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Applications of Pompeiu areolary derivative in expressing of the Gauss total negative curvature

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Abstract

This paper expresses the error of approximation of some surface with negative Gauss curvature using the areolary derivative of D. Pompeiu. New interpretations for this derivative are given.

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1 Introduction

We consider a surface with Gauss negative curvature [1] of the form:

$$(1)\qquad (S_G) : \overline{r} = (f(t),\ M(t, v),\ \ N(t, v))\,;\ (t, v) \in D^* \subset \mathbb{R}^2$$

D^* - simple connected domain functions $f(t),\ M(t,v),\ N(t,v) \in C^2(D^*)$ Using a transformation of the form:

$$(2)\qquad \begin{cases} t = t(x,y) \\ v = v(x,y) \end{cases} \text{with} \quad (x,y) \in D \subset \mathbb{R}^2 \ \text{ and } \ \frac{D(t,v)}{D(x,y)} \neq 0$$

in D – simple connected domain.

In the case of surfaces of form (1), we note:

$$(3)\qquad y = f(t)$$

and in the case of the existence of the inverted function from relation (3), with $[f^{-1}(y)]^{/} \neq 0$ in D, we will obtain from the transformation (2) the functions $M(t,v)$ and $N(t,v)$ of the form:

$$(4)\qquad \begin{aligned} M(t,v) &= M(f^{-1}(y),\ \ v(x, f^{-1}(y))) = U(x,y) \\ N(t,v) &= N(f^{-1}(y),\ \ v(x, f^{-1}(y))) = V(x,y) \end{aligned}$$

In our paper [1], we determined the function $v(x, f^{-1}(y))$ by imposing the Cauchy-Riemann monogenity conditions on functions $U(x,y)$ and $V(x,y)$ from (4).

$$(5)\qquad U_x = V_y \quad \text{and} \quad U_y = -V_x$$

System (5) allows the determination of partial derivatives v_x, v_y and the determination of function $v\,(x,y)$ is reduced to the determination of a function when its partial derivatives are known. System (5) will also give an expression of function $y = f^*(t)$, generally different from its expression given in (1) and (3). Analyzing this situation constitutes the objective of the paper [1].

In the case $f(t) \equiv f^*(t)$ when .obtained by applying the monogenity conditions (5) , then surface (1) is transformed into a surface we called monogenous of the form (S_m):

$$(S_m) : \overline{r} = (f\,(t)\,,\ U\,(x,y)\,,\ \ V\,(x,y)) \tag{6}$$

with $y = f(t) = f^*(t)$. Surfaces (1) and (6) have the same negative Gauss curvature. In this situation, the following monogenous function can be attached to surface (6):

$$F\,(z) = U(x,y) + iV(x,y) \tag{7}$$

If from the imposing of monogenity conditions to functions $U\,(x,y)$ and .$V\,(x,y)$ given in relation (4) we obtain v_x, v_y, but $y = f(t) \neq f^*(t)$, where $f\,(t)$ is initially given in relation (1), then we'll say that the surface of form:

$$(S_G) : \overline{r} = (f^*\,(t)\,,\ U\,(x,f^*(t))\,,\ \ V(x,f^*(t))) \quad \text{with} \ \ y = f^*\,(t) \tag{8}$$

is the surface which approximates the given surface (1).

In paper [1], the approximation error was calculated using the Euclidian distance between surfaces (1) and (8).

2 The areolary derivative of D. Pompeiu

In this paper we will evaluate the approximation error of a surface with negative Gauss curvature of form (1) through a monogenous surface of form (8). This subject was first introduced in [1] and [2].

In this case we'll evaluate the approximation error using the areolary derivative of Pompeiu, but also a geometric point of view.

The given surface (1), having the negative Gauss curvature (S_G) (S_G) : $\overline{r} = (f(t), M(t,v), N(t,v))$,with $(t,v) \in D^* \subset \mathbb{R}^2, D^*-$simple connected domain and the functions: $f(t), M(t,v), N(t,v) \in C^2(D^*)$ are approximated with the monogenous surface of form (8)

$$(S_m) : \overline{r} = (f^*(t), M(t^*,v), N(t^*,v)),$$

where we noted $t^* = f^{*^{-1}}(y)$.

