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Development in series of orthogonal polynomials with applications in optimization[1]

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Dedicated to Professor Ph.D. Alexandru Lupaş on his 65th anniversary

Abstract

Our aim in this paper is to find some developments in Chebyshev series and using these to prove that $\min_{Q_n \in \widetilde{\Pi}_n} \|Q_n\|_p = \|\widetilde{T}_n\|_p$, where $\widetilde{T}_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x)$ is the n-th Chebyshev monic polynomial.

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1 Introduction

Let $\widetilde{\Pi}_n$ be the set of all real monic polinomials and $L^p_w[-1,1]$, $1 < p < \infty$, the Lebesque linear space with the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$, endowed with the norm $\|f\|_p = \left[\int\limits_{-1}^{1} |f(t)|^p w(t) dt\right]^{1/p}$.

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Theorem 1.1 *The extremal problem*

$$(1) \qquad \|Q_n\|_p \to \min,\ Q_n \in \widetilde{\Pi}_n,$$

has an unique solution.

Proof. It is known that the Lebesque normed linear space $L^p_w[-1,1]$ is strict convex for $1 < p < \infty$.

Lemma 1.1 *The extremal problem (1) is equivalent with the following problem:*

$$(2) \qquad \int\limits_0^{\pi} \left| \frac{1}{2^{n-1}} \cdot \cos nt + \sum_{i=0}^{n-1} a_i \cdot \cos it \right|^p dt \to \min\ ,$$

where $a = (a_0, \ldots, a_{n-1}) \in \mathbb{R}^n$.

Proof. $\{T_0, \ldots, T_n\}$ represents a basis in $\widetilde{\Pi}_n$, hence there is $a = (a_0, \ldots, a_{n-1}) \in \mathbb{R}^n$ so that:

$$\widetilde{Q}_n(x) = \frac{1}{2^{n-1}} \cdot T_n(x) + \sum_{i=0}^{n-1} a_i \cdot T_i(x)\ .$$

Therefore

$$\|\widetilde{Q}_n\|_p = \left[\int\limits_0^{\pi} \left| \frac{1}{2^{n-1}} \cdot \cos nt + \sum_{i=0}^{n-1} a_i \cdot \cos it \right|^p dt \right]^{1/p} .$$

Further we consider the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ be the function

$$(3) \qquad \varphi(a_0, \ldots, a_{n-1}) = \int\limits_0^{\pi} \left| \frac{1}{2^{n-1}} \cdot \cos nt + \sum_{i=0}^{n-1} a_i \cdot \cos it \right|^p dt.$$

Hence

$$(4) \qquad \begin{aligned} &\frac{\partial \varphi}{\partial a_k}(a_0, \ldots, a_{n-1}) = p \cdot \int\limits_0^{\pi} \left| \frac{1}{2^{n-1}} \cdot \cos nt + \sum_{i=0}^{n-1} a_i \cdot \cos it \right|^p \cdot \\ &\cdot sgn \left[\frac{1}{2^{n-1}} \cdot \cos nt + \sum_{i=0}^{n-1} a_i \cdot \cos it \right] \cdot \cos kt\, dt\ ,\ k = 0, 1, \ldots, n-1. \end{aligned}$$

If we shall prove that $\frac{\partial \varphi}{\partial a_k}(a^*) = 0$, $k = 0, 1, \ldots, n-1$, where $a^* = (0, \ldots, 0) \in \mathbb{R}^n$, using Theorem 1.1 and Lemma 1.1 we deduce that $\widetilde{T}_n(x)$ is the unique solution of the extremal problem (1).

Easily it obtains

$$\frac{\partial \varphi}{\partial a_k}(a^*) = \frac{p}{4^{n-1}} \cdot \int_0^\pi |\cos nt|^{p-1} \cdot sgn[\cos nt] \cdot \cos kt \, dt \tag{5}$$

and we denote:

$$J_{n,k} = \int_0^\pi |\cos nt|^{p-1} \cdot sgn[\cos nt] \cdot \cos kt \, dt \tag{6}$$

where $k = 0, 1, ..., n-1$.

