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Some Dunwoody parameters and cyclic presentations[1]

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Abstract

In this paper, we found the cyclically presented groups obtained from the word w generated with some Dunwoody parameters.

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1 Introduction

Let F_n be the free group on free generators $x_0, x_1, x_2, \ldots, x_{n-1}$. Let $\theta{:}F_n \to F_n$ be the automorphism such that

$$\theta(x_i) = x_{i+1},\ i = 0, 1, \ldots, n-2,\ \theta(x_{n-1}) = x_0.$$

For $w \in F_n,\ G_n(w)$ is defined as $G_n(w) = F_n/R$ where R is the normal closure in F_n of the set

$$\{w, \theta(w), \theta^2(w), \ldots, \theta^{n-1}(w)\} \text{ [1]}.$$

For a reduced word $w \in F_n$, the cyclically presented group $G_n(w)$ is given by

$$G_n(w) = < x_0, x_1, \ldots, x_{n-1} |\ w, \theta(w), \theta^2(w), \ldots, \theta^{n-1}(w) > \text{ [2]}.$$

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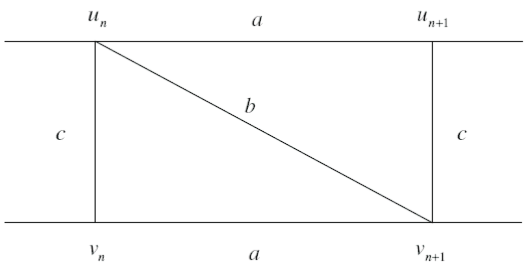

Figure 1

Definition 1. *A group G is said to have a cyclic presentation if $G \cong G_n(w)$ for some n and w [3].*

Definition 2. *A generalized Sieradski group is defined by the cyclic presentation*

$$S(r,n) =< x_1x_2, \ldots, x_n|\ x_ix_{i+2}\ldots x_{i+2r-2} = x_{i+1}x_{i+3}\ldots x_{i+2r-3} >$$

(indices are again modulo n) for any two positive integers r and $n \geq 2$. For $r = 2$, these $S(r,n)$ are the Sieradski groups [4].

Let a, b, c, n be integers such that $n > 0$, $a, b, c \geq 0$ and $a + b + c > 0$. Let $\bar{\tau}(a, b, c)$ be the graph shown in Figure 1. This is an infinite graph with an automorphism θ such that $\theta(u_n) = u_{n+1}$ and $\theta(v_n) = v_{n+1}$. The labels indicate the number of edges joining a pair of vertices. Thus, there are a edges joining u_1 and u_2. We see that the $\bar{\tau}(a, b, c)$ is d-regular where $d = 2a + b + c$. Let $\tau_n = \tau_n(a, b, c)$ denote the graph obtained from $\bar{\tau}(a, b, c)$ by identifying all edges and vertices in each orbit of θ^n. Thus τ_n has $2n$ vertices [5].

We say that the 6-tuple (a, b, c, r, s, n) has property M if it corresponds to the Heegaard diagram of a 3-manifold. An algorithm determining which 6-tuples have property M is now described. Put $d = 2a + b + c$ and let

$$X = \{-d, -d + 1, \ldots, -1, 1, 2, \ldots, d\}.$$

Let α, β be the permutations of X defined as follows:

$\alpha = (1, d)(2, d-1)\ldots(a, d-a+1)(a+1, -a-c-1)(a+2, -a-c-2)\ldots$
$(a + b, -a - c - b)(a + b + 1, -a - 1)(a + b + 2, -a - 2)\ldots$
$(a + b + c, -a - c)(-1, -d)$

and

$$\beta(j) = \begin{cases} -(j+r) & \text{if } j > 0 \text{ and } j + r \leq d \ \text{ or } \ j < 0 \ \text{ and } j + r < 0 \\ -(j+r-d) & \text{if } j + r \geq 0 \end{cases}$$

The following theorem characterizes the 6-tuples (a, b, c, r, s, n) that have property M. Detail and the proof of this theorem can be found in [5].

Theorem 1.1. *Let $d = 2a + b + c$ be odd. The 6-tuple (a, b, c, r, s, n) has property M if and only if the following two conditions hold simultaneously:*

- *$\alpha\beta$ has two cycles of length d*
- *$ps + q \equiv 0 \pmod{n}$.*

Here p is the difference between the number of arrows pointing down the page and the number of arrows pointing up, whereas q is the number of arrows pointing from left to right minus the number of arrows pointing from right to left in the oriented path determined by $\alpha\beta$. The entries in the first cycle of $\alpha\beta$ contain one vertex from each line segment of the diagram. There exists an integer s such that $ps + q \equiv 0 \pmod{n}$. The first cycle of $\alpha\beta$ and the value of s can also be used to calculate the word w of the corresponding cyclic presentation.

