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On the Iterative Combinations of Baskakov Operator [1]

Naokant Deo

Dedicated to Professor Alexandru Lupaş on the occasion
of his 65th birthday

Abstract

In the present paper, we obtain a quantitative result and asymptotic approximation of sufficiently smooth functions by Baskakov operator.

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1 Introduction and Basic Properties

Let $p_{n,k} = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k(1+x)^{-(n+k)}$, $x \in [0,\infty)$, $n \in N$. The Baskakov operator defined by

$$V_n f = V_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right) \tag{1}$$

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the operator (1) was introduced by V. A. Baskakov [2] and can be used to approximate a function f defined on $[0,\infty)$.

In the present paper, we consider iterative combination due to Micchelli [6] for the operators V_n, defined by (1) and obtain the quantitative result as well as study the asymptotic approximation of sufficiently smooth functions by these operators. Few researchers studied iterative combinations on the different operators (see [1], [3]). For $f \in C[0,\infty)$, we define the operators $T_{n,k}(f,x)$ as

$$(2)\quad T_{n,k}(f,x) = \left[I-(I-V_n)^k\right](f,x) = \sum_{k=1}^{k}(-1)^{r+1}\begin{pmatrix} k \\ r \end{pmatrix} V_n^r(f,x),$$

where V_n^r denotes the $r-th$ iterate (superposition) of the operator V_n. As far as the degree of approximation is concerned, the behavior of operator V_n is similar to the exponential type operators.

Analogous manner of [4], for $r=1,2,...$, we get

$$(3)\qquad V_n^r(1;x) = 1,$$

$$(4)\qquad V_n^r(t;x) = x,$$

$$(5)\qquad V_n^r(t^2;x) = \left(1+\frac{1}{n}\right)^r x^2 + \left\{\left(1+\frac{1}{n}\right)^r - 1\right\} x.$$

Let $m \in N_0 = \{0,1,...\}$, $p \in N = \{1,2,...\}$ and $[\beta]$ denote the integral part of β. Now, suppose that the $m-th$ order moment $\mu_{n,m}^{\{p\}}(x)$ of V_n^p be defined by

$$(6)\qquad \mu_{n,m}^{\{p\}}(x) = V_n^p\left((t-x)^m;x\right).$$

We shall write $\mu_{n,m}(x)$ for $\mu_{n,m}^{\{1\}}(x)$ and let $Q(j,k,x)$ denote the coefficient of n^{-k} in $T_{n,k}\left((t-x)^j;x\right)$.

2 A Set of Lemmas

To prove of main results we need the following results:

Lemma 2.1. *Since the degree of $\mu_{n,m}(x)$ is n^{-1}, which is less than or equal to m with*

$$\mu_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right).$$

The coefficient of n^{-m} in $\mu_{n,2m}(x)$ is $(2m-1)!!p^m(x)$, where $a!!$ is the semi-factorial of a and $p(x) = x(1+x)$. Also, the coefficient of n^{-m} in $\mu_{n,2m+1}(x)$ is $(2m+1)!!(m/3)p^m(x)p'(x)$.

Proof. Since Baskakov operator is exponential type operator, the proof of this lemma follows from May [5, Proposition 3.2].

Remark 2.1. *Now from Lemma 2.1, it is clear that the degree of $\mu_{n,m}^{\{p\}}(x)$ less than or equal to m and $\mu_{n,m}^{\{p\}}(x)$ is defined by (6).*

Lemma 2.2. *There holds the recurrence relation*

$$\mu_{n,m}^{\{p+1\}}(x) = \sum_{j=o}^{m} \binom{m}{j} \sum_{i=0}^{m-j} \frac{1}{i!} \frac{d^i}{dx^i} \left(\mu_{n,m}^{\{p\}}(x)\right) \mu_{n,i+j}(x). \tag{7}$$

Proof. By definition of $\mu_{n,m}^{\{p\}}(x)$, we have $\mu_{n,m}^{\{p+1\}}(x) = V_n\left(V_n^p\,(t-x)^m\,;x\right)$ $= \sum_{j=o}^{m} \binom{m}{j} V_n\left((u-x)^j \left(V_n^p\,(t-u)^{m-j}\,;u\right);x\right)$ $= \sum_{j=o}^{m} \binom{m}{j} V_n\left(\sum_{i=0}^{m-j} \frac{(u-x)^{i+j}}{i!} \frac{d^i}{dx^i}\left(\mu_{n,m-j}^{\{p\}}(x)\right);x\right)$. This leads to require result (7).