2 Main results

Further for $f, g \in C[0,1]$ and $\alpha > -1$ let consider the following inner product

$$\langle f, g \rangle_\alpha = \int_0^1 f(t) g(t) \frac{t^\alpha (1-t)^\alpha}{B(\alpha+1, \alpha+1)} dt.$$

Lemma 2.1 *(see [3]) If $z \in [0,1]$, $-\infty < -4\lambda < \min(0, 2\alpha+1)$, then*

$$z^\lambda = \frac{B(\lambda+\alpha+1, \alpha+1)}{B(\alpha+1, \alpha+1)} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k (-\lambda)_k}{(2\alpha+\lambda+2)_k} \cdot \gamma_k^{(\alpha)} \varphi_k^{(\alpha)}(z), \tag{7}$$

where

$B(a,b)$ is the "beta" function, $a > -1, b > -1$,

$$(c)_k = c(c+1)\ldots(c+k-1), k = 1, 2, \ldots, (c)_0 = 1, c \in \mathbb{R},$$

$$\gamma_k^{(\alpha)} = \begin{cases} 1, k = 0 \\ \frac{2k+2\alpha+1}{k+2\alpha+1} \cdot \frac{(2\alpha+2)_k}{k!}, \ k = 1, 2, \ldots \end{cases}, \tag{8}$$

(9) $\varphi_k^{(\alpha)}(z) = R_k^{(\alpha,\alpha)}(2z-1)$ *is the ultraspherical polynomials,*

with the condition that the series converges uniformly on $[0,1]$.

In addition

$$\langle \varphi_k^{(\alpha)}, \varphi_j^{(\alpha)} \rangle_\alpha = \begin{cases} 0, k \neq j \\ \frac{1}{\gamma_k^{(\alpha)}}, k = j \end{cases} . \tag{10}$$

Lemma 2.2 *If* $z \in [0,1]$, $\lambda > 0$ *then*

$$z^{\frac{\lambda}{2}} = \frac{\Gamma(\lambda+1)}{2^\lambda \Gamma\left(\frac{\lambda}{2}+1\right)} \cdot \sum_{k=0}^{\infty} \frac{\binom{\lambda/2}{k} \cdot k!}{\Gamma\left(\frac{\lambda}{2}+k+1\right)} \cdot \gamma_k \cdot T_k^*(z), \tag{11}$$

where $T_k^*(z)$ *is the* k*-th Chebyshev polynomial on* $[0,1]$ *and*

$$\gamma_k = \begin{cases} 1, k = 0 \\ 2, k = 1, 2, \ldots \end{cases} ,$$

with the condition that the series converges uniformly on $[0,1]$.

In addition

$$\langle T_k^*, T_j^* \rangle_{-\frac{1}{2}} = \begin{cases} 0, k \neq j \\ \frac{1}{\gamma_k}, k = j \end{cases} . \tag{12}$$

Proof. In (7) we consider $\lambda := \frac{\lambda}{2}$, $\alpha := -\frac{1}{2}$ and using (8)–(10) we deduce (11) and (12).

Lemma 2.3 *If* $\lambda > 0$ *then*

$$\int_0^1 z^{\frac{\lambda}{2}} \cdot T_j^*(z) \frac{dz}{\sqrt{z(1-z)}} = \frac{\pi\Gamma(\alpha+1) \cdot \binom{\lambda/2}{j} j!}{2^\lambda \Gamma\left(\frac{\lambda}{2}+1\right) \cdot \Gamma\left(\frac{\lambda}{2}+1+j\right)}, j = 0, 1, \ldots . \tag{13}$$

Proof. From (11), (12) we find

$$\left\langle z^{\frac{\lambda}{2}}, T_j^*(z) \right\rangle_{-\frac{1}{2}} = \frac{\Gamma(\lambda+1)}{2^\lambda \Gamma\left(\frac{\lambda}{2}+1\right)} \cdot \frac{\binom{\lambda/2}{j} j!}{\Gamma\left(\frac{\lambda}{2}+j+1\right)}$$

and using the definition of inner product we obtain (13).

