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On a class of multivalent functions defined by Salagean operator

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Abstract

The present paper investigates new subclasses of multivalent functions involving Salagean operator. Coefficient inequalities and other interesting properties of these classes are studied.

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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

(1) $$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic in the open disc $\mathbb{U} = \{z : |z| < 1\}$.

For $f(z) \in \mathcal{A}$, Salagean [1] introduced the following operator:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in \mathbb{N} = 1, 2, 3, ...).$$

We note that,

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \qquad (p \geq 1) \tag{2}$$

which are analytic and p-valent in the open disc $\mathbb{U}$. We can write the following equalities for the functions $f(z) \in \mathcal{A}_p$:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = \frac{z}{p} f'(z) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{j}{p}\right) a_j z^j$$

$$\vdots \qquad\qquad \vdots$$

$$D^n f(z) = D(D^{n-1} f(z)) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{j}{p}\right)^n a_j z^j \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Let $\mathcal{N}_p(m, n, \alpha, \beta)$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfies the inequality

$$Re \left\{ \frac{D^m f(z)}{D^n f(z)} \right\} > \beta \left| \frac{D^m f(z)}{D^n f(z)} - 1 \right| + \alpha.$$

for some $0 \leq \alpha < 1$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and all $z \in \mathbb{U}$.

Special cases of our classes are following:

(i)$\mathcal{N}_1(m,n,\alpha,\beta) = \mathcal{N}_{m,n}(\alpha,\beta)$ which was studied by Eker and Owa [5].

(ii)$\mathcal{N}_1(1,0,\alpha,\beta) = \mathcal{SD}(\alpha,\beta)$ which was studied by Shams at all [3].

(iii) $\mathcal{N}_1(1,0,\alpha,0) = \mathcal{S}^*(\alpha)$ and $\mathcal{N}_1(2,1,\alpha,0) = \mathcal{K}(\alpha)$ which was studied by Silverman [2].

(iv) $\mathcal{N}_1(m,n,\alpha,0) = \mathcal{K}_{m,n}(\alpha)$ which was studied by Eker and Owa [4].

2 Coefficient inequalities for classes $\mathcal{N}_p(m,n,\alpha,\beta)$

Theorem 1. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\sum_{j=2}^{\infty} \Psi_p(m,n,j,\alpha,\beta)\,|a_j| \leq 2(1-\alpha) \tag{3}$$

where

$$\Psi_p(m,n,j,\alpha,\beta) = \left|(1+\alpha)\left(\frac{j}{p}\right)^n - \left(\frac{j}{p}\right)^m\right| + \left((1-\alpha)\left(\frac{j}{p}\right)^n + \left(\frac{j}{p}\right)^m\right) + 2\beta\left|\left(\frac{j}{p}\right)^m - \left(\frac{j}{p}\right)^n\right| \tag{4}$$

for some $\alpha(0 \leq \alpha < 1), \beta \geq 0$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ then $f(z) \in \mathcal{N}_p(m,n,\alpha,\beta)$.

Proof. Suppose that (3) is true for $\alpha(0 \leq \alpha < 1), \beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$. Using the fact that $Re w > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, it suffices to show that

$$(5) \qquad \left|(1-\alpha)D^n f(z) + D^m f(z) - \beta e^{i\theta}\left|D^m f(z) - D^n f(z)\right|\right|$$

$$-\left|(1+\alpha)D^n f(z) - D^m f(z) + \beta e^{i\theta}\left|D^m f(z) - D^n f(z)\right|\right| > 0$$

Substituting for $D^n f(z)$ and $D^m f(z)$ in (5) yields,

$$\left|(1-\alpha)D^n f(z) + D^m f(z) - \beta e^{i\theta}\left|D^m f(z) - D^n f(z)\right|\right|$$
$$-\left|(1+\alpha)D^n f(z) - D^m f(z) + \beta e^{i\theta}\left|D^m f(z) - D^n f(z)\right|\right|$$
$$= \left|(2-\alpha)z^p + \sum_{j=p+1}^{\infty}\left[(1-\alpha)\left(\frac{j}{p}\right)^n + \left(\frac{j}{p}\right)^m\right]a_j z^j - \beta e^{i\theta}\left|\sum_{j=p+1}^{\infty}\left[\left(\frac{j}{p}\right)^m - \left(\frac{j}{p}\right)^n\right]a_j z^j\right|\right|$$
$$-\left|\alpha z^p + \sum_{j=p+1}^{\infty}\left[(1+\alpha)\left(\frac{j}{p}\right)^n - \left(\frac{j}{p}\right)^m\right]a_j z^j + \beta e^{i\theta}\left|\sum_{j=p+1}^{\infty}\left[\left(\frac{j}{p}\right)^m - \left(\frac{j}{p}\right)^n\right]a_j z^j\right|\right|$$