The function $y = f^*(t)$ was determined from the Cauchy-Riemann monogenity conditions applied to functions U and V, with the notations:

$$\begin{aligned} M(t^*,v) &= M(f^{*^{-1}}(y), v(x, f^{*^{-1}}(y)) = U(x,y) \\ N(t^*,v) &= N(f^{*^{-1}}(y), v(x, f^{*^{-1}}(y)) = V(x,y) \end{aligned}$$

It is known [2] that the areolary derivative of D. Pompeiu expresses the distance of monogenity of a complex function and, for a function $F(z)$, it has the expression:

$$\lim_{\delta\to 0}\frac{\oint\limits_{\gamma} F(z)dz}{\iint\limits_{D} dxdy} = \lim_{\delta\to 0}\frac{2i\iint\limits_{D}\frac{\partial F}{\partial\overline{z}}dxdy}{\iint\limits_{D} dxdy} = 2i\left(\frac{\partial F}{\partial \overline{z}}\right)_P \tag{9}$$

where D – simple connected domain; $\gamma-$ frontier of D, γ – omotop with zero, and $\delta = \sup\limits_{M\to P} d(M,P)$

The last equality from (9) will give the value of the areolary derivative $2i\left(\dfrac{\partial F}{\partial\overline{z}}\right)$ in an arbitrary point $P \in D^0$.

Using the derivation operators $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right), \dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$ in the case of a function

$$\begin{aligned} F(z) &= U(x,y) + iV(x,y) = M(t,v) + iN(t,v) = \\ &= M(f^{-1}(y), v(x, f^{-1}(y)) + iN(f^{-1}(y), v(x, f^{-1}(y)) \end{aligned}$$

we obtain:

$$(10)\qquad \begin{cases} \dfrac{\partial U}{\partial x} = M_v \cdot v_x \\[2ex] \dfrac{\partial U}{\partial y} = M_t \cdot t_y + M_v \cdot v_y = M_t \cdot [f^{-1}(y)]' + M_v \cdot v_y \\[2ex] \dfrac{\partial V}{\partial x} = N_v \cdot v_x \\[2ex] \dfrac{\partial V}{\partial y} = N_t \cdot t_y + N_v \cdot v_y = N_t [f^{-1}(y)]' + N_v \cdot v_y \end{cases}$$

and thus

$$\frac{\partial F}{\partial \overline{z}} = \frac{1}{2}\left\{ M_v \cdot v_x - N_t \cdot \left[f^{-1}(y)\right]' - N_v \cdot v_y + i\left[N_v \cdot v_x + M_t\left(f^{-1}(y)\right)' + M_v \cdot v_y\right]\right\}$$

We mention that functions $M(t,v)$; $N(t,v)$ are given, and $t = f^{-1}(y)$ supposedly exists having the derivative $[f^{-1}(y)]' \neq 0$ in $D \subset \mathbb{C}$.

Relation (11), which expresses the areolary derivative of the complex function with complex variable $F(z) = F(x,y) = U(x,y) + iV(x,y)$, is not null in the case when $.y = f(t) \neq f^*(t)$. It expresses analytically the monogenity distance of the complex function with complex variable $F(z)$

$$(12)\qquad F(z) = F(x,y) = M(f^{-1}(y), v(x,y) + iN(f^{-1}(y), v(x,y))$$

attached to surface (1) having negative Gauss curvature.

In the case when the derivative $\dfrac{\partial F}{\partial \overline{z}} = 0$, the surface (1) is identically transformed into the monogenous surface (S_m)

$$(13)\qquad (S_m) : \overline{r} = (y, M(f^{-1}(y), v(x,y)), N(f^{-1}(y), v(x,y)))$$

Surfaces (1) and (13) have the same negative Gauss curvature when $f^*(t) = f(t)$.

