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A note of the Conjectured of Sierpinski on triangular numbers[1]

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Abstract

Recently, Bennett arononled that he proved a conjecture of Sierpinski on triangular numbers. In this paper, we firstly modified the mistakes in reference [7] of Bennett and [8] of Chen and Fang, and then using Störmer's theorem of the solutions of Pell equation, and a deep result of primitive divisor of Bilu, Hanrot and Voutier, we proved that there do not exist four distinct triangular numbers in geometric progression $\{AQ^r\}_{r=1}^{\infty}$. Therefore we totally solved the question of Sierpinski on triangular numbers.

Key words: triangular number, geometric progression, the question of Sierpinski, Pell equation, primitive divisor.

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1 Introduction

Let $\mathbb{Q}, \mathbb{N}, \mathbb{P}$ be the sets of all rational number, positive integers and primes. Let $n \in \mathbb{N}$, let T_n be the nth triangular number, then

$$T_n = \frac{1}{2}n(n+1) \tag{1}$$

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The study of triangular number problem is very active so far (see [1-5]). For example, power number in triangular numbers, Fibonacci number in triangular numbers, Lucas number in triangular numbers etc (see [2,3]).

The problem of finding three such three triangular numbers in geometric progression is readily reduced to finding solutions to a Pell equation, implies that there are infinitely many such triples, the smallest of which is (T_1, T_3, T_8). In [5, D23] by Guy, it is stated that Sierpinski asked the question as following

Question. *Are there four distinct triangular numbers in geometric progression?*

Szymiczek conjecture that the answer to Sierpinski's question is negative [6]. Recently, Bennett[7] proved that there do not exist four distinct triangular numbers in geometric progression with the ratio being positive integer. Moreover Chen and Fang[8] extend Bennett's result to the rational common ratio. But their proof is not complete in reference [7] and [8]. Because they supposed that the four distinct triangular are A, AQ, AQ^2, AQ^3, where $A \in \mathbb{N}, Q \in \mathbb{Q}$. In fact, arbitrary four numbers in a geometric progression $\{AQ^r\}_{r=1}^{\infty}$, for example, AQ, AQ^2, AQ^3, AQ^6, does not in geometric progression.

In this paper, using Störmer's theorem of the solutions of Pell equation, and a deep result of primitive divisor of Bilu, Hanrot and Voutier([9,10]), we completely solved the question of Sierpinski on triangular numbers. We prove a more general result as follows.

Theorem. *There do not exist four distinct triangular numbers in geometric progression* $\{AQ^r\}_{r=1}^{\infty}$.

2 Preliminaries

We give the following lemma.

Lemma 1. *(Störmer theorem) If (x, y) is solution of the Pell equation*

$$x^2 - Dy^2 = 1, D, x, y \in \mathbb{N} \tag{2}$$

and ε is the fundamental solution of (2), then every solution of (2) can be expressed as $x + y\sqrt{D} = \varepsilon^k$, where $k \in \mathbb{N}$. If $y = y'$, $y' \mid^* D$, where $y' \mid^* D$ denotes every prime factor of y' dividing D, then

$$x + y\sqrt{D} = \varepsilon. \tag{3}$$

Proof. See Lemma 2 of [1] p122-123.

A *Lucas pair* (α, β) is a pair of algebraic integers α, β, such as that $\alpha+\beta$ and $\alpha\beta$ are nor-zero co-prime rational integers, and $\dfrac{\alpha}{\beta}$ is not a root of unity. Further, let $s = \alpha + \beta$ and $t = \alpha\beta$, then we have

$$\alpha = \frac{1}{2}(s + \lambda\sqrt{d}), \beta = (s - \lambda\sqrt{d})$$

where $d = s^2 - 4t$, $\lambda \in \{1, -1\}$. For a given Lucas pair (α, β), one defines the corresponding sequence of *Lucas numbers* by

$$u_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n = 1, 2, 3, \ldots$$

Definition. *Let p be a prime. The prime p is a* **primitive divisor** *of the Lucas number $u_n(\alpha, \beta)$, if $p \mid u_n(\alpha, \beta)$ and $p \not| (\alpha-\beta)^2 u_1(\alpha, \beta) \cdots u_{n-1}(\alpha, \beta)$.*

Lemma 2. *If $4 < n \leq 30$, $n \neq 6$, and the Lucas sequence whose nth term has no primitive divisor, then*

(1)$n = 5, (s, d) = (1, 5), (1, -7), (2, -40), (1, -11), (1, -15), (12, -76), (12, -1346)$;

(2)$n = 7, (s, d) = (1, -7), (1, -19)$;

(3)$n = 8, (s, d) = (1, -7), (2, -24)$;

(4)$n = 10, (s, d) = (2, -8), (5, -3), (5, -47)$;

(5)$n = 12, (s, d) = (1, 5), (1, -7), (1, -11), (2, -56), (1, -15), (1, -19)$;

(6)$n = 13, (s, d) = (1, -7)$.

