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On a Property of Convex Functions [1]

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Dedicated to Professor Ph.D. Alexandru Lupaş on the occasion of his 65th birthday

Abstract

In this paper, a property of convex functions and its generalization are given.

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1 Introduction

The following result was obtained in 1965 by Tiberiu Popoviciu [4]:

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Theorem 1.1. *Let f be a real continuous function, defined on a finite interval I. Then, f is convex on I if and only if*

$$f(x)+f(y)+f(z)+3\cdot f\left(\frac{x+y+z}{3}\right)\geq \tag{1}$$
$$\geq 2\cdot\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right],$$

for arbitrary x, y, z from I.

2 Main Results

We have established an analogous property, given by:

Theorem 2.1. *A real continuous function f, defined on the finite interval I, is convex on I if and only if the inequality*

$$4\cdot[f(x)+f(y)+f(z)]+6\cdot f\left(\frac{x+y+z}{3}\right)\geq 3\cdot\left[f\left(\frac{2x+y}{3}\right)+\right. \tag{2}$$
$$\left.+f\left(\frac{x+2y}{3}\right)+f\left(\frac{2y+z}{3}\right)+f\left(\frac{y+2z}{3}\right)+f\left(\frac{2z+x}{3}\right)+f\left(\frac{z+2x}{3}\right)\right]$$

holds for all $x, y, z \in I$.

Proof. The condition (2) is sufficient

Interchanging y with x, z with y, and then x with y, z with x in relation (2), we obtain:

$$f(x)+2\cdot f(y)\geq 3\cdot f\left(\frac{x+2y}{3}\right),\quad \text{for all } x,y\in I, \tag{3}$$

and

$$2\cdot f(x)+f(y)\geq 3\cdot f\left(\frac{2x+y}{3}\right),\quad \text{for all } x,y\in I. \tag{4}$$

From (3) and (4), we deduce:

$$f(x)+f(y)\geq f\left(\frac{x+2y}{3}\right)+f\left(\frac{2x+y}{3}\right),\quad \text{for all } x,y\in I,$$

or

(5)
$$f(x)+f(y)\geq f\left(\frac{x+y}{2}+\frac{y-x}{6}\right)+f\left(\frac{x+y}{2}-\frac{y-x}{6}\right),\quad \text{for all } x,y\in I.$$

Since (5) is valid for all x and y belonging to the interval I, it remains valid in case we interchange x with $\frac{x+y}{2}-\frac{y-x}{6}$ and y with $\frac{x+y}{2}+\frac{y-x}{6}$, both elements being situated between x and y. Then, we have:

$$f\left(\frac{x+y}{2}+\frac{y-x}{6}\right)+f\left(\frac{x+y}{2}-\frac{y-x}{6}\right)\geq$$
$$\geq f\left(\frac{x+y}{2}+\frac{y-x}{6\cdot 3}\right)+f\left(\frac{x+y}{2}-\frac{y-x}{6\cdot 3}\right),$$

for all $x,y\in I$. Continuing this procedure, we find:

(6) $$f(x)+f(y)\geq f\left(\frac{x+y}{2}+\frac{y-x}{6\cdot 3^{n-1}}\right)+f\left(\frac{x+y}{2}-\frac{y-x}{6\cdot 3^{n-1}}\right),$$

for all $x,y\in I$, $n\in\mathbb{N}^*$, an inequality that can be proved using complete induction.

Since f is continuous on I, considering $n\to\infty$ in the previous relation, we deduce:

(7) $$f(x)+f(y)\geq 2\cdot f\left(\frac{x+y}{2}\right),\quad \text{for all } x,y\in I.$$

The continuity of f and the relation (7) prove the convexity of f on I. The condition (2) is necessary

I. First method (inspired by an ideea from [2])

It is known that a function f is convex if and only if the second order divided difference of f at all points $x, y, z \in I$, $[x, y, z; f] \geq 0$. In order to prove the inequality (2), we should show that:

$$(8)\qquad F(x,y,z) = 4 \cdot [f(x) + f(y) + f(z)] + 6 \cdot f\left(\frac{x+y+z}{3}\right) - \\ - 3 \cdot \left[f\left(\frac{2x+y}{3}\right) + f\left(\frac{x+2y}{3}\right) + f\left(\frac{2y+z}{3}\right) + \right. \\ \left. + f\left(\frac{y+2z}{3}\right) + f\left(\frac{2z+x}{3}\right) + f\left(\frac{z+2x}{3}\right)\right] \geq 0,$$

for all $x, y, z \in I$, writing this expression as a linear combination of second order divided differences with nonnegative coefficients.

