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Some new properties in q-Calculus[1]

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Abstract

Our aim is to present new proofs of some results from q - Calculus. These results occur in many applications as physics, quantum theory, number theory, statistical mechanics, etc. Our proofs are based only on interpolation theory ([4]).

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1 Introduction

1. In the following, for $q \in \mathbb{C}\backslash\{1\}$, let us denote $[\mathbf{n}]_\mathbf{q} = \frac{q^n-1}{q-1}$, and for $n \in \mathbb{N}$

$$(1) \qquad [\mathbf{n}]_\mathbf{q}! = \begin{cases} 1 & \text{if} \quad n = 0 \\ [1]_q[2]_q \dots [n]_q & \text{if} \quad n = 1, 2, \dots, \end{cases},$$

$$(2) \qquad \begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}_\mathbf{q} = \frac{[n]_q!}{[k]_q![n-k]_q!} \text{ for } k \in \{0, 1, \dots, n\}.$$

The numbers $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}_q$, $0 \le k \le n$, are called Gaussian - coefficients. These coefficients satisfy the q - Pascal identities

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$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q = \left[\begin{array}{c} n \\ k \end{array}\right]_q q^k + \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q = \left[\begin{array}{c} n \\ k \end{array}\right]_q + \left[\begin{array}{c} n \\ k-1 \end{array}\right]_q q^{n+1-k} \tag{3}$$

where $k \in \{1, 2, \ldots, n\}$, $1 \leq k \leq n$.

Let q be an arbitrary complex number, $q \neq 1$, and $\mathcal{D} = \mathcal{D}_q \subseteq \mathbb{C}$ such that $x \in \mathcal{D}$ implies $qx \in \mathcal{D}$.

Definition 1.1 *Any function $f : \mathcal{D}_q \to \mathbb{C}$ is called q - derivable, under restriction that if $0 \in \mathcal{D}_q$, there is $f'(0)$.*

Definition 1.2 *A function $f : \mathcal{D}_q \to \mathbb{C}$ is q - derivable of order n, iff $0 \in \mathcal{D}_q$, implies that $f^{(n)}(0)$ exists.*

For a function $f : \mathcal{D}_q \to \mathbb{C}$ which is q - derivable its q - derivative $D_q f$ was defined in 1908 by F.H. Jackson [3], in the following way

$$(\mathbf{D_q f})(\mathbf{x}) = \frac{f(x) - f(qx)}{(1-q)x}, \quad q \in \mathbb{C} \backslash \{1\}. \tag{4}$$

For instance, $D_q(x^n) = [n]_q x^{n-1}$, and the linear operator $f \to D_q f$ satisfies (see [1])

$$(D_q fg)(x) = g(x)(D_q f)(x) + f(qx)(D_q g)(x). \tag{5}$$

$$\left(D_q \frac{f}{g}\right)(x) = \frac{g(x)(D_q f)(x) - f(x)(D_q g)(x)}{g(x)g(qx)}, \quad g(x)g(qx) \neq 0.$$

In the following by $[x_0, x_1, \ldots, x_n; f]$ we denote the divided differences at a system of distinct points $x_0, x_1, \ldots, x_n$. More precisely

$$[\mathbf{x_0}, \mathbf{x_1}, \ldots, \mathbf{x_n}; \mathbf{f}] = \\ = \sum_{k=0}^{n} \frac{f(x_k)}{(x_k - x_0) \cdot \ldots \cdot (x_k - x_{k-1})(x_k - x_{k+1}) \cdot \ldots \cdot (x_k - x_n)}. \tag{6}$$

2 Main theoretical results

We consider the points $x_k = q^k x$, $k = 0, 1, \ldots, n$, ($x_0 = x,\ x_1 = qx, \ldots, x_n = q^n x$). In this case, we have

Theorem 2.1 *Let $q \in \mathbb{C}\backslash\{1\}$ and $f : \mathcal{D}_q \to \mathbb{C}$. Then on the knots $x_k = q^k x$, we have the representation*

$$[\mathbf{x}, \mathbf{qx}, \ldots, \mathbf{q^{n-1}x}, \mathbf{q^n x}; \mathbf{f}] = \tag{7}$$
$$= \frac{1}{q^{\frac{n(n-1)}{2}}[n]_q! x^n (q-1)^n} \sum_{k=0}^{n} (-1)^k \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} f(q^{n-k}x).$$

