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Duffin-Schaeffer type inequalities [1]

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Dedicated to Professor Alexandru Lupaş
on the occasion of his 65th birthday

Abstract

We give estimation for the weighted L^2-norm of the k-th derivative of polynomial provided $|p_{n-1}(x)|$ is bounded at a set of n points, which are related in a certain way with the weight.

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1 Introduction

The following problem was raised by P.Turán (*Varna,1970*).

Let $\varphi(x) \geq 0$ *for* $-1 \leq x \leq 1$ *and consider the class* $P_{n,\varphi}$ *of all polynomials of degree* n *such that*

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$|p_n(x)| \leq \varphi(x)$ *for* $-1 \leq x \leq 1$. *How large can* $max_{[-1,1]} \left| p_n^{(k)}(x) \right|$ *be if* p_n *is arbitrary in* $P_{n,\varphi}$ *?*

The aim of this paper is to consider the solution in the weighted L^2 norm for the majorant
$\varphi(x) = \frac{\alpha-\beta x}{(1-x)\sqrt{1+x}}$, $0 \leq \beta \leq \alpha$.

Let as denote by

$$(1) \qquad x_i = \cos\frac{2i\pi}{2n+1}, \text{the zeros of } \; W_n(x) = \frac{\sin[(2n+1)\theta/2]}{\sin(\theta/2)},$$

$x = \cos\theta$, the Chebyshev polynomial of the fourth kind,
$x_i^{(k)}$ the zeros of $W_n^{(k)}(x)$ and

$$(2) \qquad G_{n-1}(x) = \frac{\sqrt{2}}{2n(2n+1)}\left[(2n\alpha+\beta)W_n'(x) - (2n+1)\beta W_{n-1}'(x)\right]$$

Let $Z_{\alpha,\beta}^{W,\varphi}$ be the class of polynomials p_{n-1}, of degree $\leq n-1$ such that

$$(3) \qquad |p_{n-1}(x_i)| \leq \frac{\alpha-\beta x_i}{(1-x_i)\sqrt{1+x_i}}, i = 1, 2, ..., n,$$

where the x_i's are given by (1) and $0 \leq \beta \leq \alpha$.

Remark 1.1. *From* $|G_{n-1}(x_i)| = \frac{\alpha-\beta x_i}{(1-x_i)\sqrt{1+x_i}}$ *it follows:*

- $P_{n-1,\varphi} \subset Z_{\alpha,\beta}^{W,\varphi}$
- $G_{n-1} \in Z_{\alpha,\beta}^{W,\varphi}$
- *If* $p_{n-1} \in Z_{\alpha,\beta}^{W,\varphi}$ *and* $i = 1, 2, ..., n,$

$$(4) \qquad |p_{n-1}(x_i)| \leq |G_{n-1}(x_i)|$$

2 Main results

Theorem 2.1. *If* $p_{n-1} \in Z_{\alpha,\beta}^{W,\varphi}$ *then we have*

$$(5) \quad \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[p_{n-1}(x)\right]^2 dx \le \frac{2\pi n \left[2\left(\alpha^2+\beta^2+\alpha\beta\right)(n+1)-3\beta^2\right]}{3(2n+1)}$$

$$(6) \quad \int_{-1}^{1} (1-x)\sqrt{1-x^2}\left[p'_{n-1}(x)\right]^2 dx \le$$

$$\le \frac{2\pi\left(n^3-n\right)}{15(2n+1)}[2(2\alpha^2+\alpha\beta+2\beta^2)n+8\alpha^2+4\alpha\beta-7\beta^2]$$

with equality for $p_{n-1} = G_{n-1.}$

Two cases are of special interest:

I. Case $\alpha = \beta = 1$, $\varphi(x) = \frac{1}{\sqrt{1+x}}$,

$G_{n-1} = \frac{\sqrt{2}}{2n}\left[W'_n(x) - W'_{n-1}(x)\right] = \frac{\sqrt{2}}{n}T'_n(x) = \sqrt{2}U_{n-1}(x)$,

$U_{n-1}(x) = \sin n\theta/\sin\theta, x = \cos\theta$, the Chebyshev polynomial of the second kind.