Without loss of generality, we may assume $x < y < z$. The following cases can be distinguished:

i) $y - x \geq z - y$;

ii) $y - x < z - y$.

We will only consider case i), since the proof in case ii) is similar.

i) Denoting $\alpha = z - y$, then exists $t \in \mathbb{R}_+$ such that $y = x + \alpha + t$ and $z = x + 2\alpha + t$. There are two subcases:

i_1) $t \leq \alpha$;

i_2) $t > \alpha$.

i_1) Let $t \leq \alpha$. Denoting $z_1 = x$, $z_2 = \dfrac{2x+y}{3}$, $z_3 = \dfrac{2x+z}{3}$, $z_4 = \dfrac{2y+x}{3}$, $z_5 = \dfrac{x+y+z}{3}$, $z_6 = y$, $z_7 = \dfrac{2z+x}{3}$, $z_8 = \dfrac{2y+z}{3}$, $z_9 = \dfrac{2z+y}{3}$, $z_{10} = z$, we deduce:

$$z_1 \leq z_2 \leq z_3 \leq z_4 \leq z_5 \leq z_6 \leq z_7 \leq z_8 \leq z_9 \leq z_{10}.$$

We require the following representation for $F(x,y,z)$:

$$\begin{aligned}(9)\quad F(x,y,z) = {} & A\cdot[z_1,z_2,z_3;f]+B\cdot[z_2,z_3,z_4;f]+C\cdot[z_3,z_4,z_5;f]+\\ & +D\cdot[z_4,z_5,z_6;f]+E\cdot[z_5,z_6,z_7;f]+F\cdot[z_6,z_7,z_8;f]+\\ & +G\cdot[z_7,z_8,z_9;f]+H\cdot[z_8,z_9,z_{10};f],\quad \text{for all } x,y,z\in I.\end{aligned}$$

From (8) and (9), we obtain:

$$(10)\qquad \begin{cases} \dfrac{9A}{(\alpha+t)(2\alpha+t)}=4\\ -\dfrac{9A}{\alpha(\alpha+t)}+\dfrac{9B}{\alpha(\alpha+t)}=-3\\ \dfrac{9A}{\alpha(2\alpha+t)}-\dfrac{9B}{\alpha t}+\dfrac{9C}{t(\alpha+t)}=-3\\ \dfrac{9B}{t(\alpha+t)}-\dfrac{9C}{\alpha t}+\dfrac{9D}{\alpha(\alpha+t)}=-3\\ \dfrac{9C}{\alpha(\alpha+t)}-\dfrac{9D}{\alpha t}+\dfrac{9E}{\alpha t}=6\\ \dfrac{9D}{t(\alpha+t)}-\dfrac{9E}{t(\alpha-t)}+\dfrac{9F}{\alpha(\alpha-t)}=4\\ \dfrac{9E}{\alpha(\alpha-t)}-\dfrac{9F}{t(\alpha-t)}+\dfrac{9G}{t(\alpha+t)}=-3\\ \dfrac{9F}{\alpha t}-\dfrac{9G}{\alpha t}+\dfrac{9H}{2\alpha^2}=-3\\ \dfrac{9G}{\alpha(\alpha+t)}-\dfrac{9H}{\alpha^2}=-3\\ \dfrac{9H}{2\alpha^2}=4 \end{cases}$$

with the unique, nonnegative solution:

$$(11)\qquad \begin{cases} A = \dfrac{4(\alpha+t)(2\alpha+t)}{9} \\ B = \dfrac{(\alpha+t)(5\alpha+4t)}{9} \\ C = \dfrac{(\alpha+t)(5\alpha+2t)}{9} \\ D = \dfrac{2t(\alpha+t)}{9} \\ E = \dfrac{3\alpha t}{9} \\ F = \dfrac{\alpha(5\alpha-2t)}{9} \\ G = \dfrac{5\alpha(\alpha+t)}{9} \\ H = \dfrac{8\alpha^2}{9}. \end{cases}$$

