

Torseforming Vector fields in a 3-dimensional Contact Metric Manifold¹

C.S. Bagewadi, D.G. Prakasha and Venkatesha

Abstract

The object of the present paper is to study torseforming vector field in a 3-dimensional contact metric manifold with $Q\varphi = \varphi Q$. Here we prove that the torseforming vector field in a 3-dimensional contact metric manifold with $Q\varphi = \varphi Q$ is a concircular vector field.

2000 Mathematics Subject Classification: 53D15, 53C25.

Key words: 3-dimensional contact metric manifold, Concircular vector field, Torseforming vector field.

1 Introduction

It is known that the set of metrics associated to the contact form η is very large in contact metric manifolds $(M^{2n+1}, (\varphi, \xi, \eta, g))$. Also one does not know a complete classification, even if the structure is η -Einstein. And also for $n = 1$, we know very little about the geometry of these manifolds [6]. On the other hand if the structure is Sasakian, the Ricci operator Q commutes with φ [3], i.e., $Q\varphi = \varphi Q$, but in general $Q\varphi \neq \varphi Q$. The problem of the characterisation of contact metric manifold with $Q\varphi = \varphi Q$ is open for study. S. Tanno[9], defined a special family of contact metric manifolds by the requirement that ξ belong to the k -nullity distribution of g . The study of torseforming vector fields has a long history starting in 1925

¹Received 3 August 2007

Accepted for publication (in revised form) 10 December 2007

by the work of H.W.Brinkmann [5], P.A. Shirokov [8] and K.Yano [10,11]. These vector fields have been used in many areas of differential geometry, for example in conformal mappings and transformations, geodesic, almost geodesic and holomorphically projective mapping and transformation, and others. With the above background we plan to study the torseforming vector field in Contact metric manifold with $Q\varphi = \varphi Q$ and prove that they are of concircular type.

2 Preliminaries and Known results

Let $M = M^{2n+1}$ be a connected differentiable manifold with contact form η , i.e., a tensor field of type (0,1) satisfying $\eta \wedge (d\eta)^n \neq 0$. It is well known that such a manifold admits a vector field ξ , called the characteristic vector field such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$, for every $X \in \chi(M)$, where $\chi(M)$ being the Lie algebra of the vector fields of M . Moreover, if M admits a Riemannian metric g and a tensor field of type (1,1) such that [2]

$$(1) \quad \varphi^2 = I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \quad \varphi\xi = 0, \quad g(X, \varphi Y) = d\eta(X, Y)$$

we then say that (φ, ξ, η, g) is a contact metric structure. As a consequence of these relation, one has

$$(2) \quad g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \varphi\xi = 0, \quad \eta \cdot \varphi = 0.$$

Denoting by L and R the Lie differentiation and the curvature tensor respectively, we define the operators l and h by

$$(3) \quad lX = R(X, \xi)\xi, \quad hX = \frac{1}{2}(L_\xi X)$$

The (1,1) tensors l and h and self-adjoint and satisfy

$$(4) \quad h\xi = 0, \quad l\xi = 0, \quad trh = trh\varphi = 0.$$

Since h anticommutes with φ , if λ is an eigenvalue of h it corresponds to the eigenvalue $-\lambda$ of φ . If ∇ denotes the Luri-Civita connection and Q is the Ricci operator of g , then

$$(5) \quad \nabla_x \xi = -\varphi X - \varphi hX,$$

$$(6) \quad Tr.l = g(Q\xi, \xi) = 2n - Trh^2,$$

(where Q is the Ricci operator and ∇ the Riemannian connection of g).
A contact manifold is said to be η -Einstein if

$$(7) \quad Q = aI + b\eta \otimes \xi,$$

where a, b are smooth function on M . The sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a ξ -section curvature, while the sectional curvature $K(X, \varphi X)$ is called a φ -sectional curvature. The (k, μ) -nullity distribution of a contact metric manifold for the pair $(k, \mu) \in R^2$, is a distribution

$$(8) \quad \begin{aligned} N(k, \mu) : P &\longrightarrow N_P(k, \mu) = \{Z \in T_P M | R(X, Y)Z \\ &= (kI + \mu h)[g(Y, Z)X - g(X, Z)Y]\} \end{aligned}$$

So, if the characteristic vector field ξ belongs to the (k, μ) -nullity distribution we have,

$$(9) \quad R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

By definition, the Weyl conformal curvature tensor C is given by

$$(10) \quad \begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2}[g(Y, Z)QX - g(X, Z)QY \\ &+ g(QY, Z)X - g(QX, Z)Y] \\ &- \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] \end{aligned}$$

where r is the scalar curvature[1].