2 Materials and Methods

We can now state our theorems:

Theorem 2.1. *The cyclically presented groups obtained from the word w generated with Dunwoody parameters $(1, b, 0, 2)$ are isomorphic to the groups $S((d+1)/2, d)$ when b is an odd positive integer and $d = 2a + b + c$.*

Proof. Suppose for now that $d > 3$. In this case, there are 2 horizontal arcs and b diagonal arcs. Thus, the terms in the first cycle of $\alpha\beta$ for 6-tuple $(1, b, 0, 2, s, n)$ have the following form

$$(1, -2, -4, \ldots, -d+1, -1, b, b-2, b-4, \ldots, 3).$$

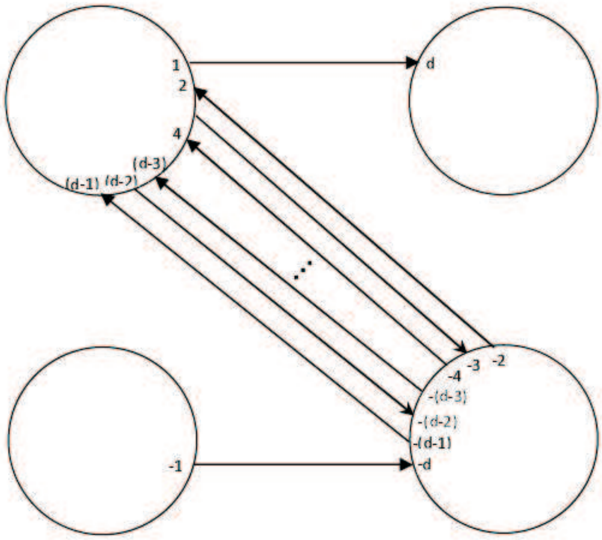

Figure 2. Heegaard diagram for the 6-tuble $(1, b, 0, 2, s, n)$

According to Figure 2, for the 6-tuple $(1, b, 0, 2, s, n)$, since always $p = -1$ and $q = 1$, from $ps + q \equiv 0 \pmod{n}$, s always takes value 1. Thus, the defining word w corresponding to the 6-tuple $(1, b, 0, 2, s, n)$, calculated using the first cycle of $\alpha\beta$ and the value of s, has the following form

$$x_{d-1}^{-1} x_{d-3}^{-1} x_{d-5}^{-1} \cdots x_{d-b}^{-1} x_0^{-1} x_1 x_3 x_5 \cdots x_b \tag{1}$$

For this reduced word w, the cyclically presented group $G_d(w)$ is

$$\begin{aligned} G_d(w) &= G_d(x_{d-1}^{-1} x_{d-3}^{-1} x_{d-5}^{-1} \cdots x_{d-b}^{-1} x_0^{-1} x_1 x_3 x_5 \cdots x_b) \\ &= < x_1 x_2, \ldots, x_d | \ x_i x_{i+2} \cdots x_{i+d-5} x_{i+d-3} x_{i+d-1} \\ &= x_{i+1} x_{i+3} x_{i+5} \cdots x_{i+b} > \end{aligned}$$

where subscripts are understood to be reduced modulo d to lie in the set $\{1, 2, \ldots, d\}$. It can be easily seen that the groups $G_d(w)$ have exactly the same presentation as the groups, where

$$S(r, n) =< x_1 x_2, \ldots, x_n | \ x_i x_{i+2} \ldots x_{i+2r-2} = x_{i+1} x_{i+3} \ldots x_{i+2r-3} >$$

(indices are modulo n) for any two positive integers r and $n \geq 2$, given by Definition 2. We get

$$x_{i+d-1} = x_{i+2r-2}$$

from corresponding terms of $G_d(w)$ and $S(r,n)$ and also $i+d-1 = i+2r-2$. Thus, $r = (d+1)/2$.
Assume now that $d = 3$. Fort his case, the terms in the first cycle of $\alpha\beta$ of the 4-tuple $(1,1,0,2)$ are

$$(1, -2, -4, \dots, -d+1, -1).$$

Notice that when $d = 3$, from (1), the defining word w can be written as $w = x_2^{-1}x_0^{-1}x_1$. For this reduced word w, the cyclically presented group $G_3(w)$ is

$$G_3(w) = G_3(x_2^{-1}x_0^{-1}x_1) =< x_0, x_1, x_2 | x_i x_{i+2} = x_{i+1},\ i \equiv 0 \pmod 3 > .$$

This is $S((d+1)/2, d)$, so the proof is complete.