3 The geometric interpretation of the situation $y = f^*(t) \neq f(t)$.

We consider again surfaces (1) and (8) in the system

$$(14)\quad \begin{cases} (S_G) : \overline{r}(f(t), M(t,v), N(t,v)\,, & y = f(t) \\ (S_m) : \overline{r}(f^*(t), M(t^*,v), N(t^*,v)); & t^* = f^{*-1}(y) \text{ and } y = f^*(t) \end{cases}$$

The parameter x is the same in both cases of surfaces from (14), only y differs from (S_G) to (S_m). The expressions of functions M and N are the same in the system of surfaces (14).

From geometric point of view, for each $x = x_0 =$ constant, the surfaces from (14) have the same curve of coordinates, which we obtain by replacing x with x_0 in (14)

(15)

$$\begin{cases} \overline{r}_G(x_0, y) = \overline{r}_G(y) = (y, M(f^{-1}(y), v(x_0, y)), N(f^{-1}(y), v(x_0, y))) \\ \overline{r}_m(x_0, y) = \overline{r}_m(y) = (y, M(f^{*-1}(y), v(x_0, y)), N(f^{-1}(y), v(x_0, y))) \end{cases}$$

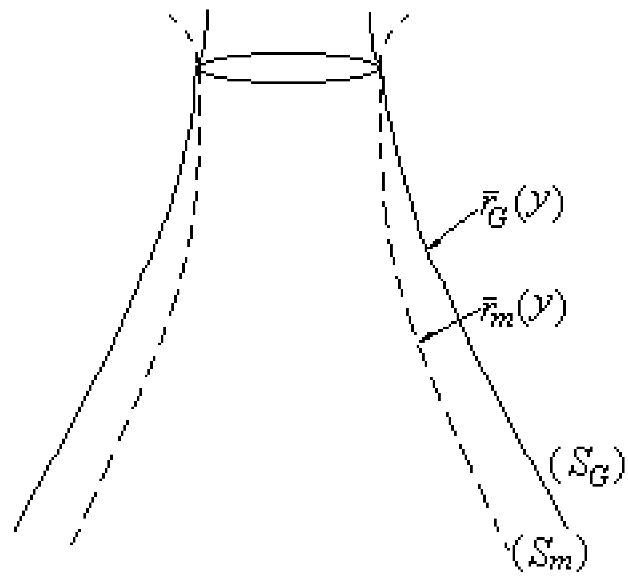
Geometrical characteristics (see [1], [2], [3]-it verifies immediately)

1. The surfaces from (14) have different negative Gauss curvature.

2. The coordinate curves of surfaces from relation (15) are rectangular.

3. The coordinate curve $x = x_0$ is plane (respectively).

In the figure below, we sketched the surfaces (S_G) and (S_m) in a plane section. In the plane of the sketch, the coordinate curvatures from relation (15) appear, noted with $\overline{r}_G(y)$. respectively $\overline{r}_m(y)$.



The approximation error of surface (S_G) through a monogenous surface (S_m) will be evaluated as follows by comparing the curvatures of the coordinate curves from system (15).

Generally [4], for a curvature in space of form

$$\overline{r} = \overline{r}(x(t), y(t), z(t)), \qquad t \in I \tag{16}$$

the curvature can be calculated using the known formula:

$$K = \frac{1}{R^2} = \frac{A^2 + B^2 + C^2}{(x'^2 + y'^2 + z'^2)^2} \tag{17}$$

In the case of coordinate curves $x = x_0 =$ constant, we will replace

(x, y, z) from (16) with

$$(18) \qquad \begin{cases} x \to y \\ y \to M(f^{-1}(y) \text{ respectively } M(f^{*-1}(y) \\ z \to N(f^{-1}(y)), \text{ respectively } N(f^{*-1}(y) \end{cases}$$

and thus system (15) becomes of form:

$$(19) \qquad \begin{cases} \overline{r}_G(y) = (y, M(f^{-1}(y)), N(f^{-1}(y))), \text{ with } t = f^{-1}(y) \\ \overline{r}_m(y) = (y, M(f^{*-1}(y)), N(f^{*-1}(y))), \text{ with } t^* = f^{*-1}(y) \end{cases}$$

The variable parameter on the curvature will be noted y instead of t. The curves of the curvatures given in (19) will be noted with K_G, respectively K_m.