Note that the positivity of the solution (11) and that of the second order divided differences imply the positivity of (8). The conclusion then follows immediately.

i_2) Let $t > \alpha$. Denoting $z_1 = x$, $z_2 = \dfrac{2x+y}{3}$, $z_3 = \dfrac{2x+z}{3}$, $z_4 = \dfrac{2y+x}{3}$, $z_5 = \dfrac{x+y+z}{3}$, $z_6 = \dfrac{x+2z}{3}$, $z_7 = y$, $z_8 = \dfrac{2y+z}{3}$, $z_9 = \dfrac{y+2z}{3}$, $z_{10} = z$, we have:

$$z_1 \le z_2 \le z_3 \le z_4 \le z_5 \le z_6 \le z_7 \le z_8 \le z_9 \le z_{10}.$$

The proof of i_2) is similar to that of i_1), with small obvious changes.

II. Second Method

The proof is based on the following theorem (a particular case of Theorem 1 from [3]):

Theorem 2.2. *Let $A : C[a,b] \to \mathbb{R}$ be a bounded linear functional. If:*

i) $A(e_0) = A(e_1) = 0$, $A(e_2) > 0$, *where* $e_0(t) = 1$, $e_i(t) = t^i$, $i = 1, 2, \ldots$, $t \in [a,b]$;

ii) $A(\varphi_\lambda) \geq 0$, *for all* $\lambda \in [a,b]$, *where* $\varphi_\lambda(t) = |t - \lambda|$, $t \in [a,b]$,

then $A(f) \geq 0$ *for all convex functions* f *defined on* $[a,b]$.

We first state the following inequality:

$$(12)\quad 4 \cdot (|u| + |v| + |w|) + 2 \cdot |u + v + w| \geq |2u + v| + |u + 2v| + \\ + |2v + w| + |v + 2w| + |2w + u| + |w + 2u|, \quad \text{for all}\, u, v, w \in \mathbb{R}.$$

In order to do that, we use the well known Hlawkaş inequality:

$$(13)\quad |u|+|v|+|w|+|u+v+w| \geq |u+v|+|v+w|+|w+u|, \quad \text{for all}\, u, v, w \in \mathbb{R},$$

and the obvious relations:

(14)

$$\begin{aligned} |u| + |v| + 2 \cdot |u + v| &\geq |2u + v| + |u + 2v|, && \text{for all } u, v \in \mathbb{R} \\ |v| + |w| + 2 \cdot |v + w| &\geq |2v + w| + |v + 2w|, && \text{for all } v, w \in \mathbb{R} \\ |w| + |u| + 2 \cdot |w + u| &\geq |2w + u| + |w + 2u|, && \text{for all } w, u \in \mathbb{R}. \end{aligned}$$

Using (13) and (14), we deduce (12).

Denote:

$$A(f) = 4 \cdot [f(x) + f(y) + f(z)] + 6 \cdot f\left(\frac{x + y + z}{3}\right) - \\ -3 \cdot \left[f\left(\frac{2x + y}{3}\right) + f\left(\frac{x + 2y}{3}\right) + f\left(\frac{2y + z}{3}\right) + \right. \\ \left. +f\left(\frac{y + 2z}{3}\right) + f\left(\frac{2z + x}{3}\right) + f\left(\frac{z + 2x}{3}\right) \right], \quad \text{for all } x, y, z \in I.$$

