About the use of a result of Professor Alexandru Lupaş to obtain some properties in the theory of the number e [1]

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Dedicated to Professor Alexandru Lupaş on his 65th anniversary

Abstract

A very elegant result of Professor Alexandru Lupaş gives us that the point c" of the mean value theorem (of Lagrange) applied to the logarithmic function on an interval $[a, b] \subset (0, \infty)$ has the property that $\sqrt{ab} < c < (a + b)/2$.

In this paper we show that the result of Professor Alexandru Lupaş is decisively useful to establish some other nice results in the theory of the number e and in some related question.

1. The work of Professor *Alexandru Lupaş* contains numerous, various and elegant results especially in the following domains: Real functions, Inequalities, Sequences and Series, Special functions, Numerical Analysis, Umbral Calculus, Approximation Theory, but also Computer Sciences, Computational Mathematics and Experimental Mathematics.

All these result can be found in his numerous papers and books.

In this paper we consider one of his most elegant results, namely concerning the position of the point „c" of the mean value theorem (of *Lagrange*) applied to the logarithmic function on an interval $[a, b] \subset (0, \infty)$. This point, given by the equation:

$$(1) \qquad f(b) - f(a) = f'(c) \cdot (b - a),$$

for $f(x) = \ln x$, $x \in [a, b] \subset (0, \infty)$ and unique because of the injectivity of the derivative has the property:

$$(2) \qquad \sqrt{ab} < c < \frac{a + b}{2}.$$

This property refines the relation $c \in (a, b)$, because, of course

$$\left(\sqrt{ab}, (a + b)/2 \right) \subset (a, b).$$

So it gives an answer from a celebrated question proposed by *D. Pompeiu*: to find for the point „c" smaller intervals as (a, b) (see [8], at the pages 201, 273, 293; also, see [1]).

2. To prove the two-sided inequality (2) consider the following
Lemma ([7], page 273). *For any $x > 0$, we have:*

$$(3) \qquad \frac{2}{2x + 1} < \ln \left(1 + \frac{1}{x} \right) < \frac{1}{\sqrt{x(x + 1)}}.$$

Proof. For the left part, consider the function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \ln \left(1 + \frac{1}{x} \right) - \frac{1}{2x+1}$. Because of the equality

$$f'(x) = -\frac{1}{x(x - 1)(2x + 1)^2} < 0,$$

the function f is strictly decreasing. The comparison with $\lim_{x \to \infty} f(x) = 0$, closes the proof.

Analogously, for the right part, consider the function $g : (0, \infty) \to \mathbb{R}$, $g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{\sqrt{x(x+1)}}$. We obtain

$$g'(x) = \frac{2}{2x(x+1)\sqrt{x(x+1)}} \left(\frac{2x+1}{2} - \sqrt{x(x+1)}\right) > 0$$

and so, because of the comparison with $\lim\limits_{x \to \infty} g(x) = 0$, we obtain the right part.

Theorem 1. *For any interval* $(a, b) \subset (0, \infty)$, $a < b$, *considering the point* „c" *previously defined in the section 1, we have the inequality (2).*

Proof. The formula (1), written for the logarithmic function becomes $\ln b - \ln a = \frac{b-a}{c}$ and (2) is equivalent with:

$$(4) \qquad \qquad \frac{2(b-a)}{b+a} < \ln\frac{b}{a} < \frac{b-a}{\sqrt{ab}},$$

or with:

$$(5) \qquad \qquad \frac{2(t-1)}{t+1} < \ln t < \frac{t-1}{\sqrt{t}},$$

where $t = b/a$ and $t > 1$. If we put $e = 1 + 1/s$, with $s > 0$, the inequality (5) becomes

$$\frac{e}{2s+1} < \ln\left(1 + \frac{1}{s}\right) < \frac{1}{\sqrt{s(s+1)}},$$

which is exactly (3), written in the variable s in the place of x, therefore it is true.