$$\begin{aligned} K_G &= \frac{\left|M'_t \cdot t' \left[N''_{t^2}(t')^2 + N'_t t''\right] - N'_t t'[M''_{t^2}(t')^2 + M'_t \cdot t'']\right|}{[(M'_t \cdot t')^2 + N'_t \cdot (t')^2]^2} = \\ &= \frac{1}{(t')^3} \cdot \frac{\left|(t')^2[M'_t N''_{t^2} - N'_t M''_{t^2}] + t''[M'_t \cdot N'_t - N'_t \cdot M'_t]\right|}{[M'^2_t + N'^2_t]^2} = \\ &= \frac{\left|M'_t \cdot N''_{t^2} - N'_t \cdot M''_{t^2}\right|}{|t'| \, [M'_{t^2} + N'_{t^2}]^2} = \text{of form} \quad \frac{A}{B\,|t'|} \end{aligned}$$

The structure of curvature K_m expression will be of form

$$(21) \qquad K_m = \frac{A}{B\,|t^{*\prime}|}$$

because the expressions of functions M, N are the same

$$(22) \qquad \frac{K_G}{K_m} = \left|\frac{t^{*\prime}}{t'}\right|$$

Ratio (22) will give an image of the "distances" of the coordinate curves from the surfaces in a point of contact (figure 1). Formula (22) is useful for evaluating the approximation error of a surface (S_G) through a monogenous surface (S_m). We can formulate theorems:

Theorem 1 *For any surface having the Gauss negative curvature of form*

$$(S_G) : \overline{r} = (f(t), M(t,v), N(t,v)),\ (t,v) \in D^2 \subset \mathbb{R}^2, \quad t = f^{-1}(y)$$

approximated with a monogenous surface of form

$$(S_m) : \overline{r} = (f^*(t), M(t^*,v), N(t^*,v)),\ (t^*,v) \in D^2 \subset \mathbb{R}^2, \quad t = f^{*-1}(y)$$

the coordinate curves $x = x_0$ *have the ratio of curvatures given by relation*

$$\frac{K_G}{K_m} = \left| \frac{t^{*\prime}}{t'} \right|$$

4 The areolary derivative in a particular case: $x = x_0$

We consider the difference function $\phi(z) = F(x,y) - F^*(x,y)$, where the complex functions of complex variable F and F^*. are respectively:

$$\begin{aligned} F(x,y) &= U(x,y) + iV(x,y)\,, \qquad y = f(t) \\ F^*(x,y) &= U(x,y^*) + iV(x,y^*)\,,\ \ y^* = f^*(t) \end{aligned} \tag{23}$$

Function $F^*(x,y)$ is monogenous and $\dfrac{\partial F^*}{\partial \overline{z}} = 0$ in all the points of monogenity.

We remind that the determination of function $.f^*(t)$ was done by imposing the monogenity conditions. Relations (10) and (11) for the evaluation of the areolary derivative in the general case are laborious. In the particular case of the difference function $\phi(x,y)$, we will use the advantage of the common expressions of functions M and N form (19).

For the case $x = x_0 =$ constant, the areolary derivative of function ϕ becomes:

$$\frac{\partial \phi}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\phi = \frac{i}{2}\frac{\partial \phi}{\partial y} \tag{24}$$

(25)
$$\left|\frac{\partial \phi}{\partial \overline{z}}\right| = \frac{1}{2}\left|M'(t'_y - t_y^{*-1}) + N'(t_y^{-1} - t_y^{*-1})\right| = \frac{1}{2}\left|M' + N'\right| \cdot \left|(t_y^{-1} - t_y^{*-1})\right|$$

Relation (25) is especially interesting. We can formulate the following theorem:

Theorem 2 *Let there be surfaces (S_G) and (S_m) having negative Gauss curvatures*

$$(S_G) : \overline{r} = (f(t), M(t,v), N(t,v)), (t,v) \in D^* \subset \mathbb{R}^2, t = f^{-1}(y)$$

$$(S_m) : \overline{r} = (f^*(t), M(t^*,v), N(t^*,v)), t^* = f^{*-1}(y)\,, (t^*,v) \in D^* \subset \mathbb{R}^2$$