Further for $f, g \in C[-1,1]$ we use the following inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\frac{dt}{\sqrt{1-t^2}} \ .$$

We consider the following development in Chebyshev series

$$|z|^\lambda = \sum_{k=0}^{\infty} c_k(\lambda)\cdot T_k(z), z \in [-1,1], \lambda > 0, \tag{14}$$

where

$$c_k(\lambda) = \frac{1}{\pi}\gamma_k \cdot \left\langle |z|^\lambda, T_k(z)\right\rangle = \frac{1}{\pi}\gamma_k \cdot \int_{-1}^{1} |z|^\lambda T_k(z)\frac{dz}{\sqrt{1-z^2}} \ . \tag{15}$$

Substituting $z = -t$ in (15) and using the relation

$$T_k(-t) = (-1)^k T_k(t),$$

we find

$$c_{2j+1}(\lambda) = 0, k = 0, 1, \ldots, \tag{16}$$

hence

$$|z|^\lambda = \sum_{k=0}^{\infty} c_{2k}(\lambda)\cdot T_{2k}(z), \tag{17}$$

where

$$c_{2k}(\lambda) = \frac{2}{\pi}\gamma_k \cdot \int_{0}^{1} z^\lambda T_{2k}(z)\frac{dz}{\sqrt{1-z^2}} \ . \tag{18}$$

It is known that

$$T_{mn}(z) = T_m(T_n(z)),\ m, n \in N,$$

hence

$$T_{2k}(z) = T_k(T_2(z)) = T_k(2z^2-1) \ . \tag{19}$$

Using (19) in (18) and substituting $z^2 = t$ we obtained

$$c_{2k}(\lambda) = \frac{1}{\pi}\gamma_k \cdot \int\limits_0^1 t^{\frac{\lambda}{2}} T_k^*(t) \frac{dt}{\sqrt{t(1-t)}} \ . \tag{20}$$

From equalities (13), (17) and (20) we conclude with

Lemma 2.4 *If* $\lambda > 0$, $z \in [-1, 1]$ *we have the following development in Chebyshev series*

$$|z|^\lambda = \sum_{k=0}^{\infty} c_{2k}(\lambda) \cdot T_{2k}(z), \tag{21}$$

where

$$c_{2k}(\lambda) = \frac{\gamma_k \cdot \Gamma(\lambda+1) \cdot \binom{\lambda/2}{k} \cdot k!}{2^\lambda \cdot \Gamma\left(\frac{\lambda}{2}+1\right) \cdot \Gamma\left(\frac{\lambda}{2}+1+k\right)}, k = 0, 1, \ldots \ . \tag{22}$$

The previous result allow us to obtain:

Theorem 2.1 *If* $1 < p < \infty$ *the following equality holds*

$$|T_n(x)|^{p-1} = a_0(p) + \sum_{j=1}^{\infty} a_j(p) \cdot T_{2jn}(x), \ x \in [-1, 1] \ , \tag{23}$$

$$|\cos nt|^{p-1} = a_0(p) + \sum_{j=1}^{\infty} a_j(p) \cdot \cos(2jnt), \ t \in [0.\pi] \ , \tag{24}$$

where

$$a_0(p) = \frac{\Gamma(p)}{2^{p-1}\Gamma^2\left(\frac{p+1}{2}\right)}, a_j(p) = \frac{\Gamma(p) \cdot \binom{\frac{p-1}{2}}{j} \cdot j!}{2^{p-2}\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}+j\right)}, j = 1, 2, \ldots \tag{25}$$

Proof. In (21) we consider $\lambda := p-1$, $1 < p < \infty$, $z := T_n(x)$, $x \in [-1, 1]$ and we find (23). If we consider in (23) $t := \arccos x$, $t \in [0, \pi]$ we deduce (24).