For every 3-dimensional Riemannian manifold $C = 0$. So, for the curvature tensor R of 3-dimensional Riemannian manifolds can be written the following formula:

$$(11) \quad \begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X \\ &- g(QX, Z)Y - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Using $\varphi Q = Q\varphi$, (6) and $\varphi\xi = 0$ we have that

$$(12) \quad Q\xi = (Trl)\xi.$$

From (11), using (3) and (12) we have for any X ,

$$(13) \quad lX = QX + (Trl - \frac{S}{2})X + (\frac{S}{2} - 2Trl)\eta(X)\xi.$$

For any X in a contact metric manifold with $Q\varphi = \varphi Q$ we have[4]

$$(14) \quad lX = -\frac{1}{2}(Trl)\varphi^2 X.$$

Substituting (14) in (13) we get

$$(15) \quad QX = \frac{1}{2}(r - Trl)X + \frac{1}{2}(3Trl - r)\eta(X)\xi.$$

From this it follows that

$$(16) \quad S(X, Y) = g(QX, Y) = \frac{1}{2}(r - Trl)g(X, Y) + \frac{1}{2}(3Trl - r)\eta(X)\eta(Y).$$

Now, substituting (15) in (11) we obtain

$$(17) \quad \begin{aligned} R(X, Y)Z &= \{\alpha g(Y, Z) + b\eta(Y)\eta(Z)\}X \\ &\quad - \{\alpha g(X, Z) + b\eta(X)\eta(Z)\}Y \\ &\quad + b\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\xi \end{aligned}$$

where $\alpha = \frac{1}{2} - Trl$.

A vector field ρ defined by $g(X, \rho) = \omega(X)$ for any vector field X is said to be a torseforming vector field ([7],[10]) if

$$(18) \quad (\nabla_X \omega)(Y) = kg(X, Y) + \pi(X)\omega(Y)$$

where k is a non-zero scalar and π is a non-zero 1-form.

3 Torseforming Vector field in a 3-Dimensional Contact Metric manifold with $Q\varphi = \varphi Q$

We consider a unit torseforming vector field $\tilde{\rho}$ corresponding to the vector field ρ . Suppose $g(X, \tilde{\rho}) = T(X)$, then

$$(19) \quad T(X) = \frac{\omega(X)}{\sqrt{\omega(\rho)}}$$

From (18) we get

$$(20) \quad \frac{(\nabla_X \omega)(Y)}{\sqrt{\omega(\rho)}} = \frac{k}{\sqrt{\omega(\rho)}} g(X, Y) + \frac{\pi}{\sqrt{\omega(\rho)}} \omega(Y).$$

Using (19) in the above, we obtain

$$(21) \quad (\nabla_X T)(Y) = \lambda g(X, Y) + \pi(X)T(Y),$$

where $\lambda = \frac{k}{\sqrt{\omega(\rho)}}$

$Y = (\tilde{\rho})$ in (21), we obtain

$$(22) \quad (\nabla_X T)(\tilde{\rho}) = \lambda g(X, \tilde{\rho}) + \pi(X)T(\tilde{\rho})$$

As $T(\tilde{\rho}) = g(\tilde{\rho}, \tilde{\rho}) = 1$, equation (22) reduces to

$$(23) \quad \pi(X) = -\lambda T(X)$$

and hence (21) can be written in the form

$$(24) \quad (\nabla_X T)(Y) = \lambda[g(X, Y) - T(X)T(Y)]$$

which implies T is closed.

Taking covariant differentiation of (24) and using Ricci identity we get

$$-T(R(X, Y)Z) = (X\lambda)[g(Y, Z) - T(Y)T(Z)] - (Y\lambda)[g(X, Z) - T(X)T(Z)]$$

$$(25) \quad +\lambda^2[g(Y, Z)T(X) - g(X, Z)T(Y)]$$

Putting $Z = \xi$ in (25) and using (16), we obtain

$$(26) \quad \begin{aligned} -T \left[\frac{Trl}{2}(\eta(Y)X - \eta(X)Y) \right] &= (X\lambda) = [\eta(Y) - T(Y)T(\xi)] \\ &\quad - (Y\lambda)[\eta(X) - T(X)T(\xi)] \\ &\quad + \lambda^2[\eta(Y)T(X) - \eta(X)T(Y)] \end{aligned}$$

Since $T(\xi) = g(\xi, \tilde{\rho}) = \eta(\tilde{\rho})$, (26) reduces to

$$(27) \quad \begin{aligned} \left[\lambda^2 + \frac{Trl}{2} \right] [\eta(Y)T(X) - \eta(X)T(Y)] &+ [(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] + \\ &+ \eta(\tilde{\rho})[(Y\lambda)T(X) - (X\lambda)T(Y)] = 0. \end{aligned}$$

Putting $X = \tilde{\rho}$ in (27) and as $T(\tilde{\rho}) = g(\tilde{\rho}, \tilde{\rho}) = 1$, we get

$$(28) \quad \left[\lambda^2 + \frac{Trl}{2} + (\tilde{\rho}\lambda) \right] [\eta(Y) - \eta(\tilde{\rho})T(Y)] = 0.$$

Thus we have the following

Lemma 3.1 *If a 3-dimensional contact metric manifold M^3 admits a torse-forming vector field, then the following cases occur*

$$(29) \quad [\eta(Y) - \eta(\tilde{\rho})T(Y)] = 0$$

$$(30) \quad \left[\lambda^2 + \frac{Trl}{2} + (\tilde{\rho}\lambda) \right] = 0$$