(7)$n = 18, (s, d) = (1, -7)$.

(8)$n = 30, (s, d) = (1, -7)$.

Proof. See Theorem 1 of [5].

Lemma 3. *For any integer $n > 30$, every n-th term of any Lucas sequence has a primitive divisor.*

Proof. See Theorem 1.4 of [10].

3 Proof of Theorem

Suppose that there exist four distinct triangular numbers T_x, T_y, T_z, T_w in a geometric progression $\{AQ^r\}_{r=1}^{\infty}$, let T_x be smallest, and the ratio is $Q = \frac{q}{p}$, $\gcd(q, p) = 1$, $q > p$. Let $8T_x = A$, then there exist $1 \leq r_1 < r_2 < r_3 \in \mathbb{N}$, satisfy

$$8T_y = A\left(\frac{q}{p}\right)^{r_1}, 8T_z = A\left(\frac{q}{p}\right)^{r_2}, 8T_w = A\left(\frac{q}{p}\right)^{r_3} \quad (4)$$

where $x, y, z, w, q, p, A \in \mathbb{N}$. Because $8T_w$ is positive integers, we have $p^{r_3}|A$. Let $A = ap^{r_3}$, $a \in \mathbb{N}$. By (1) and (4) we have

$$ap^{r_3} + 1 = u^2, ap^{r_3-r_1}q^{r_1} + 1 = v^2, ap^{r_3-r_2}q^{r_2} + 1 = w^2, aq^{r_3} + 1 = m^2 \quad (5)$$

where $u, v, w, m \in \mathbb{N}$.

Case 1: $p = 1$

If there are two odd numbers among r_1, r_2 and r_3. Without loss of generality, we may assume that $2 \nmid r_1, 2 \nmid r_2$, $r_2 > r_1 \geq 1$. By (5) we have aq is not a perfect square, then by (5) we get,

$$aq(q^{\frac{r_1-1}{2}})^2 + 1 = v^2, aq(q^{\frac{r_2-1}{2}})^2 + 1 = w^2, \quad (6)$$

So Pell equation $x^2 - aqy^2 = 1$ have solutions $(x, y) = (v, q^{\frac{r_1-1}{2}}), (w, q^{\frac{r_2-1}{2}})$. Since $q \mid^* aq$, by Lemma 1, we get

$$u + q^{\frac{r_1-1}{2}}\sqrt{Aq} = \varepsilon, v + q^{\frac{r_2-1}{2}}\sqrt{Aq} = \varepsilon \quad (7)$$

where ε is the fundamental solution of $x^2 - aqy^2 = 1$. But $r_2 > r_1$, it is impossible.

If there are two odd numbers among r_1, r_2 and r_3, we may assume that $2 \mid r_1, 2 \mid r_2$, $r_2 > r_1 \geq 2$, by (5), we get the Pell equation $x^2 - ay^2 = 1$ have solutions $(x, y) = (u, q^{\frac{r_1}{2}}), (v, q^{\frac{r_2}{2}})$. By (5), we have $a + 1 = u^2$, then the fundamental solution of $x^2 - ay^2 = 1$ is $\varepsilon = a + \sqrt{a^2 - 1}$. Let $\bar{\varepsilon} = a - \sqrt{a^2 - 1}$, thus there exist $k_1 < k_2 \in \mathbb{N}$, satisfy that

$$q^{\frac{r_1}{2}} = \frac{\varepsilon^{k_1} - \bar{\varepsilon}^{k_1}}{\varepsilon - \bar{\varepsilon}}, q^{\frac{r_2}{2}} = \frac{\varepsilon^{k_2} - \bar{\varepsilon}^{k_2}}{\varepsilon - \bar{\varepsilon}} \tag{8}$$