$$\geq (2-\alpha)|z|^p - \sum_{j=p+1}^{\infty}\left|(1-\alpha)\left(\frac{j}{p}\right)^n + \left(\frac{j}{p}\right)^m\right||a_j|\,|z|^j - \beta\left|e^{i\theta}\right|\sum_{j=p+1}^{\infty}\left|\left(\frac{j}{p}\right)^m - \left(\frac{j}{p}\right)^n\right||a_j|\,|z|^j$$

$$-\alpha|z|^p - \sum_{j=p+1}^{\infty}\left|(1+\alpha)\left(\frac{j}{p}\right)^n - \left(\frac{j}{p}\right)^m\right||a_j|\,|z|^j - \beta\left|e^{i\theta}\right|\sum_{j=p+1}^{\infty}\left|\left(\frac{j}{p}\right)^m - \left(\frac{j}{p}\right)^n\right||a_j|\,|z|^j$$

$$\geq 2(1-\alpha) - \sum_{j=p+1}^{\infty}\left[\left|(1+\alpha)\left(\frac{j}{p}\right)^n - \left(\frac{j}{p}\right)^m\right| + \left((1-\alpha)\left(\frac{j}{p}\right)^n + \left(\frac{j}{p}\right)^m\right) + 2\beta\left|\left(\frac{j}{p}\right)^m - \left(\frac{j}{p}\right)^n\right|\right]|a_j|$$

$$\geq 0$$

Example 1. *The function $f(z)$ given by*

$$f(z) = z^p + \sum_{j=p+1}^{\infty}\frac{2(p+1+\delta)(1-\alpha)\epsilon_j}{(j+\delta)(j+1+\delta)\Psi_p(m,n,j,\alpha,\beta)}z^j$$

belongs to the class $\mathcal{N}_p(m,n,\alpha,\beta)$ for $\delta > -p-1$, $0 \leq \alpha < 1$, $\beta \geq 0$, $\epsilon_j \in \mathbb{C}$ and $|\epsilon_j| = 1$.

3 Relation for $\widetilde{\mathcal{N}}_p(m,n,\alpha,\beta)$

In view of Theorem 1, we now introduce the subclass $\widetilde{\mathcal{N}}_p(m,n,\alpha,\beta)$ which consist of functions $f(z) \in \mathcal{A}_p$ whose Taylor-Maclaurin coefficients satisfy the inequality (3). By the coefficient inequality for the class $\widetilde{\mathcal{N}}_p(m,n,\alpha,\beta)$ we see,

Theorem 2. *If $f(z) \in \mathcal{A}_p$, then*

$$\widetilde{\mathcal{N}}_p(m,n,\alpha,\beta_2) \subset \widetilde{\mathcal{N}}_p(m,n,\alpha,\beta_1)$$

for some β_1 and β_2, $0 \leq \beta_1 \leq \beta_2$.

Proof. For $0 \leq \beta_1 \leq \beta_2$ we obtain

$$\sum_{j=p+1}^{\infty} \Psi_p(m,n,j,\alpha,\beta_1)\,|a_j| \;\leq\; \sum_{j=p+1}^{\infty} \Psi_p(m,n,j,\alpha,\beta_2)\,|a_j|\,.$$

Therefore, if $f(z) \in \widetilde{\mathcal{N}}_p(m,n,\alpha,\beta_2)$, then $f(z) \in \widetilde{\mathcal{N}}_p(m,n,\alpha,\beta_1)$. Hence we get the required result.

4 Extreme points

The determination of the extreme points of a family F of univalent functions enables us to solve many extremal problems for F.

Theorem 3. *Let $f_p(z) = z^p$ and*

$$f_j(z) = z^p + \frac{2(1-\alpha)\epsilon_j}{\Psi_p(m,n,j,\alpha,\beta)} z^j \quad (j = p+1, p+2, ... \;;\; |\epsilon_j| = 1).$$

Then $f \in \widetilde{\mathcal{N}}_p(m, n, \alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z),$$

where $\lambda_j > 0$ *and* $\lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j$.