Note that $P_{n-1,\varphi} \subset Z_{1,1}^{W,\varphi}$, $\sqrt{2}U_{n-1} \notin P_{n-1,\varphi}$,
$\sqrt{2}U_{n-1} \in Z_{1,1}^{W,\varphi}$.

Corollary 2.1. *If* $p_{n-1} \in Z_{1,1}^{W,\varphi}$ *then we have*

$$(7) \quad \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[p_{n-1}(x)\right]^2 dx \le 2\pi n$$

$$\int_{-1}^{1} (1-x)\sqrt{1-x^2}\left[p'_{n-1}(x)\right]^2 dx \le \frac{2\pi\left(n^3-n\right)}{3}$$

with equality for $p_{n-1} = \sqrt{2}U_{n-1}$.

II. Case $\alpha = 1$, $\beta = 0$, $\varphi(x) = \frac{1}{(1-x)\sqrt{1+x}}$,
$G_{n-1} = \frac{\sqrt{2}}{2n+1} W_n'(x)$

Note that $P_{n-1,\varphi} \subset Z_{1,0}^{W,\varphi}$, $\frac{\sqrt{2}}{2n+1} W_n'(x) \in P_{n-1,\varphi}$, $\frac{\sqrt{2}}{2n+1} W_n'(x) \in Z_{1,0}^{W,\varphi}$.

Corollary 2.2. *If* $p_{n-1} \in Z_{1,0}^{W,\varphi}$ *then we have*

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[p_{n-1}(x)\right]^2 dx \leq \frac{4\pi n (n+1)}{3(2n+1)} \tag{8}$$

$$\int_{-1}^{1} (1-x)\sqrt{1-x^2} \left[p_{n-1}'(x)\right]^2 dx \leq \frac{8\pi (n^3-n)(n+2)}{15(2n+1)}$$

with equality for $p_{n-1} = \frac{\sqrt{2}}{2n+1} W_n'(x)$.

In this second case we have a more general result:

Theorem 2.2. *If* $p_{n-1} \in Z_{1,0}^{W,\varphi}$ *and* $0 \leq |b| \leq a$ *then we have*

$$\int_{-1}^{1} (a+bx)(1-x)^{k+1/2}(1+x)^{k-1/2} \left[p_{n-1}^{(k)}(x)\right]^2 dx \leq$$

$$\leq \frac{2\pi (n+k+1)!\,(2ak+2a-b)}{(n-k-1)!\,(2n+1)(2k+1)(2k+3)}$$

$k = 1, ..., n-2$ *with equality for* $p_{n-1} = \frac{\sqrt{2}}{2n+1} W_n'(x)$.

Setting $a = 1$, $b \in \{-1, 0, 1\}$ one obtains the following

Corollary 2.3. *If* $p_{n-1} \in Z_{1,0}^{W,\varphi}$ *then we have*

$$(9)\qquad \int_{-1}^{1}(1-x)^{k+3/2}(1+x)^{k-1/2}\left[p_{n-1}^{(k)}(x)\right]^{2}dx \leq \frac{2\pi(n+k+1)!}{(n-k-1)!\,(2n+1)(2k+1)}$$

$$(10)\qquad \int_{-1}^{1}(1-x)^{k+1/2}(1+x)^{k-1/2}\left[p_{n-1}^{(k)}(x)\right]^{2}dx \leq \frac{4\pi(n+k+1)!\,(k+1)}{(n-k-1)!\,(2n+1)(2k+1)(2k+3)}$$

$$(11)\qquad \int_{-1}^{1}\left(1-x^{2}\right)^{k+1/2}\left[p_{n-1}^{(k)}(x)\right]^{2}dx \leq \frac{2\pi(n+k+1)!}{(n-k-1)!\,(2n+1)(2k+3)}$$

$k=1,...,n-2$ *with equality for* $p_{n-1}=\frac{\sqrt{2}}{2n+1}W_n'(x)$.

3 Auxiliary results

Here we state some lemmas which help us in proving the above theorems.