Let $\alpha = a + \sqrt{a^2 - 1}, \beta = a - \sqrt{a^2 - 1}$, then $\alpha + \beta$ and $\alpha\beta$ are nor-zero co-prime rational integers, and $\frac{\alpha}{\beta}$ is not a root of unity, then (α, β) is a Lucas pair. Let the sequence of *Lucas numbers* is

$$u_n = u_n(\varepsilon, \bar{\varepsilon}) = \frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}} \tag{9}$$

by (8), we have

$$u_{k_1}(\varepsilon, \bar{\varepsilon}) \mid u_{k_2}(\varepsilon, \bar{\varepsilon}) \tag{10}$$

then $u_{k_2}(a + \sqrt{a^2 - 1}, a - \sqrt{a^2 - 1})$ has no primitive divisor. By Lemma 2, Lemma 3, we have $k_2 = 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 18, 30$. But by Lemma 2, k_2 can not be $5, 7, 8, 10, 12, 13, 18, 30$, then $k_2 = 2, 3, 4, 6$.

By (9), we get $u_1 = 1$, $u_2 = 2a$, $u_3 = 4a^2 - 1$, $u_4 = 8a^3 - 4a$, $u_5 = 16a^4 - 12a^2 + 1$, $u_6 = 32a^5 - 32a^3 + 6a$. If $k_2 = 2$, then $k_1 = 1$, but by (5), (8), it is impossible. If $k_2 = 3$, then $k_1 = 2$, then by (5), (8), we have $q^{\frac{r_1}{2}} = 2a, q^{\frac{r_2}{2}} = 4a^2 - 1$, but $\gcd(2a, 4a^2 - 1) = 1$, it is also impossible.

If $k_2 = 4$, then $k_1 = 3, 2$. If $k_1 = 3$, then $\gcd(u_3, u_4) = \gcd(4a^2 - 1, 8a^3 - 4a) = \gcd(4a^2 - 1, 2a) = 1$, but by (10), it is impossible. If $k_1 = 2$, by (10), we have $q^{\frac{r_1}{2}} = 2a$, $q^{\frac{r_2}{2}} = 8a^3 - 4a$, then $q^{\frac{3r_1}{2}} - 2q^{\frac{r_1}{2}} = q^{\frac{r_2}{2}}$, then

$$q^{r_1} - 2 = q^{\frac{r_2 - r_1}{2}} \tag{11}$$

then by (11), we get $q=2$, $r_1=2$, $r_2=4$, then $a=1$, but by (5), it is also impossible.

If $k_2=6$, then $k_1=5,4,3,2$. Since $\gcd(u_6,u_5)=\gcd(32a^5-32a^3+6a,16a^4-12a^2+1)=1$, $\dfrac{u_6}{u_4}=\dfrac{16a^4-16a^2+3}{4a^2-2}$, but $2\nmid(16a^4-16a^2+3)$, $\dfrac{16a^4-16a^2+3}{4a^2-2}\notin\mathbf{N}$, $\gcd\left(\dfrac{u_6}{u_3},u_3\right)=\gcd(8a^3-6a,4a^2-1)=1$, then if $k_1=5,4,3$, by (8), it is impossible. If $k_1=2$, $\gcd\left(\dfrac{u_6}{u_2},u_2\right)=\gcd(16a^4-16a^2+3,2a)=1,3$, then $a=3$, by (8), it is also impossible.

Case 2: $p>1$

If $2\nmid r_3$, $2\nmid r_2$, then by (5), we get

$$aq(p^{\frac{r_3-r_2}{2}}q^{\frac{r_2-1}{2}})^2+1=w^2, aq(q^{\frac{r_3-1}{2}})^2+1=m^2 \tag{12}$$

we have aq is not a perfect square. Then Pell equation $x^2-aqy^2=1$ have solutions $(x,y)=(w,p^{\frac{r_3-r_2}{2}}q^{\frac{r_2-1}{2}})$, $(m,q^{\frac{r_3-1}{2}})$. Since $q^{\frac{r_3-1}{2}}\ |^*\ aq$, by Lemma 1, we have $\varepsilon=m+q^{\frac{r_3-1}{2}}\sqrt{aq}$ is the fundamental solution of Pell equation $x^2-aqy^2=1$, which is impossible, because $m>w$.