Proof. Suppose that

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z) = z^p + \sum_{j=p+1}^{\infty} \lambda_j \frac{2(1-\alpha)\epsilon_j}{\Psi_p(m, n, j, \alpha, \beta)} z^j$$

Then

$$\sum_{j=p+1}^{\infty} \Psi_p(m, n, j, \alpha, \beta) \left| \frac{2(1-\alpha)\epsilon_j}{\Psi_p(m, n, j, \alpha, \beta)} \lambda_j \right| = \sum_{j=p+1}^{\infty} 2(1-\alpha)\lambda_j$$

$$= 2(1-\alpha) \sum_{j=p+1}^{\infty} \lambda_j$$

$$= 2(1-\alpha)(1-\lambda_p)$$

$$\leq 2(1-\alpha)$$

Thus, $f(z) \in \widetilde{\mathcal{N}}_p(m, n, \alpha, \beta)$ from the definition of the class of $\widetilde{\mathcal{N}}_p(m, n, \alpha, \beta)$.

Conversely, suppose that $f(z) \in \widetilde{\mathcal{N}}_p(m, n, \alpha, \beta)$. Since

$$|a_j| \leq \frac{2(1-\alpha)}{\Psi_p(m, n, j, \alpha, \beta)} \qquad (j = p+1, p+2, ...),$$

we may set

$$\lambda_j = \frac{\Psi_p(m, n, j, \alpha, \beta)}{2(1-\alpha)\epsilon_j} a_j \quad (|\epsilon_j| = 1)$$

and

$$\lambda_p = 1 - \sum_{j=p+1}^{\infty} \lambda_j.$$

Then,

$$f(z) = \lambda_p f_p(z) + \sum_{j=p+1}^{\infty} \lambda_j f_j(z).$$

This completes the proof of theorem.

Corollary 1. *The extreme points of* $\widetilde{\mathcal{N}}_p(m, n, \alpha, \beta)$ *are the functions* $f_p(z) = z^p$ *and*

$$f_j(z) = z^p + \frac{2(1-\alpha)\epsilon_j}{\Psi_p(m,n,j,\alpha,\beta)} z^j \quad (j = p+1, p+2, \ldots \ ; \ |\epsilon_j| = 1). \tag{6}$$

5 Integral means inequalities

Definition 1. (Subordination Principle) *For two functions* f *and* g*, analytic in* $\mathbb{U}$*, we say that the function* $f(z)$ *is subordinate to* $g(z)$ *in* $\mathbb{U}$*, and write*

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$*, analytic in* $\mathbb{U}$ *with*

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \ ,$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In 1925, Littlewood [6] proved the following subordination theorem. (See also Duren [7])

Theorem 4. (Littlewood [6]) *If f and g are analytic in $\mathbb{U}$ with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu} \, d\theta.$$

We will make use of Theorem 5 to prove

Theorem 5. *Let $f(z) \in \widetilde{\mathcal{N}}_p(m, n, \alpha, \beta)$ and supposed that$f_j(z)$ is defined by (6). If there exists an analytic function $w(z)$ given by*

$$\{w(z)\}^{j-p} = \frac{\Psi_p(m, n, j, \alpha, \beta)}{2(1-\alpha)\epsilon_j} \sum_{j=p+1}^{\infty} a_j z^{j-p},$$

then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| f_j(re^{i\theta}) \right|^{\mu} d\theta \qquad (\mu > 0).$$

Proof We must show that

$$\int_0^{2\pi} \left| 1 + \sum_{j=p+1}^{\infty} a_j z^{j-p} \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| 1 + \frac{2(1-\alpha)\epsilon_j}{\Psi_p(m, n, j, \alpha, \beta)} z^{j-p} \right|^{\mu} d\theta.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1 + \sum_{j=p+1}^{\infty} a_j z^{j-p} \prec 1 + \frac{2(1-\alpha)\epsilon_j}{\Psi_p(m, n, j, \alpha, \beta)} z^{j-p}.$$

By setting

$$1 + \sum_{j=p+1}^{\infty} a_j z^{j-p} = 1 + \frac{2(1-\alpha)\epsilon_j}{\Psi_p(m, n, j, \alpha, \beta)} \{w(z)\}^{j-p}$$

we find that

$$\{w(z)\}^{j-p} = \frac{\Psi_p(m, n, j, \alpha, \beta)}{2(1-\alpha)\epsilon_j} \sum_{j=p+1}^{\infty} a_j z^{j-p}$$

which readily yields $w(0) = 0$.

Furthermore, using (3), we obtain

$$
\begin{aligned}
|\{w(z)\}|^{j-p} &= \left| \frac{\Psi_p(m,n,j,\alpha,\beta)}{2(1-\alpha)\epsilon_j} \sum_{j=p+1}^{\infty} a_j z^{j-p} \right| \\
&\leq \frac{\Psi_p(m,n,j,\alpha,\beta)}{2(1-\alpha)\,|\epsilon_j|} \sum_{j=p+1}^{\infty} |a_j|\,|z|^{j-p} \\
&\leq |z| < 1.
\end{aligned}
$$

This completes the proof of the theorem.

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