Lemma 3.1. (Duffin-Schaeffer) *If* $q(x)=c\prod_{i=1}^{n}(x-x_i)$ *is a polynomial of degree* n *with* n *distinct real zeros and if* $p\in P_n$ *such that*

$|p'(x_i)|\leq|q'(x_i)| \quad (i=1,2,...,n)$,

then for $k=1,2,...,n$

$\left|p^{(k+1)}(x)\right|\leq\left|q^{(k+1)}(x)\right|$ *whenever* $q^{(k)}(x)=0$.

Lemma 3.2. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \leq \frac{\alpha-\beta x_i}{(1-x_i)\sqrt{1+x_i}}$,
$i=1,2,...,n,$ *where the* $x_i's$ *are given by* (1)*. Then we have*

$$\left|p'_{n-1}(x_j^{(1)})\right| \leq \left|G'_{n-1}(x_j^{(1)})\right|, j=1,...,n-1, \tag{12}$$

Proof. By the Lagrange interpolation formula based on the zeros of W_n and using $W'_n(x_i) = \frac{(-1)^{i+1}(2n+1)}{(1-x_i)\sqrt{2(1+x_i)}}$,
we can represent any algebraic polynomial p_{n-1} by
$p_{n-1}(x) = \frac{\sqrt{2}}{2n+1}\sum\limits_{i=1}^{n}\frac{W_n(x)}{x-x_i}(-1)^{i+1}(1-x_i)\sqrt{1+x_i}p_{n-1}(x_i)$.
From $G_{n-1}(x_i) = (-1)^{i+1}\frac{\alpha-\beta x_i}{(1-x_i)\sqrt{1+x_i}}$ we have
$G_{n-1}(x) = \frac{\sqrt{2}}{2n+1}\sum\limits_{i=1}^{n}\frac{W_n(x)}{x-x_i}[\alpha-\beta x_i]$.
Differentiating with respect to x we obtain

$$p'_{n-1}(x) = \frac{\sqrt{2}}{2n+1}\sum_{i=1}^{n}\frac{W'_n(x)(x-x_i)-W_n(x)}{(x-x_i)^2} \times(-1)^{i+1}(1-x_i)\sqrt{1+x_i}p_{n-1}(x_i).$$

On the roots of $W'_n(x)$ and using (3) we find

$$\left|p'_{n-1}\left(x_j^{(1)}\right)\right| \leq \frac{\sqrt{2}}{2n+1}\sum_{i=1}^{n}\frac{\left|W_n\left(x_j^{(1)}\right)\right|}{\left(x_j^{(1)}-x_i\right)^2}[\alpha-\beta x_i]$$
$$= \frac{\sqrt{2}\left|W_n\left(x_j^{(1)}\right)\right|}{2n+1}\sum_{i=1}^{n}\frac{\alpha-\beta x_i}{\left(x_j^{(1)}-x_i\right)^2} = \left|G'_{n-1}\left(x_j^{(1)}\right)\right|.$$

Lemma 3.3. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \leq \frac{1}{(1-x_i)\sqrt{1+x_i}}$,
$i=1,2,...,n,$*where the* $x_i's$ *are given by (1). Then we have*

$$\left|p_{n-1}^{(k)}(x_j^{(k)})\right| \leq \frac{\sqrt{2}}{2n+1}\left|W_n^{(k+1)}(x_j^{(k)})\right|, \tag{13}$$

whenever $W_n^{(k)}(x_j^{(k)}) = 0,\ k=0,1,...,n-1.$

Proof. For $\alpha=1$, $\beta=0$, $G_{n-1} = \frac{\sqrt{2}}{2n+1}W'_n$
and (12) give $\left|p'_{n-1}(x_j^{(1)})\right| \leq \frac{\sqrt{2}}{2n+1}\left|W''_n(x_j^{(1)})\right|$.

Now the proof is concluded by applying Duffin-Schaeffer Lemma.