Lemma 2.3. *Again from the definition of $\mu_{n,m}^{\{p\}}(x)$, we have*

$$\mu_{n,m}^{\{p\}}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right). \tag{8}$$

Proof. From Lemma 2.1, this result is true for $p = 1$. Now we assuming the result for a certain p, we have to prove it for $p+1$. Since $\mu_{n,m-j}^{\{p\}}(x) = O\left(n^{-\left[\frac{m-j+1}{2}\right]}\right)$ and the degree of $\mu_{n,m-j}^{\{p\}}(x) \leq m-j$, it follows that

$$\frac{d^i}{dx^i}\left(\mu_{n,m-j}^{\{p\}}(x)\right) = O\left(n^{-\left[\frac{m-j+1}{2}\right]}\right).$$

Thus, from (7), we have

$$\begin{aligned} \mu_{n,m}^{\{p+1\}}(x) &= O\left(\sum_{j=0}^{m}\sum_{i=0}^{m-j} n^{-\left(\left[\frac{m-j+1}{2}\right]+\left[\frac{i+j+1}{2}\right]\right)}\right) \\ &= O\left(\sum_{j=0}^{m}\sum_{i=0}^{m-j} n^{-\left[\frac{m+i+1}{2}\right]}\right), \end{aligned}$$

the result (8) is immediate.

Lemma 2.4. *The $l-th$ moment $(l \in N)$ of $T_{n,k}$, we have*

$$T_{n,k}\left((t-x)^l; x\right) = O\left(n^{-k}\right). \tag{9}$$

Proof. From Lemma 2.1, the result easily holds for $k = 1$. Now assuming that the result (9) holds, for certain values of k, then from Lemma 2.2 and 2.3, it may be seen that it also holds for $k+1$.

3 Main Results

In this section, we first prove the following result.

Theorem 3.1. *Let $f \in C[0,\infty)$ and $[0,b] \subseteq [0,\infty)$, then for all n*

$$\|T_{n,k}(f) - f\| \leqslant \omega(f; n^{-1/2})\left[(2^k - 1) + n\xi\left\{\left(2+\frac{1}{n}\right)^k - 2^k\right\}\right] \tag{10}$$

where $\omega(f,.)$ is the modulus of continuity of f and $\|.\|$ is the supremum norm.

Proof. From (2) and (3), we obtain

$$T_{n,k}(f;x)-f(x)=\sum_{r=1}^{k}(-1)^{r+1}\left(\begin{array}{c}k\\ r\end{array}\right)V_n^r\left(f(t)-f(x);x\right).$$

Hence,

$$|T_{n,k}(f;x)-f(x)|\leq\sum_{r=1}^{k}\left(\begin{array}{c}k\\ r\end{array}\right)V_n^r\left(|f(t)-f(x)|;x\right).$$

It has been shown in [7] that for all $t,x\in[0,\infty)$ and $\delta>0$

$$|f(t)-f(x)|\leqslant\left(1+\frac{(t-x)^2}{2}\right)\omega(f;\delta). \tag{11}$$

Using (3)-(5) and (11), we get

$$|T_{n,k}(f;x)-f(x)|\leqslant\sum_{r=1}^{k}\left(\begin{array}{c}k\\ r\end{array}\right)V_n^r\left(\left(1+\tfrac{(t-x)^2}{2}\right)\omega(f;\delta);x\right)$$

$$\leqslant\omega(f;\delta)\left[\sum_{r=1}^{k}\left(\begin{array}{c}k\\ r\end{array}\right)V_n^r(1;x)+\tfrac{1}{\delta^2}\sum_{r=1}^{k}\left(\begin{array}{c}k\\ r\end{array}\right)V_n^r\left((t-x)^2;x\right)\right]$$

$$=\omega(f;\delta)\left[(2^k-1)+\tfrac{1}{\delta^2}\left\{\left(2+\tfrac{1}{n}\right)^k-2^k\right\}b(1+b)\right].$$

Finally, we get

$$\|T_{n,k}(f)-f\|\leqslant\omega(f;\delta)\left[(2^k-1)+\frac{\xi}{\delta^2}\left\{\left(2+\frac{1}{n}\right)^k-2^k\right\}\right],$$

where $\xi=b(1+b)$. Choosing $\delta=n^{-1/2}$ in above, we arrive at required result.