It is easy to verify that:

$$\begin{aligned}
\text{i)}\quad A(e_0) &= 4\cdot 3+6-3\cdot 6=0, \quad \text{for all } x,y,z\in I;\\
A(e_1) &= 4\cdot(x+y+z)+6\cdot\frac{x+y+z}{3}-3\cdot\left(\frac{2x+y}{3}+\frac{x+2y}{3}+\right.\\
&\left.+\frac{2y+z}{3}+\frac{y+2z}{3}+\frac{2z+x}{3}+\frac{z+2x}{3}\right)=0, \quad \text{for all } x,y,z\in I;\\
A(e_2) &= \frac{4}{3}\cdot(x^2+y^2+z^2-xy-yz-zx)\geq 0, \quad \text{for all } x,y,z\in I.\\
\text{ii)}\quad A(\varphi_\lambda) &= 4\cdot(|x-\lambda|+|y-\lambda|+|z-\lambda|)+6\cdot\left|\frac{x+y+z}{3}-\lambda\right|-\\
&-3\cdot\left(\left|\frac{2x+y}{3}-\lambda\right|+\left|\frac{x+2y}{3}-\lambda\right|+\left|\frac{2y+z}{3}-\lambda\right|+\right.\\
&\left.+\left|\frac{y+2z}{3}-\lambda\right|+\left|\frac{2z+x}{3}-\lambda\right|+\left|\frac{z+2x}{3}-\lambda\right|\right).
\end{aligned}$$

The inequality $A(\varphi_\lambda)\geq 0$, for all $x,y,z\in I$ is equivalent to the following:

(15)

$$\begin{aligned}
&4\cdot(|x-\lambda|+|y-\lambda|+|z-\lambda|)+2\cdot|x+y+z-3\lambda|\geq\\
&\geq|2x+y-3\lambda|+|x+2y-3\lambda|+|2y+z-3\lambda|+\\
&+|y+2z-3\lambda|+|2z+x-3\lambda|+|z+2x-3\lambda|, \quad \text{for all } x,y,z\in I.
\end{aligned}$$

By denoting $u=x-\lambda$, $v=y-\lambda$, and $w=z-\lambda$ in the above inequality, we obtain (12).

Finally, according to the considered theorem, we find that $A(f)\geq 0$ for all convex functions defined on I, namely what we had to prove.

III. Third Method

Without loss of generality, we may assume that $x<y<z$ (the equality cases generate obvious relations). Again, we distinguish two situations:

a) $y - x \geq z - y = \alpha$, $\alpha \in (0, +\infty)$;

b) $\alpha = y - x < z - y$, $\alpha \in (0, +\infty)$.

a) If $y - x \geq z - y = \alpha$, $\alpha \in (0, +\infty)$, then $(\exists)\, t \in [0, +\infty)$ such that $y = x + \alpha + t$ and $z = x + 2\alpha + t$. Then, we have:

$$x \leq \frac{x+y+z}{3} \leq y \leq z$$

and

$$\frac{2x+y}{3} \leq \frac{2x+z}{3} \leq \frac{2y+x}{3} + \frac{2z+x}{3} \leq \frac{2y+z}{3} + \frac{y+2z}{3}.$$

Denoting:

$$a_1 = a_2 = a_3 = a_4 = z, \quad a_5 = a_6 = a_7 = a_8 = y,$$
$$a_9 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = \frac{x+y+z}{3}, \quad a_{15} = a_{16} = a_{17} = a_{18} = x,$$

and

$$b_1 = b_2 = b_3 = \frac{y+2z}{3}, \quad b_4 = b_5 = b_6 = \frac{2y+z}{3}, \quad b_7 = b_8 = b_9 = \frac{2z+x}{3},$$
$$b_{10} = b_{11} = b_{12} = \frac{2y+x}{3}, \quad b_{13} = b_{14} = b_{15} = \frac{2x+z}{3}, \quad b_{16} = b_{17} = b_{18} = \frac{2x+y}{3},$$

we can easily verify that:

$$\text{(16)} \qquad \begin{aligned} &\text{i)} \quad a_1 \geq a_2 \geq \ldots \geq a_{18} \ \text{ and } \ b_1 \geq b_2 \geq \ldots \geq b_{18}; \\ &\text{ii)} \quad a_1 + a_2 + \ldots + a_k \geq b_1 + b_2 + \ldots + b_k, \quad k \in \{1, 2, \ldots, 17\}; \\ &\text{iii)} \quad a_1 + a_2 + \ldots + a_{18} = b_1 + b_2 + \ldots + b_{18}. \end{aligned}$$