Proof. For $0 \le j < k$ we have $x_k - x_j = xq^j(q^{k-j} - 1)$. Therefore $x_k - x_j = xq^j(q-1)[k-j]_q$, $0 \le j \le k-1$. When $k < j \le n$ $x_k - x_j = xq^k(1-q)[j-k]_q$. Further

$$\prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j) = \prod_{j=0}^{k-1} (x_k - x_j) \cdot \prod_{j=k+1}^{n} (x_k - x_j) =$$

$$= x^n q^{\frac{k(k-1)}{2}} (q-1)^k [k]_q [k-1]_q \cdot \ldots \cdot [1]_q \cdot\ q^{n-k} (1-q)^{k(n-k)} [1]_q \cdot \ldots \cdot [n-k]_q =$$

$$\overset{(1)}{=} (-1)^k (1-q)^n x^n q^{k(n-k)+\frac{k(k-1)}{2}} [k]_q! [n-k]_q!.$$

Replacing the product in the equality (6) and using (2) we obtain

$$[x_0, x_1, \ldots, x_n; f] = \frac{1}{[n]_q! x^n (q-1)^n} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{f(xq^k)}{(-1)^{n-k} q^{k(n-k)-\frac{k(k-1)}{2}}}.$$

This completes the proof.

If $D_q^0 = I$, $D_q^1 = D_q$, $D_q^k = D^q D_q^{k-1}$, a representation of operator D_q^n is given by

Theorem 2.2 *Let $f : \mathcal{D}_q \to \mathbb{C}$ is q - derivable of order n, then*

$$(D_q^n f)(x) = (q-1)^{-n} x^{-n} q^{-\binom{n}{2}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q (-1)^k q^{\binom{k}{2}} f(q^{n-k}x). \tag{8}$$

Proof. Let us remark that

$$(D_q f)(x) = [x, qx; f] = [1]_q [x, qx; f],$$

and

$$(D_q^2 f)(x) = \frac{(D_q f)(x) - (D_q f)(qx)}{(1-q)x} = \frac{qf(x) - (q+1)f(qx) + f(q^2x)}{(1-q)^2 q x^2} =$$

$$= \frac{(q+1)}{q(1-q)^2x^2}\left[\frac{q}{q+1}f(x) - f(qx) + \frac{1}{q+1}f(q^2x)\right] =$$

$$= (q+1)\left[\frac{f(x)}{x^2(1-q)(1-q^2)} - \frac{f(qx)}{x^2q(1-q)^2} + \frac{f(q^2x)}{x^2q(1-q)(1-q^2)}\right] =$$

$$= [2]_q [x, qx, q^2x; f].$$

By induction, let us prove that $(D_q^n f)(x) = [n]_q [x, qx, \ldots, q^n x; f]$. Assume that the formula is proved for $n = m$. Then it is also true for $n = m+1$, because

$$(D_q^{m+1} f)(x) = (D_q(D_q^n f))(x) \overset{by\ (4)}{=} \frac{(D_q^m f)(x) - (D_q^m f)(qx)}{(1-q)x} =$$

$$= \frac{[m]_q! [x, qx, \ldots, q^m x; f] - [m]_q! [qx, q^2x, \ldots, q^{m+1}x; f]}{(1-q)x} =$$

$$= [m]_q! \frac{q^{m+1}x - x}{(1-q)x} [x, qx, \ldots, q^{m+1}x; f] = [m+1]_q! [x, qx, \ldots, q^{m+1}x; f],$$

where we used formula

$$[x_0, x_1, \ldots, x_n; \cdot] = \frac{[x_0, \ldots, x_{n-1}; \cdot] - [x_1, \ldots, x_n; \cdot]}{x_n - x_0}.$$

On the other hand

$$[x, qx, \ldots, q^{n-1}x, q^n x; f] =$$
$$= \frac{1}{q^{\frac{n(n-1)}{2}} [n]_q! x^n (q-1)^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} f(q^{n-k}x).$$

Therefore

$$(D_q^n)(x) = [n]_q [x, qx, \ldots, q^n x; f] =$$
$$= \frac{1}{q^{\binom{n}{2}} x^n (q-1)^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(q^{n-k}x). \tag{9}$$

