

for which $g(t) = |t - \gamma|^p$ in Theorem 2.1 will procure the corresponding bounds in terms of m and M with the property that $0 < m \leq f(t) \leq M$ for a.e. $t \in [a,b]$. The details are omitted.

References

[1] F.V. Atkinson, *An inequality*, Univ. Beograd Publ. Elektr. Fak. Ser. Mat. Fiz., No. 357-380 (1971), 5-6.

[2] S.S. Dragomir, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure & Appl. Math., **3**(2) (2002), Art. 31

[3] A. Lupaş, *An integral inequality for convex functions*, Univ. Beograd Publ. Elektr. Fak. Ser. Mat. Fiz., No. 381-409 (1972), 17-19.

[4] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1992.

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