In accordance with the well known majorization theorem due to Hardy, Littlewood and Polyá [1], we obtain:

$$\text{(17)} \qquad f(a_1) + f(a_2) + \ldots + f(a_{18}) \geq f(b_1) + f(b_2) + \ldots + f(b_{18}),$$

namely the inequality (2).

b) If $\alpha = y - x < z - y$, $\alpha \in (0, +\infty)$, then $(\exists)\, t \in (0, +\infty)$ such that $y = x + \alpha$ and $z = x + 2\alpha + t$. Note that:

$$x \le y \le \frac{x+y+z}{3} \le z$$

and

$$\frac{2x+y}{3} \le \frac{2y+x}{3} \le \frac{2x+z}{3} \le \frac{2y+z}{3} \le \frac{2z+x}{3} \le \frac{2z+y}{3}.$$

Denoting:

$$a_1 = a_2 = a_3 = a_4 = z, \;\; a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = \frac{x+y+z}{3},$$

$$a_{11} = a_{12} = a_{13} = a_{14} = y, \;\; a_{15} = a_{16} = a_{17} = a_{18} = x,$$

and

$$b_1 = b_2 = b_3 = \frac{2z+y}{3}, \;\; b_4 = b_5 = b_6 = \frac{2z+x}{3}, \;\; b_7 = b_8 = b_9 = \frac{2y+z}{3},$$
$$b_{10} = b_{11} = b_{12} = \frac{2x+z}{3}, \;\; b_{13} = b_{14} = b_{15} = \frac{2y+x}{3}, \;\; b_{16} = b_{17} = b_{18} = \frac{2x+y}{3},$$

we can immediately verify that relations (16) hold. The conclusion is obvious.

IV. Fourth Method (inspired by an ideea from [5])

Without loss of generality, we may assume $x \le y \le z$. Note that:

$$x \le \frac{2x+y}{3} \le \frac{2x+z}{3} \le \frac{x+y+z}{3}$$

and

$$\frac{x+y+z}{3} \le \frac{2z+x}{3} \le \frac{2z+y}{3} \le z.$$

Since $\dfrac{2x+y}{3}, \ \dfrac{2x+z}{3} \in \left[x, \dfrac{x+y+z}{3}\right]$, it follows that exists $p, q \in [0,1]$ such that:

$$\frac{2x+y}{3} = p \cdot x + (1-p) \cdot \frac{x+y+z}{3} \tag{18}$$

and

$$\frac{2x+z}{3} = q \cdot x + (1-q) \cdot \frac{x+y+z}{3}. \tag{19}$$

From (18) and (19), we deduce:

$$\frac{4x+y+z}{3} = (p+q) \cdot x + \frac{2(x+y+z)}{3} - (p+q) \cdot \frac{x+y+z}{3}$$

or

$$\frac{2x-y-z}{3} = (p+q) \cdot \frac{2x-y-z}{3}.$$

The two equalities above imply:

$$p+q = 1. \tag{20}$$

Since f is convex, we find:

$$f\left(\frac{2x+y}{3}\right) \le p \cdot f(x) + (1-p) \cdot f\left(\frac{x+y+z}{3}\right) \tag{21}$$

and

$$f\left(\frac{2x+z}{3}\right) \le q \cdot f(x) + (1-q) \cdot f\left(\frac{x+y+z}{3}\right). \tag{22}$$

Relations (21), (22), and (20) give us:

$$f\left(\frac{2x+y}{3}\right) + f\left(\frac{2x+z}{3}\right) \le f(x) + f\left(\frac{x+y+z}{3}\right). \tag{23}$$