Other proof may be performed in the following way. For $n = 1$ we find formula (4), that is the definition of q - difference operator. We presume that formula (8) is true for $n = m$. Then it is true for $n = m + 1$ too, because

$$(D_q^{m+1} f)(x) = D_q \left((q-1)^{-m} x^{-m} q^{-\binom{m}{2}} \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array} \right]_q (-1)^k q^{\binom{k}{2}} f(q^{m-k}x) \right) \overset{by\ (5)}{=}$$

$$= (q-1)^{-m} q^{-\binom{m}{2}} \left(-\frac{q^m - 1}{q-1} q^{-m} x^{-m-1} \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array} \right]_q (-1)^k q^{\binom{k}{2}} f(q^{m-k}x) + \right.$$

$$\left. + q^{-m} x^{-m} \sum_{k=0}^{m} \left[\begin{array}{c} m+1 \\ k \end{array} \right]_q (-1)^k q^{\binom{k}{2}} f(q^{m+1-k}x) \right) \overset{by\ (3)}{=}$$

$$= (q-1)^{-m-1} x^{-m-1} q^{-\binom{m}{2}} \sum_{k=0}^{m+1} \left[\begin{array}{c} m+1 \\ k \end{array} \right]_q (-1)^k q^{\binom{k}{2}} f(q^{m+1-k}x).$$

Theorem 2.3 *Let $q \in \mathbb{C} \backslash \{1\}$ and k be fixed in $\{2, 3, \ldots, p\}$, If $q^k = 1$ and $f : \mathcal{D}_q \to \mathbb{C}$ is q - derivable of order p, then for all x, $(D_q^p f)(x) = 0$.*

Proof. From (9) we have

$$(D_q^p f)(x) = [p]_q! [x_0, x_1, \ldots, x_p; f],$$

where $x_j = q^j x$, $j = 0, 1, \ldots, p$. According to our hypothesis we have $q^k = 1$ for k fixed in $\{2, 3, \ldots, p\}$. Because

$$[p]_q! = \frac{(1-q)(1-q^2) \cdot \ldots \cdot (1-q^k) \cdot \ldots \cdot (1-q^p)}{(1-q)^p}$$

we have $[p]_q! = 0$, and the proposition is proved.

Theorem 2.4 *Let $f : \mathcal{D}_q \to \mathbb{C}$ is q - derivable of order n. Then we have the following representation*

$$f(q^n x) = \sum_{k=0}^{n} (q-1)^k x^k q^{\binom{k}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_q (D_q^k f)(x). \qquad (10)$$

Proof. Starting from Newton's formula we have:

$$f(z)=\sum_{k=0}^{n-1}(z-x_0)(z-x_1)\dots(z-x_{k-1})[x_0,x_1,\dots,x_k;f]+$$

$$+(z-x_0)(z-x_1)\dots(z-x_{n-1})[x_0,x_1,\dots,x_{n-1},z;f].$$

Replacing $x_k=q^kx$, $k=0,1,\dots,n-1$ in the above formula and $z=q^nx$ and using formula (7) we obtain succesively

$$f(q^nx)=\sum_{k=0}^{n-1}(q^nx-x)(q^nx-qx)\cdot\ldots\cdot(q^nx-q^{k-1})\frac{(D_q^kf)(x)}{(-1)^k[k]_q!}+$$

$$+(q^nx-x)(q^nx-qx)\cdot\ldots\cdot(q^nx-q^{n-1})\frac{(D_q^nf)(x)}{(-1)^n[n]_q!}=$$

$$=\sum_{k=0}^{n}(q^nx-x)(q^nx-qx)\cdot\ldots\cdot(q^nx-q^{k-1})\frac{(D_q^kf)(x)}{(-1)^k[k]_q!}=$$

$$=\sum_{k=0}^{n}x^kq^{\binom{k}{2}}\frac{(-1)^n\dfrac{(q^n-1)\cdot\ldots\cdot(q-1)}{(1-q)^n}}{(-1)^{n-k}\dfrac{(q^{n-k}-1)\cdot\ldots\cdot(q-1)}{(1-q)^{n-k}(1-q)^k}[k]_q!}(D_q^kf)(x)=$$

$$=\sum_{k=0}^{n}(q-1)^kx^kq^{\binom{k}{2}}\frac{[n]_q!}{[n-q]_q![k]_q!}(D_q^kf)(x).$$