We first consider the case where (29) holds good. From (29) we get

$$(31) \quad \eta(Y) = \eta(\tilde{\rho})T(Y).$$

Now $Y = \xi$ implies $1 = (\eta(\tilde{\rho}))^2$ and thus $\eta(\rho) = \pm 1$. So

$$(32) \quad \eta(Y) = \pm T(Y).$$

Using (32) in (5) and then in view of (24), we have

$$(33) \quad -g(\varphi X, Y) - g(\varphi hX, Y) = \pm \lambda[g(X, Y) - T(X)T(Y)].$$

This implies that $\lambda = \pm C$, where C is constant (say). Hence (23) reduces to

$$(34) \quad \pi(X) = \pm CT(X).$$

Since T is closed, π is also closed. Hence we can state:

Lemma 3.2 *The equation (29) implies that the vector field $\tilde{\rho}$ is a concircular vector field*

We next assume the case (30). Then

$$(35) \quad \eta(Y) - \eta(\tilde{\rho})T(Y) \neq 0.$$

From (25), we get

$$(36) \quad -T(QX) = (X\lambda) + (\tilde{\rho}\lambda)T(X) + 2\lambda^2T(X),$$

where $g(QX, Y) = S(X, Y)$.

Put $X = \xi$ in (36) and using (12), we obtain

$$(37) \quad \xi\lambda = -\eta(\tilde{\rho}) \left[\lambda^2 + \frac{Trl}{2} \right].$$

Putting $Y = \xi$ in (27), in virtue of (37) and $T(\xi) = \eta(\tilde{\rho})$ we get

$$(38) \quad X\lambda = - \left[\lambda^2 + \frac{Trl}{2} \right] T(X).$$

From(23) it follows that

$$(39) \quad Y\pi(X) = -[(Y\lambda)T(X) + \lambda(YT(X))].$$

Using (38) in the above equation, we get

$$(40) \quad Y\pi(X) = - \left[- \left\{ \lambda^2 + \frac{Trl}{2} \right\} T(Y)T(X) + \lambda[YT(X)] \right].$$

Also

$$(41) \quad X\pi(Y) = - \left[- \left\{ \lambda^2 + \frac{Trl}{2} \right\} T(X)T(Y) + \lambda[XT(Y)] \right]$$

and

$$(42) \quad \pi([X, Y]) = -\lambda T([X, Y]).$$

From (40), (41) and (42), we obtain

$$(43) \quad d\pi(X, Y) = -\lambda[(dT)(X, Y)].$$

Since T is closed, π is also closed. Thus we have

Lemma 3.3 *The equation (32) implies that the vector field $\tilde{\rho}$ is a Concircular vector field.*

Thus from Lemma 3.2 and Lemma 3.3, we can state the following:

Theorem 3.1 *A torseforming vector field in a 3-Dimensional Contact metric manifold M^3 with $Q\varphi = \varphi Q$ is a Concircular vector field*

References

- [1] C.S. Bagewadi and Venkatesha, *Torseforming vector field in a Trans-Sasakian manifold*, Differential Geometry-Dynamical System., 8 (2006), 23-28
- [2] C.S. Bagewadi, D.G. Prakasha and Venkatesha, *On Pseudo projective curvature tensor of a Contact Metric Manifold*, In Press, SUT J.Math.
- [3] D.E. Blair, *Contact manifold in Riemannian Geometry*, Lecture Notes in Mathematics, 509, Springer-Verlag, Berlin, (1976).
- [4] D.E. Blair, T. Koufogiorgos and R. Sharma, *Clasification of 3-dimensional contact metric manifolds with $Q\varphi = \varphi Q$* , Kodai Math. J., 13(1990),391-401
- [5] H.W. Brinkmann *Enstein spaces wich mapped conformally on each other* Math. Ann., 94(1925)
- [6] D. Perrone, *Torsion and critical metrics on contact three manifolds*, Kodai Math. J., 13(1990), 88-100.

- [7] J.A. Scouten, *Ricci calculus*, Springer-Verlag, 2nd edition, 1954.
- [8] P.A. Shirokov, *Collected works of geometry* Kazan Univ. Press, 1966.
- [9] S. Tanno, *Ricci Curvature of contact Riemannian manifolds*, Tohoku Math. J.,40(1988), 441-448.
- [10] K. Yano, *On the torseforming direction in Riemannian spaces*, Proc. Imp. Acad. Tokyo., 20(1994), 340-345
- [11] K. Yano, *Concircular geometry I*, Proc. Imp. Acad. Tokyo., 16,(1940), 195-200.

Department of Mathematics and Computer Science,
Kuvempu University,
Jnana Sahyadri-577 451,
Shimoga, Karnataka,
INDIA.

E-mail: prof_bagewadi@yahoo.co.in,

E-mail: prakashadg@gmail.com

E-mail: vens.2003@rediffmail.com.