We need the following quadrature formulae:

Lemma 3.4. *For any given n and k, $0 \leq k \leq n-1$, let $x_i^{(k+1)}$, $i = 1, ..., n-k-1$, be the zeros of $W_n^{(k+1)}$. Then the quadrature formula*

$$\int_{-1}^{1} \left(1-x^2\right)^k \sqrt{\frac{1-x}{1+x}} f(x)\,dx = Af(-1) + Bf(1) + \tag{14}$$

$$+ \sum_{i=1}^{n-k-1} s_i f\left(x_i^{(k+1)}\right),$$

$$A = \frac{2^{2k}(2n+1)\,\Gamma(k+1/2)\,\Gamma(k+3/2)\,(n-k-1)!}{(n+k+1)},$$

$$B = \frac{2^{2k+2}\Gamma(k+3/2)\,\Gamma(k+5/2)\,(n-k-1)!}{(2n+1)\,(n+k+1)!},$$

$s_i > 0$ *have algebric degree of precision* $2n-2k-1$.

Proof. The quadrature formula (14) is the Bouzitat formula of the second kind [3, *formula* (4.8.1)], for the zeros of $W_n^{(k+1)} = cP_{n-k-1}^{\left(k+\frac{3}{2},k+\frac{1}{2}\right)}$. Setting $\alpha = k+1/2$, $\beta = k-1/2$, $m = n-k-1$ in [3, *formula* (4.8.5)] we find A, B and $s_i > 0$ (cf. [3, *formula* (4.8.4)]).

4 Proof of the Theorems

Because $W_n(x) = \frac{(2n)!!}{(2n-1)!!} P_n^{\left(\frac{1}{2},\frac{-1}{2}\right)}(x)$ we recall the formulae:

$$\frac{d}{dx} P_m^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+m+1}{2} P_{m-1}^{(\alpha+1,\beta+1)}(x), \tag{15}$$

$$P_m^{(\alpha,\beta)}(1) = \frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(m+1)},$$

$$P_m^{(\alpha,\beta)}(-1) = \frac{(-1)^m \Gamma(m+\beta+1)}{\Gamma(\beta+1)\Gamma(m+1)}$$

Setting $\alpha = k + 1/2$, $\beta = k - 1/2$, $m = n - k$ in [3, *formula* (4.8.5)] on obtains the Gauss formula for the zeros of $W_n^{(k)} = cP_{n-k}^{\left(k+\frac{1}{2},k-\frac{1}{2}\right)}$

$$(16) \qquad \int_{-1}^{1} \left(1-x^2\right)^k \sqrt{\frac{1-x}{1+x}} f(x)\,dx = \sum_{i=1}^{n-k} H_i^{(k)} f\left(x_i^{(k)}\right),$$

with algebric degree of precision $2n - 2k - 1$ and

$$(17) \qquad H_i^{(k)} > 0 \quad (\text{cf.}[3, \textit{formula}(4.8.4)]).$$

Proof of Theorem 2.1

For $k = 0$ in (16) we obtain

$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} f(x)\,dx = \sum_{i=1}^{n} H_i^{(0)} f(x_i)$ of degree $2n-1$.

According to this quadrature formula and using (4) and (17) we have

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[p_{n-1}(x)\right]^2 dx = \sum_{i=1}^{n} H_i^{(0)} \left(p_{n-1}(x_i)\right)^2$$

$$\le \sum_{i=1}^{n} H_i^{(0)} \left(G_{n-1}(x_i)\right)^2 = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[G_{n-1}(x)\right]^2 dx.$$

Using the following formula ($k = 0$ in (14))

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} f(x)\,dx = \frac{\pi(2n+1)}{2n(n+1)} f(-1) + \frac{3\pi}{2n(n+1)(2n+1)} f(1) + \sum_{i=1}^{n-1} s_i f\left(x_i^{(1)}\right)$$

we find $\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left[G_{n-1}(x)\right]^2 = \frac{2\pi n\left[2\left(\alpha^2+\beta^2+\alpha\beta\right)(n+1)-3\beta^2\right]}{3(2n+1)}$.

Setting $k = 1$ in (16) we get

$\int_{-1}^{1} (1-x)\sqrt{1-x^2} f(x)\,dx = \sum_{i=1}^{n-1} H_i^{(1)} f\left(x_i^{(1)}\right)$ of degree $2n-3$.