In a similar way, from $\dfrac{2z+x}{3}, \dfrac{2z+y}{3} \in \left[\dfrac{x+y+z}{3}, z\right]$, it follows that exists $r, s \in [0,1]$ such that:

$$\frac{2z+x}{3} = r \cdot z + (1-r) \cdot \frac{x+y+z}{3} \tag{24}$$

and

$$\frac{2z+y}{3} = s \cdot z + (1-s) \cdot \frac{x+y+z}{3}. \tag{25}$$

By adding the last two inequalities, we obtain:

$$\frac{2z-x-y}{3} = (r+s) \cdot \frac{2z-x-y}{3}$$

or

$$r + s = 1. \tag{26}$$

Using the convexity of f and considering relations (24) and (25), we may write:

$$f\left(\frac{2z+x}{3}\right) \leq r \cdot f\left(\frac{x+y+z}{3}\right) + (1-r) \cdot f(z) \tag{27}$$

and

$$f\left(\frac{2z+y}{3}\right) \leq s \cdot f\left(\frac{x+y+z}{3}\right) + (1-s) \cdot f(z). \tag{28}$$

The last three relations give us the inequality:

$$f\left(\frac{2z+x}{3}\right) + f\left(\frac{2z+y}{3}\right) \leq f\left(\frac{x+y+z}{3}\right) + f\left(z\right). \tag{29}$$

By the fact that f is convex, we deduce:

$$f\left(\frac{2y+x}{3}\right) \leq \frac{2}{3} \cdot f(y) + \frac{1}{3} \cdot f(x) \tag{30}$$

and

$$f\left(\frac{2y+z}{3}\right) \leq \frac{2}{3} \cdot f(y) + \frac{1}{3} \cdot f(z). \tag{31}$$

Finally, inequalities (23), (29), (30) and (31) imply (2).

The following result is a generalization of Theorem 2.1.

Theorem 2.3. *If a real continuous function f, defined on the finite interval I, is convex, then the inequality:*

$$f(x) + f(y) + f(z) + \frac{3n}{m} \cdot f\left(\frac{x+y+z}{3}\right) \geq$$

$$\geq \frac{m+n}{2m} \cdot \left[f\left(\frac{mx+ny}{m+n}\right) + f\left(\frac{nx+my}{m+n}\right) + f\left(\frac{my+nz}{m+n}\right) + \right.$$

$$\left. + f\left(\frac{ny+mz}{m+n}\right) + f\left(\frac{mz+nx}{m+n}\right) + f\left(\frac{nz+mx}{m+n}\right)\right],$$

holds for all $x, y, z \in I$ and all $m, n \in (0, \infty)$, $m \geq n$.

Proof. In order to prove this generalization, we will consider the theorem used for proving the previous result by the second method.

First, we state the inequality:

$$2m \cdot (|u| + |v| + |w|) + 2n \cdot |u+v+w| \geq |mu+nv| + |nu+mv| + \\ + |mv+nw| + |nv+mw| + |mw+nu| + |nw+mu|, \tag{32}$$

for all $u, v, w \in \mathbb{R}$, for all $m, n \in (0, +\infty)$, $m \geq n$.

The well known Hlawka's inequality (13), together with the obvious inequalities:

(33)

$$\begin{aligned}(m-n)\cdot|u|+(m-n)\cdot|v|+2n\cdot|u+v| &\geq |mu+nv|+|nu+mv|,\\ (m-n)\cdot|v|+(m-n)\cdot|w|+2n\cdot|v+w| &\geq |mv+nw|+|nv+mw|,\\ (m-n)\cdot|w|+(m-n)\cdot|u|+2n\cdot|w+u| &\geq |mw+nu|+|nw+mu|,\end{aligned}$$

for all $u, v, w \in \mathbb{R}$, for all $m, n \in (0, +\infty)$, $m \geq n$, prove (32).