Theorem 2.2. *The cyclically presented group obtained from the word w generated with Dunwoody parameters $(1, b, 0, d-2)$ has the cyclic presentation*

$$< x_1 x_2, \dots, x_d |\ x_{i+d-1} x_{i+d-3} x_{i+d-5} \cdots x_{i+2} x_i = x_{i+b} x_{i+b-2} \cdots x_{i+5} x_{i+3} x_{i+1} >$$

when b is an odd positive integer and $d = 2a + b + c$.

Proof. In this case, there are 2 horizontal arcs and b diagonal arcs. Thus, the terms in the first cycle of $\alpha\beta$ for 6-tuple $(1, b, 0, d-2, s, n)$ have the following form

$$(1, -b, -b+2, -b+4, \dots, -3, -1, 2, 4, \dots, d-1).$$

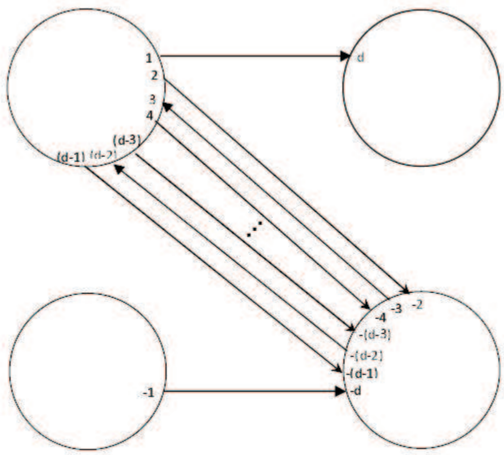

Figure 3. Heegaard diagram for the 6-tuble $(1, b, 0, d-2, s, n)$

According to Figure 3, for the 6-tuple $(1, b, 0, d-2, s, n)$, since always $p = 1$ and $q = 3$, from $ps + q \equiv 0 \pmod{n}$, s always takes value -3. Thus, the defining word w corresponding to the 6-tuple $(1, b, 0, d-2, s, n)$, calculated using the first cycle of $\alpha\beta$ and the value of s, has the following form

$$x_1^{-1}x_3^{-1}x_5^{-1}\cdots x_b^{-1}x_{d-1}x_{d-3}x_{d-5}\cdots x_{d-b}x_0.$$

For this reduced word w, the cyclically presented group $G_d(w)$ is

$$\begin{aligned} G_d(w) &= G_d(x_1^{-1}x_3^{-1}x_5^{-1}\cdots x_b^{-1}x_{d-1}x_{d-3}x_{d-5}\cdots x_{d-b}x_0) \\ &= < x_1x_2,\ldots,x_d|\ x_{i+d-1}x_{i+d-3}x_{i+d-5}\cdots x_{i+2}x_i = \\ &= x_{i+b}x_{i+b-2}\cdots x_{i+5}x_{i+3}x_{i+1} > \end{aligned}$$

where all indices are modulo d. This completes the proof.

It is easy to see that the cases $(1, b, 0, 2)$ and $(1, 0, b, 2)$, and $(1, b, 0, d-2)$ and $(1, 0, b, d-2)$, where b is an odd positive integer, are really the same.

Theorem 2.3. *The cyclically presented group obtained from the word w generated with Dunwoody parameters $(a, 1, 0, a)$ has the cyclic presentation*

$$< x_1x_2,\ldots,x_d|\ x_{i+d-1}x_i = x_{i+d-2}x_{i+d-3}^{-1}x_{i+d-4}\cdots x_{i+3}x_{i+2}^{-1}x_{i+1} >$$

when a is a positive integer and $d = 2a + b + c$.

Proof . In this case, there are $2a$ horizontal arcs and 1 diagonal arc. Thus, the terms in the first cycle of $\alpha\beta$ for the 6-tuple $(a, 1, 0, a, s, n)$ have the following form

$$(1, -a, -a+1, 3, -a+2, \ldots, -2, d-a-1, -1, d-a).$$

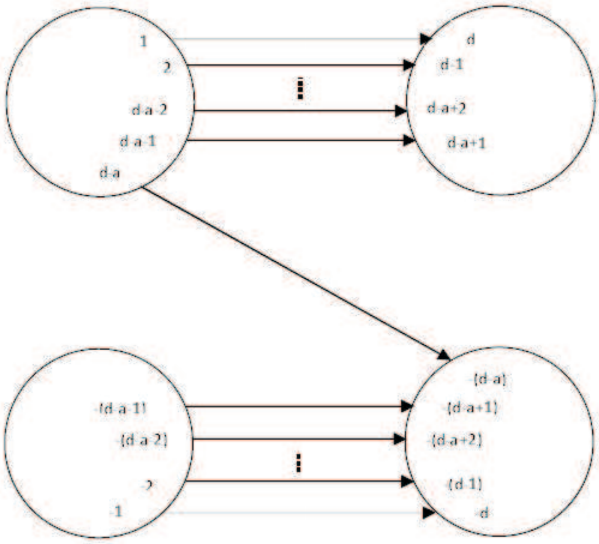

Figure 4. Heegaard diagram for the 6-tuble $(a, 1, 0, a, s, n)$

According to Figure 4, for the 6-tuple $(a, 1, 0, a, s, n)$, since always $p = 1$ and $q = d$, from $ps+q \equiv 0 \pmod{n}$, s always takes value -d. Thus, the defining word w corresponding to the 6-tuple $(a, 1, 0, a, s, n)$, calculated using the first cycle of $\alpha\beta$ and the value of s, has the following form