Further we consider the following situation:

1. Suppose that

n is even, $n = 2s$, $s \in N^*$, and k is odd, $k = 2m+1$, $m = \overline{0, s-1}$,
or n is odd, $n = 2s+1$, $s \in N$, and k is even, $k = 2m$, $m = \overline{0, s}$.

Substituting in (6) $t = \pi - x$ we deduce

Theorem 2.2 *If $n = 2s,\ s \in N^*$, and $k = 2m + 1,\ m = \overline{0, s-1}$, or $n = 2s + 1,\ s \in N$, and $k = 2m,\ m = \overline{0, s}$ we have*

$$J_{2s,2m+1} = 0 \text{ and } J_{2s+1,2m} = 0. \tag{26}$$

2. Suppose that n is even, $n = 2s$, $s \in N^*$, and k is even, $k = 2m$, $m = \overline{0, s-1}$.

Let $t_i = \frac{(2i-1)\pi}{4s}$, $i = \overline{1, 2s}$, be the zeros of $\cos(2st)$ on $[0, \pi]$ and $t_0 = 0$, $t_{2s+1} = \pi$.

It is known that

$$\cos(2st) > 0, \text{ for } t \in \bigcup_{i=0}^{s} (t_{2i}, t_{2i+1}) \tag{27}$$

and

$$\cos(2st) < 0, \text{ for } t \in \bigcup_{i=1}^{s} (t_{2i-1}, t_{2i}) \ . \tag{28}$$

Therefore

$$J_{2s,2m} = J^+_{2s,2m} + J^-_{2s,2m} \tag{29}$$

where

$$J^+_{2s,2m} = \sum_{i=0}^{s} \int_{t_{2i}}^{t_{2i+1}} (\cos 2st)^{p-1} \cdot \cos 2mt\, dt, \tag{30}$$

$$J^-_{2s,2m} = -\sum_{i=1}^{s} \int_{t_{2i-1}}^{t_{2i}} (-\cos 2st)^{p-1} \cdot \cos 2mt\, dt. \tag{31}$$

From (24) it follows that

Lemma 2.5 *If* $m = 0, 1, \ldots, s-1,\ s \in N^*,$ *then*

$$(32)\qquad \begin{aligned} J^+_{2s,2m} &= a_0(p) \cdot \sum_{i=0}^{s} \int_{t_{2i}}^{t_{2i+1}} \cos(2mt)dt+ \\ &+\sum_{j=1}^{\infty} a_j(p) \sum_{i=0}^{s} \int_{t_{2i}}^{t_{2i+1}} \cos(4jst)\cdot\cos(2mt)dt\ , \end{aligned}$$

$$(33)\qquad \begin{aligned} J^-_{2s,2m} &= -a_0(p) \cdot \sum_{i=1}^{s} \int_{t_{2i-1}}^{t_{2i}} \cos(2mt)dt- \\ &-\sum_{j=1}^{\infty} a_j(p) \cdot \sum_{i=1}^{s} \int_{t_{2i-1}}^{t_{2i}} \cos(4jst)\cdot\cos(2mt)dt\ . \end{aligned}$$

After an easily computation we obtain the results.

Lemma 2.6 *If* $i = 0, \ldots, s,\ m = 0, \ldots, s-1,\ s \in N^*,$ *then*

$$(34)\int_{t_{2i}}^{t_{2i+1}} \cos(2mt)dt = \begin{cases} \frac{\pi}{4s},\ i = 0, s,\ m = 0 \\ \frac{\pi}{2s},\ i = \overline{1, s-1},\ m = 0 \\ \frac{1}{2m}\cdot\sin\frac{m\pi}{2s},\ i = 0, s,\ m = \overline{1, s-1} \\ \frac{1}{m}\cdot\sin\frac{m\pi}{2s}\cdot\cos\frac{2im\pi}{s},\ i = \overline{1, s-1},\ m = \overline{1, s-1} \end{cases}$$