Let's denote

$$A(f) = f(x) + f(y) + f(z) + \frac{3n}{m} \cdot f\left(\frac{x+y+z}{3}\right) - \\ - \frac{m+n}{2m} \cdot \left[f\left(\frac{mx+ny}{m+n}\right) + f\left(\frac{nx+my}{m+n}\right) + f\left(\frac{my+nz}{m+n}\right) + \right. \\ \left. + f\left(\frac{ny+mz}{m+n}\right) + f\left(\frac{mz+nx}{m+n}\right) + f\left(\frac{nz+mx}{m+n}\right) \right],$$

for all $x, y, z \in I$ for all $m, n \in (0, \infty)$, $m \geq n$.

We can easily verify that:

i) $A(e_0) = 3 + \frac{3n}{m} - 6 \cdot \frac{m+n}{2m} = 0, \ (\forall)\, x, y, z \in I, \text{ for all } m, n \in (0, \infty), \ m \geq n;$

$$A(e_1) = x + y + z + \frac{3n}{m} \cdot \frac{x+y+z}{3} - \frac{m+n}{2m} \cdot \left(\frac{mx+ny}{m+n} + \frac{nx+my}{m+n} + \right. \\ \left. + \frac{my+nz}{m+n} + \frac{ny+mz}{m+n} + \frac{mz+nx}{m+n} + \frac{mx+nz}{m+n} \right) = 0, \\ \text{for all } x, y, z \in I, \text{ for all } m, n \in (0, \infty), \ m \geq n;$$

$$A(e_2) = x^2 + y^2 + z^2 + \frac{n(x+y+z)^2}{3m} - \frac{1}{2m(m+n)} \cdot \left[(mx+ny)^2 + (nx+my)^2 + \right. \\ \left. + (my+nz)^2 + (ny+mz)^2 + (mz+nx)^2 + (nz+mx)^2 \right] = \\ = \frac{n(2m-n)}{3m(m+n)} \cdot \left[(x-y)^2 + (y-z)^2 + (z-x)^2 \right] \geq 0,$$

$$\text{for all } x, y, z \in I, \text{ for all } m, n \in (0, \infty), \ m \geq n;$$

$$\text{ii)} A(\varphi_\lambda) = |x-\lambda| + |y-\lambda| + |z-\lambda| + \frac{3n}{m} \cdot \left| \frac{x+y+z}{3} - \lambda \right| -$$

$$-\frac{m+n}{2m} \cdot \left(\left| \frac{mx+ny}{m+n} - \lambda \right| + \left| \frac{nx+my}{m+n} - \lambda \right| + \left| \frac{my+nz}{m+n} - \lambda \right| + \right.$$

$$\left. + \left| \frac{ny+mz}{m+n} - \lambda \right| + \left| \frac{mz+nx}{m+n} - \lambda \right| + \left| \frac{nz+mx}{m+n} - \lambda \right| \right).$$

The inequality $A(\varphi_\lambda) \geq 0$, for all $x, y, z \in I$ for all $m, n \in (0, \infty)$, $m \geq n$ is equivalent to the following:

$$2m \cdot (\, |x-\lambda| + |y-\lambda| + |z-\lambda| \,) + 2n|x+y+z-3\lambda| \geq$$

$$\geq |m(x-\lambda) + n(y-\lambda)| + |n(x-\lambda) + m(y-\lambda)| + |m(y-\lambda) + n(z-\lambda)| +$$

$$+ |n(y-\lambda) + m(z-\lambda)| + |m(z-\lambda) + n(x-\lambda)| + |n(z-\lambda) + m(x-\lambda)|,$$

$$\text{for all } x, y, z \in I, \text{ for all } m, n \in (0, \infty), \ m \geq n.$$

Denoting $u = x - \lambda$, $v = y - \lambda$, $w = z - \lambda$ in the last inequality, we obtain (33). It follows that $A(f) \geq 0$, for all $x, y, z \in I$, for all $m, n \in (0, \infty)$, $m \geq n$. Thus, the proof is finished.

Remark 2.1. *a) Considering* $m = n = 1$ *in Theorem 2.3, we find the inequality* (1).

b) Considering $m = 2, n = 1$ *in Theorem 2.3, we find the inequality* (2).

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