$$x_1^{-1}x_2x_3^{-1}\cdots x_{d-4}^{-1}x_{d-3}x_{d-2}^{-1}x_{d-1}x_0.$$

For this reduced word w, the cyclically presented group $G_d(w)$ is

$$\begin{aligned} G_d(w) &= G_d(x_1^{-1}x_2x_3^{-1}\cdots x_{d-4}^{-1}x_{d-3}x_{d-2}^{-1}x_{d-1}x_0) \\ &= < x_1x_2,\ldots,x_d|\ x_{i+d-1}x_i = x_{i+d-2}x_{i+d-3}^{-1}x_{i+d-4}\cdots x_{i+3}x_{i+2}^{-1}x_{i+1} > \end{aligned}$$

where all indices are modulo d. We are done.

Theorem 2.4. *The cyclically presented group obtained from the word w generated with Dunwoody parameters $(a, 1, 0, a+1)$ has the cyclic presentation*

$$< x_1x_2,\ldots,x_d|\ x_{i+1}x_{i+2}^{-1}x_{i+3}\cdots x_{i+d-4}x_{i+d-3}^{-1}x_{i+d-2} = x_ix_{i+d-1} >$$

when a is a positive integer and $d = 2a+b+c$.

Proof. In this case, there are $2a$ horizontal arcs and 1 diagonal arc. Thus, the terms in the first cycle of $\alpha\beta$ for the 6-tuple $(a, 1, 0, a + 1, s, n)$ have the following form

$$(1, -a - 1, -1, a, -2, a - 1, \ldots, 3, -a + 1, 2, -a).$$

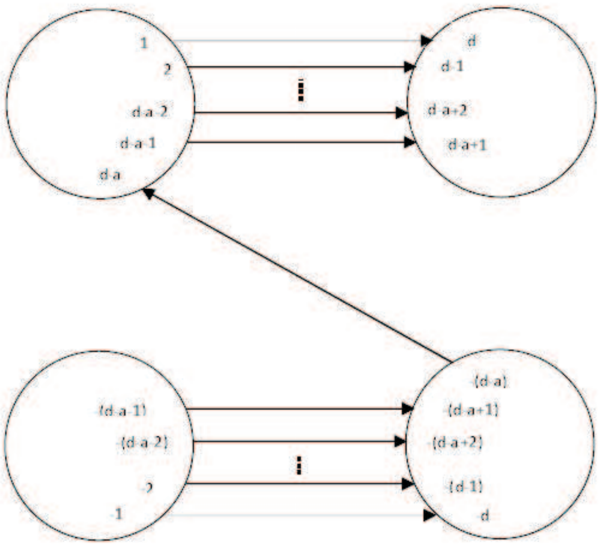

Figure 5. Heegaard diagram for the 6-tuble $(a, 1, 0, a + 1, s, n)$

According to Figure 5, for the 6-tuple $(a, 1, 0, a+1, s, n)$, since always $p = -1$ and $q = d-2$, from $ps+q \equiv 0 \pmod{n}$, s always takes value $d-2$. Thus, the defining word w corresponding to the 6-tuple $(a, 1, 0, a + 1, s, n)$, calculated using the first cycle of $\alpha\beta$ and the value of s, has the following form

$$x_{d-1}^{-1} x_0^{-1} x_1 x_2^{-1} x_3 \cdots x_{d-4} x_{d-3}^{-1} x_{d-2}.$$

For this reduced word w, the cyclically presented group $G_d(w)$ is

$$\begin{aligned} G_d(w) &= G_d(x_{d-1}^{-1} x_0^{-1} x_1 x_2^{-1} x_3 \cdots x_{d-4} x_{d-3}^{-1} x_{d-2}.) \\ &= < x_1 x_2, \ldots, x_d |\ x_{i+1} x_{i+2}^{-1} x_{i+3} \cdots x_{i+d-4} x_{i+d-3}^{-1} x_{i+d-2} = x_i x_{i+d-1} > \end{aligned}$$

where all indices are modulo d. This completes the proof.

It is easy to see that the cases $(a, 1, 0, a)$ and $(a, 0, 1, a)$, and $(a, 1, 0, a+1)$ and $(a, 0, 1, a + 1)$, where a is a positive integer, are really the same.

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