According to this formula and using (12) and (17) we have

$$\int_{-1}^{1} (1-x)\sqrt{1-x^2} \left[p'_{n-1}(x)\right]^2 dx = \sum_{i=1}^{n-1} H_i^{(1)} \left(p_{n-1}\left(x_i^{(1)}\right)\right)^2$$

$\leq \sum\limits_{i=1}^{n-1} H_i^{(1)} \left(G_{n-1}\left(x_i^{(1)}\right)\right)^2 = \int\limits_{-1}^{1} (1-x)\sqrt{1-x^2}\left[G'_{n-1}(x)\right]^2 dx.$

From (14) with $k=1$ we find

$\int\limits_{-1}^{1} (1-x)\sqrt{1-x^2}\left[G'_{n-1}(x)\right]^2 dx$

$= \frac{2\pi\left(n^3-n\right)}{15(2n+1)}[2(2\alpha^2+\alpha\beta+2\beta^2)n+8\alpha^2+4\alpha\beta-7\beta^2].$

Proof of Theorem 2.2

If we replace $f(x)$ with $(a+bx)f(x)$, $0\leq|b|\leq a$ in (16) we get

$\int\limits_{-1}^{1} (a+bx)(1-x^2)^k\sqrt{\frac{1-x}{1+x}}f(x)\,dx$

$= \sum\limits_{i=1}^{n-k}\left(a+bx_i^{(k)}\right)H_i^{(k)}f\left(x_i^{(k)}\right)$ of degree $2n-2k-2$

According to this formula and using (13) and (17) we have

$\int\limits_{-1}^{1} (a+bx)(1-x^2)^k\sqrt{\frac{1-x}{1+x}}\left[p_{n-1}^{(k)}(x)\right]^2 dx$

$= \sum\limits_{i=1}^{n-k}\left(a+bx_i^{(k)}\right)H_i^{(k)}\left[p_{n-1}^{(k)}\left(x_i^{(k)}\right)\right]^2$

$\leq \frac{2}{(2n+1)^2}\sum\limits_{i=1}^{n-k}\left(a+bx_i^{(k)}\right)H_i^{(k)}\left[W_n^{(k+1)}\left(x_i^{(k)}\right)\right]^2$

$= \frac{2}{(2n+1)^2}\int\limits_{-1}^{1}(a+bx)(1-x^2)^k\sqrt{\frac{1-x}{1+x}}\left[W_n^{(k+1)}(x)\right]^2 dx$

If we replace $f(x)$ with $(a+bx)f(x)$, $0\leq|b|\leq a$ in (14) we get

$\int\limits_{-1}^{1} (a+bx)(1-x^2)^k\sqrt{\frac{1-x}{1+x}}f(x)\,dx = A(a-b)f(-1)+B(a+b)f(1)$

$+\sum\limits_{i=1}^{n-k-1}\left(a+bx_i^{(k)}\right)s_i f\left(x_i^{(k+1)}\right)$ of degree $2n-2k-2$.

In order to complete the proof we apply this formula to $f=\frac{2}{(2n+1)^2}\left[W_n^{(k+1)}(x)\right]^2$.

Having in mind $W_n^{(k+1)}\left(x_i^{(k+1)}\right)=0$ and the following relations deduced from (15)

$W_n^{(k+1)}(-1)=\frac{(-1)^{n-k-1}(n+k+1)!}{(n-k-1)!(2k+1)!!}$, $\left(W_n^{(k+1)}(1)\right)^2=\frac{(2n+1)^2}{(2k+3)^2}\left(W_n^{(k+1)}(1)\right)^2,$

we find

$$\int_{-1}^{1} (a+bx)\,(1-x)^{k+1/2}\,(1+x)^{k-1/2}\,\frac{2}{(2n+1)^2}\left[W_n^{(k+1)}(x)\right]^2 dx$$
$$= \frac{2}{(2n+1)^2} A\,(a-b)\left[W_n^{(k+1)}(-1)\right]^2 + \frac{2}{(2n+1)^2} B\,(a+b)\left[W_n^{(k+1)}(1)\right]^2$$
$$= \frac{2\pi(n+k+1)!(2ak+2a-b)}{(n-k-1)!(2n+1)(2k+1)(2k+3)}.$$

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