The complex function of complex variable, attached to surface (S_G) differs from the monogenous function attached to surface (S_m) along a curve of coordinates $x = x_0$ (not the same on (S_m) and (S_G)). The error of monogenity distance along the curvature $x = x_0$ is evaluated with the module of the areolary derivative, being directly proportional with $\left|t_y^{-1} - t_y^{*-1}\right|$

$$\left|\frac{\partial \phi}{\partial \overline{z}}\right| = \frac{1}{2}\left|M' + N'\right| \cdot \left|(t_y^{-1} - t_y^{*-1})\right|$$

Conclusion:

The evaluations of the approximation error of surface (S_G) through monogenous surfaces (S_m) depend, in both cases, on the approach, on the derivatives of functions $t = f^{-1}(y)\,,\ t^* = f^{*-1}(y)$.

Example

The surfaces (S_G) and (S_m) are figured in a particular case, of the form:

$$(S_G) : \overline{r} = (a \log(t + \sqrt{t^2 - a^2}), t \cos x, t \sin x), a > 0$$

or using

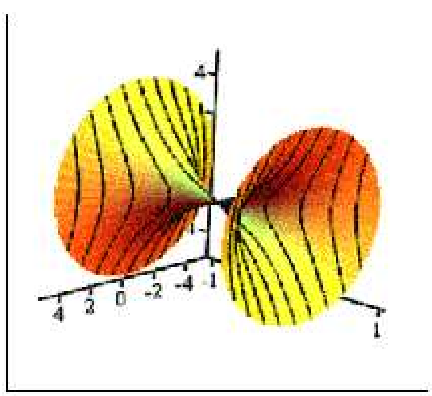
$$y = f(t) = a \log(t + \sqrt{t^2 - a^2})$$

$$(S_G) : \overline{r} = \left(y, \frac{1}{2} \left(e^{\frac{y}{a}} + a^2 e^{-\frac{y}{a}} \right) \cos x, \frac{1}{2} \left(e^{\frac{y}{a}} + a^2 e^{-\frac{y}{a}} \right) \sin x \right)$$

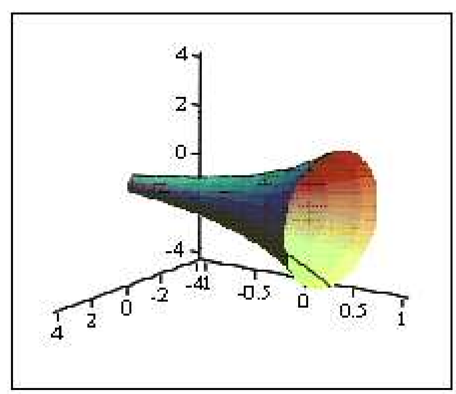
$$(S_m) : \overline{r} = \big(y, e^{K_1 y + K_2} \cdot \cos(-K_1 x + b), e^{K_1 y + K_2} \cdot \sin(-K_1 x + b) \big)$$

(S_m) is the monogenous surface which approximates surface (S_G).

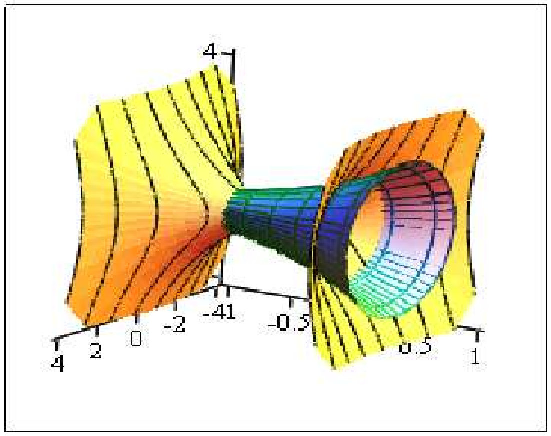
The representations, noted with M for (S_G) and respectively with N for (S_m), were realized by my colleague, computer expert Gheorghe Ardelean, from the Department of Mathematics – Informatics from the North University of Baia Mare, whom I wish to offer my gratitude for his cooperation.



N



M



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