If $2\nmid r_3$, $2|r_2$, then by (5), we get

$$ap(p^{\frac{r_3-1}{2}})^2+1=u^2, ap(p^{\frac{r_3-r_2-1}{2}}q^{\frac{r_2}{2}})^2+1=w^2 \tag{13}$$

Because ap is not a perfect square, then Pell equation $x^2-apy^2=1$ have solutions $(x,y)=(u,p^{\frac{r_3-1}{2}})$, $(w,p^{\frac{r_3-r_2-1}{2}}q^{\frac{r_2}{2}})$. Since $p^{\frac{r_3-1}{2}}\ |^*\ ap$, by Lemma 1, we have $\varepsilon=u+p^{\frac{r_3-1}{2}}\sqrt{ap}$ is the fundamental solution of Pell equation $x^2-apy^2=1$, let $\bar{\varepsilon}=u-p^{\frac{r_3-1}{2}}\sqrt{ap}$, then there exist $k\in\mathbb{N}$, satisfy that

$$p^{\frac{r_3-r_2-1}{2}}q^{\frac{r_2}{2}}=\frac{\varepsilon^k-\bar{\varepsilon}^k}{2\sqrt{ap}}=\frac{\varepsilon^k-\bar{\varepsilon}^k}{\varepsilon-\bar{\varepsilon}}p^{\frac{r_3-1}{2}} \tag{14}$$

But $r_2>1$, $\gcd(p,q)=1$, which is impossible.

If $2|r_3$, $2\nmid rr_2$, then by (5), we get

$$apq(p^{\frac{r_3-r_1-1}{2}}q^{\frac{r_1-1}{2}})^2+1=v^2, apq(p^{\frac{r_3-r_2-1}{2}}q^{\frac{r_2-1}{2}})^2+1=w^2 \tag{15}$$

Because apq is not a perfect square, then Pell equation $x^2 - apqy^2 = 1$ have solutions $(x, y) = (v, p^{\frac{r_3-r_1-1}{2}} q^{\frac{r_1-1}{2}})$, $(w, p^{\frac{r_3-r_2-1}{2}} q^{\frac{r_2-1}{2}})$. Since $q^{\frac{r_3-1}{2}} \mid^* aq$, by Lemma 1, we have $\varepsilon = v + p^{\frac{r_3-r_1-1}{2}} q^{\frac{r_1-1}{2}} \sqrt{ap} = w + p^{\frac{r_3-r_2-1}{2}} q^{\frac{r_2-1}{2}}$ is the fundamental solution of Pell equation $x^2 - apqy^2 = 1$, which is impossible.

If $2|r_3$, $2|r_1 r_2$, without loss of generality, we may assume that $2|r_1$, then by (5), we get

$$a(p^{\frac{r_3}{2}})^2 + 1 = u^2, a(p^{\frac{r_3-r_1}{2}} q^{\frac{r_1}{2}})^2 + 1 = v^2, a(q^{\frac{r_3}{2}})^2 + 1 = m^2 \tag{16}$$

By (16) we have a is not a perfect square. Let $\varepsilon_1 = x_0 + y_0\sqrt{a}$ be the fundamental solution of Pell equation $x^2 - ay^2 = 1$, then there exist $k_1, k_2, k_3 \in \mathbb{N}$, satisfy that

(17)

$$p^{\frac{r_3}{2}} = \frac{\varepsilon_1^{k_1} - \bar{\varepsilon}_1^{k_1}}{2\sqrt{a}} = \frac{\varepsilon_1^{k_1} - \bar{\varepsilon}_1^{k_1}}{\varepsilon_1 - \bar{\varepsilon}_1} y_0, p^{\frac{r_3-r_1}{2}} q^{\frac{r_1}{2}} = \frac{\varepsilon_1^{k_2} - \bar{\varepsilon}_1^{k_2}}{\varepsilon_1 - \bar{\varepsilon}_1} y_0, q^{\frac{r_3}{2}} = \frac{\varepsilon_1^{k_3} - \bar{\varepsilon}_1^{k_3}}{\varepsilon_1 - \bar{\varepsilon}_1} y_0$$

Where $\bar{\varepsilon}_1 = x_0 - y_0\sqrt{a}$, $k_1 < k_2 < k_3$. But $\gcd(p, q) = 1$, by (17) we have $y_0 = 1$, then $x_0^2 = a + 1$, $a = x_0^2 - 1$, so $\varepsilon_1 = x_0 + \sqrt{x_0^2 - 1}$.

Since $k_3 > k_2$, $\dfrac{r_1}{2} \geq 1$, $q > p$, then from (17) we obtain that $u_{k_3}(x_0 + \sqrt{x_0^2 - 1}, x_0 - \sqrt{x_0^2 - 1})$ has no primitive divisor. Same method of consideration as case 1, we get it is also impossible.

The proof is complete.

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