We remark now that, conversely, (3) implies (4). Indeed, putting in (3) $x = \frac{b}{a} - 1$, it gives us (4). So we have obtained the

Theorem 2. *The following affirmation are equivalent:*

(a) The point „c" *of the mean theorem for the logarithmic function, applied on an interval* $[a, b] \subset (0, \infty)$ *satisfies the inequality (2).*

(b) The inequality (3) is true.

3. The inequality (4) can be also interpreted in two manners (for $0 < a < b$):

(a) It is a reffinement of the classical inequality of *Napier*:

$$\frac{b-a}{b} < \ln\frac{b}{a} < \frac{b-a}{a}.$$

(b) The logaritmical mean of a and b, namely $\frac{b-a}{\ln b/a}$ is placed between the geometric and the arithmetic means of a and b.

4. The property of the point „c" of the logarithmic function, given by Professor *Alexandru Lupaş* permits us to obtain other nice results.

(a) We have the inequality (from [9]):

$$(6) \qquad \frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}.$$

For a short proof, based on (3), see [13].

(b) If $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$ and $\gamma = \lim_{n\to\infty} \gamma_m = 0,577\ldots$ is the Euler's constant, we have:

$$(7) \qquad \frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}.$$

A proof based on (3) can be given in [12].

(c) It is known that the unique rational solutions of the equation:

$$x^y = y^x \qquad (0 < x < y)$$

are given by:

$$x = e_n = \left(1 + \frac{1}{n}\right)^n, \quad y = f_n = \left(1 + \frac{1}{n}\right)^{n+1} \qquad (n = 1, 2, 3 \ldots)$$

(see [15], pp. 242-256). This means that replacing these solutions in the equation, we obtain a true equality, namely:

$$e_n^{f_n} = f_n^{e_n} \left(= \left(\frac{n+1}{n}\right)^{\frac{(n+1)^{n+1}}{n^n}} \right).$$

Let $z_n = \left(\frac{n+1}{n}\right)^{\frac{(n+1)^{n+1}}{n^n}}$ be. The sequence is strictly decreasing. For the proof, see [14]. This proof uses again (3).

A similar situation is valid for the rational solutions of the equation $u^u = v^v$ $(0 < v < u)$. We have $u = \frac{1}{e_n}$, $v = \frac{1}{f_n}$ and so:

$$\left(\frac{1}{e_n}\right)^{\frac{1}{e_n}} = \left(\frac{1}{f_n}\right)^{\frac{1}{f_n}} = \left(\frac{n}{n+1}\right)^{\frac{(n+1)^{n+1}}{n^n}}.$$

Let $w_n = \left(\dfrac{n}{n+1}\right)^{\frac{(n+1)^{n+1}}{n^n}}$ be. The sequence is also strictly decreasing; the proof of [14] uses, again, (3).

(d) The problem [6] of Professor *Alexandru Lupaş* is the following: „Let x and y be two different real numbers $x > 0$, $y > 0$, $x \neq$ e so that $x^y = y^x$. Show that $x^y >$ e$^{\text{e}}$." The solution, published in the same journal in 2006, no. 3, pp. 233-236, also uses (3).

(e) In the paper [2], the sequences of general term $\eta_n = e_n \ln(E_n) = = E_n \ln(e_n)$ and $\lambda_n = \ln(e_n) \cdot \ln(E_n)$, where $E_n = f_n = \left(1 + \dfrac{1}{n}\right)^{n+1}$, are studied. In the proof of a part of its properties, (3) is also used.

(f) In the problem [3] the sequence $(x_n)_n$ is defined by the equality $\left(1 + \dfrac{1}{n}\right)^{n+x_n} =$ e. The monotony and the convergence are requested. To solve the problem the inequality (3) are again necesary.

A similar situation is related to [10], where the properties of the sequence $(x_n)_n$ given by the equality $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} - \ln(x + x_n) = \gamma$ are requested. The inequality (3) are again used. Also se [11].

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