Remark 3.1. *For all k and n, we can easily verify that*

$$\frac{\left(2+\frac{1}{n}\right)^k-2^k}{2^k-1}\leqslant\frac{k}{n}. \tag{12}$$

Applying (12) in (10), we obtain

$$\|T_{n,k}(f)-f\|\leqslant(1+\xi k)\left(2^k-1\right)\omega\left(f;n^{-1/2}\right). \tag{13}$$

Theorem 3.2. *Suppose $f \in C[0,\infty)$ and if $f^{(2k)}$ exists at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n^k \left[T_{n,k}(f;x) - f(x)\right] = \sum_{j=2}^{2k} \tfrac{f^{(j)}(x)}{j!} Q(j,k,x) \tag{14}$$

and

$$\lim_{n\to\infty} n^k \left[T_{n,k+1}(f;x) - f(x)\right] = 0 \ . \tag{15}$$

Proof. Using Taylor expansion of $f(t)$, when $t = x$, we obtain that

$$n^k \left[T_{n,k}(f;x) - f(x)\right] = n^k \sum_{j=1}^{2k} \frac{f^{(j)}(x)}{j!} T_{n,k}\left((t-x)^j;x\right) +$$

$$+ \, n^k \sum_{r=1}^{k} (-1)^{r+1} \tbinom{r}{k} V_n^r\left(\varepsilon(t,x)(t-x)^{2k};x\right) = E_1 + E_2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

First, we estimate E_1. Since $T_{n,k}(t;x)$ and using Lemma 2.4, we get

$$E_1 = n^k \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} T_{n,k}\left((t-x)^j;x\right) = \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} Q(j,k,x) + O(1).$$

For a given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|\varepsilon(t,x)| < \varepsilon, \ \ \textit{whenever} \ \ 0 < |t-x| < \delta.$$

If $\chi_\delta(t)$ is characteristic function of the interval $(t-\delta, t+\delta)$, then we get

$$|E_2| \leq n^k \sum_{r=1}^{k} \left(\begin{array}{c} k \\ r \end{array}\right) V_n^r\left(|\varepsilon(t,x)|\,(t-x)^{2k}\chi_\delta(t);x\right)$$

$$+ \, n^k \sum_{r=1}^{k} \left(\begin{array}{c} k \\ r \end{array}\right) V_n^r\left(|\varepsilon(t,x)|\,(t-x)^{2k}\left(1-\chi_\delta(t)\right);x\right)$$

$$= E_{2.1} + E_{2.2}.$$

Finally, we estimate $E_{2.1}$ and $E_{2.2}$, in view of Lemma 2.3

$$E_{2.1} \leq \left(\sup_{|t-x|<\delta} |\varepsilon(t,x)|\right) n^k \sum_{r=1}^{k} \left(\begin{array}{c} k \\ r \end{array}\right) V_n^r\left((t-x)^{2k};x\right) < \varepsilon C_1,$$

for an arbitrary $v > k$, by Lemma 2.3, we get

$$|E_{2.2}| \leq n^k \sum_{r=1}^{k} \begin{pmatrix} k \\ r \end{pmatrix} V_n^r \left(\frac{C_2(t-x)^{2v}}{\delta^2(v-k)}; x \right) < \frac{C_3}{n^{v-k}} = O(1).$$

Hence, due to arbitrariness of $\varepsilon > 0$, we conclude that $E_2 = O(1)$.

Now by appropriate estimates of E_1 and E_2, this leads to (14).

Proceeding along the similar lines by noting that

$$T_{n,k+1}\left((t-x)^j ; x\right) = O\left(n^{-(k+1)}\right),$$

the assertion (15) can be proved.

Finally, the last assertion follows since uniform continuity of $f^{(2k)}$ on $[0, \infty)$ and the uniformness of $O(1)$ term in the estimate of E_1. This complete the proof of Theorem 3.2.

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Department of Applied Mathematics,
Delhi College of Engineering,
Bawana Road, Delhi-110042, India
E-mail address: *dr_naokant_deo@yahoo.com*