Lemma 2.7 *If* $i = 0, \ldots, s,\ m = 0, \ldots, s-1,\ s \in N^*,\ j = 1, 2, \ldots,$ *then*

$$(35)\qquad \begin{aligned} &\int_{t_{2i}}^{t_{2i+1}} \cos(4jst)\cdot\cos(2mt)dt = \\ &= \begin{cases} 0,\ i = \overline{0, s},\ m = 0 \\ \dfrac{(-1)^{j+1}\cdot m}{2(4j^2s^2 - m^2)}\cdot\sin\dfrac{m\pi}{2s},\ i = 0, s,\ m = \overline{1, s-1} \\ \dfrac{(-1)^{j+1}m}{4j^2s^2 - m^2}\cdot\sin\dfrac{m\pi}{2s}\cdot\cos\dfrac{2im\pi}{s},\ i = \overline{1, s-1},\ m = \overline{1, s-1}. \end{cases} \end{aligned}$$

Lemma 2.8 *If* $i = 1, \ldots, s,\ m = 0, \ldots, s-1,\ s \in N^*$, *then*

$$(36)\int_{t_{2i-1}}^{t_{2i}} \cos(2mt)dt = \begin{cases} \frac{\pi}{2s},\ i = \overline{1,s},\ m = 0 \\ \frac{1}{m} \cdot \sin\frac{m\pi}{2s} \cdot \cos\frac{(2i-1)m\pi}{s},\ i = \overline{1,s},\ m = \overline{1,s-1}. \end{cases}$$

Lemma 2.9 *If* $i = 1, \ldots, s,\ m = 0, \ldots, s-1,\ s \in N^*,\ j = 1, 2, \ldots$, *then*

$$(37)\quad \int_{t_{2i-1}}^{t_{2i}} \cos(4jst) \cdot \cos(2mt)dt =$$

$$= \begin{cases} 0,\ i = \overline{1,s},\ m = 0 \\ \dfrac{(-1)^{j+1} \cdot m}{4j^2s^2 - m^2} \cdot \sin\dfrac{m\pi}{2s} \cdot \cos\dfrac{(2i-1)m\pi}{s}, i = \overline{1,s},\ m = \overline{1,s-1}. \end{cases}$$

Lemma 2.10 *If* $m = 0, 1, \ldots, s-1,\ s \in N^*$, *then*

$$(38)\qquad \sum_{i=1}^{s} \cos\frac{2im\pi}{s} = 0 \ \text{and}\ \sum_{i=1}^{s} \cos\frac{(2i-1)m\pi}{s} = 0.$$

Taking into account the equalities (34)–(38) from Lemma 2.5 it follows that.

Theorem 2.3 *If* $m = 0, 1, \ldots, s-1,\ s \in N^*$, *then*

$$(39)\qquad J^{+}_{2s,2m} = \begin{cases} \dfrac{\pi}{2} \cdot a_0(p),\ m = 0 \\ 0,\ m = \overline{1,s-1} \end{cases},$$

$$(40)\qquad J^{-}_{2s,2m} = \begin{cases} -\dfrac{\pi}{2} \cdot a_0(p),\ m = 0 \\ 0,\ m = \overline{1,s-1} \end{cases}.$$

Using the results from the Theorem 2.3 we conclude with

Theorem 2.4 *If* $m = 0, 1, \ldots, s-1,\ s \in N^*$, *then*

$$(41)\qquad J_{2s,2m} = 0\ .$$

3. Suppose that n is odd, $n = 2s+1$, $s \in N^*$, and k is odd, $k = 2m+1$, $m = 0, 1, \ldots, s-1$.

Likewise that in section 2 it follows that

Theorem 2.5 *If* $m = 0, 1, \ldots, s-1$, $s \in N^*$, *then*

$$J_{2s+1,2m+1} = 0 \ . \tag{42}$$

From Theorem 2.2, 2.4, 2.5 and relation (5) we conclude with

Theorem 2.6 $\widetilde{T}_n(x)$ *is the unique solution of the extremal problem (1).*

References

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