Corollary 2.5 *Let $q\in\mathbb{C}\backslash\{1\}$ and $f:\mathcal{D}_q\to\mathbb{C}$ is q - derivable of order n. Then the following identity holds*

$$f(x)=\sum_{k=0}^{n}(1-q)^kx^kq^{\binom{k+1}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\left(D_q^kf\right)\left(q^{n-k}x\right)\tag{11}$$

Proof. We start with following equalities ($k\in\{0,1,\dots,n\}$):

$$[x]_{\frac{1}{q}}=\frac{q^{\frac{1}{x}}-1}{\frac{1}{q}-1}=q^{1-x}[x]_q,$$

$$[n]_{\frac{1}{q}}! = [1]_{\frac{1}{q}} \dots [n]_{\frac{1}{q}} =$$

$$= q^{1-1}[1]_q q^{1-2}[2]_q \dots q^{1-n}[n]_q = q^{n-(1+2+\ldots+n)}[n]_q! = q^{-\binom{n}{2}}[n]_q!,$$

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{\frac{1}{q}} = \frac{[n]_{\frac{1}{q}}!}{[k]_{\frac{1}{q}}![n-k]_{\frac{1}{q}}!} = \frac{q^{-\binom{n}{2}}}{q^{-\binom{k}{2}}q^{-\binom{n-k}{2}}} \left[\begin{array}{c} n \\ k \end{array}\right]_q,$$

On the other hand

$$(D_{\frac{1}{q}}f)(x) = \frac{f\left(\frac{x}{q}\right) - f(x)}{(1-q)x} q = q(D_q f)\left(\frac{x}{q}\right),$$

$$(D^2_{\frac{1}{q}}f)(x) = \frac{q\left[q(D_q f)\left(\frac{x}{q}\right) - q(D_q f)\left(\frac{x}{q^2}\right)\right]}{(q-1)x} = q^2(D_q^2 f)\left(\frac{x}{q^2}\right).$$

By induction, let us prove that

$$(D^n_{\frac{1}{q}}f)(x) = q^n(D_q^n f)\left(\frac{x}{q^n}\right). \tag{12}$$

Assume that the formula is proved for $n = m$. Then

$$(D^{m+1}_{\frac{1}{q}}f)(x) = (D_{\frac{1}{q}}(D^m_{\frac{1}{q}}f))(x) = \frac{(D^m_{\frac{1}{q}}f)(x) - (D^m_{\frac{1}{q}}f)\left(\frac{x}{q}\right)}{\left(1-\frac{1}{q}\right)x} =$$

$$= \frac{q}{(q-1)x}\left[q^m(D_q^m f)\left(\frac{x}{q^m}\right) - q^m(D_q^m f)\left(\frac{x}{q^{m+1}}\right)\right] =$$

$$= q^{m+1}(D_q^{m+1}f)\left(\frac{x}{q^{m+1}}\right).$$

In (10) replacing $q \to \frac{1}{q}$

$$f\left(\frac{x}{q^n}\right) = \sum_{k=0}^{n} \left(\frac{1}{q} - 1\right)^k \frac{1}{q^{\binom{k}{2}}} \frac{q}{q^{-\binom{k}{2}}q^{-\binom{n-k}{2}}} \left[\begin{array}{c} n \\ k \end{array}\right]_q (D^k_{\frac{1}{q}}f)(x)$$

if in the above formula $x \rightsquigarrow q^n x$ we obtain

$$f(x) = \sum_{k=0}^{n} \frac{(1-q)^k}{q^k} q^{nk} x^k q^{-kn+\frac{k^2}{2}+\frac{k}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(D^k_{\frac{1}{q}} f \right) (q^n x) =$$

$$= \sum_{k=0}^{n} (1-q)^k x^k q^{\binom{k}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(D^k_{\frac{1}{q}} f \right) (q^n x) \overset{by\ (12)}{=}$$

$$= \sum_{k=0}^{n} (1-q)^k x^k q^{\binom{k}{2}} q^k \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(D^k_q f \right) (q^{n-k} x).$$

Therefore we have obtained representation (11).

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