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Ostrowski Grüss Čebyšev type inequalities for functions whose modulus of second derivatives are convex[1]

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Abstract

In this paper, we establish Ostrowski Grüss Čebyšev type inequalities involving functions whose derivatives are bounded and whose modulus of second derivatives are convex.

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1 Introduction

In 1938, A. M. Ostrowski proved the following classical inequality [7]:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be continous on $[a,b]$ and diffrentiable on (a,b), whose first derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $|f'(x)| \leq M < \infty$. Then,*

$$(1)\qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M,$$

for all $x \in [a,b]$, where M is a constant.

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For two absolutely continuous functions $f, g : [a, b] \to \mathbb{R}$, consider the functional

$$(2) \qquad T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

provided, the involved integrals exist.

In 1882, P. L. Čebyšev proved that [6], if $f', g' \in L_\infty[a,b]$, then

$$(3) \qquad T(f,g) \le \frac{(b-a)^2}{12}\,\|f'\|_\infty\,\|g'\|_\infty\,.$$

In 1934, G. Grüss showed that [6],

$$(4) \qquad T(f,g) \le \frac{1}{8}(M-m)(N-n)\,,$$

provided, m, M, n, and N are real numbers satisfying the condition $-\infty < m \le f(x) \le M < \infty$, $-\infty < n \le g(x) \le N < \infty$, for all $x \in [a, b]$.

During the past few years, many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see $[1-12]$, and the references cited therein. Motivated by the recent results given in $[1-3, 11]$, in the present paper, we establish some inequalities similar to those given by Ostrowski, Grüss, Čebyšev and Pachpatte, involving functions whose derivatives are bounded and whose modulus of second derivatives are convex. The analysis used in the proofs is elementary and based on the use of integral identities proved in $[1-2]$.

2 Statement of results

Let I be a suitable interval of the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y),$$

for all $x, y \in I$ and $\lambda \in [0,1]$. A function $f : I \to (0, \infty)$ is said to be log-convex if

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda} [f(y)]^{1-\lambda},$$

for all $x, y \in I$ and $\lambda \in [0,1]$ (see[11]). We need the following identity, i.e., the identity (15):

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)dt + (x - \frac{a+b}{2})f'(x) \\ &- \frac{1}{b-a}\int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda) f''((1-\lambda)x + \lambda t)\, d\lambda\right] dt \\ &= \frac{1}{b-a}\int_a^b f(t)dt + (x - \frac{a+b}{2})f'(x) \\ &- \frac{1}{2(b-a)}\left[(x-a)^3 \int_0^1 \lambda^2 f''((1-\lambda)a + \lambda x)\, d\lambda \right. \\ &+ \left. (b-x)^3 \int_0^1 \lambda^2 f''((1-\lambda)b + \lambda x)\, d\lambda\right], \end{aligned}$$

for $x \in [a,b]$, where $f : I \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ and $\lambda \in [0,1]$. We use the following notation to simplify the details of presentation:

$$\begin{aligned} S_s(f,g) &= f(x)g(x) - \frac{1}{2}(x - \frac{a+b}{2})[f(x)g'(x) + g(x)f'(x)] \\ &- \frac{1}{2(b-a)}\left[f(x)\int_a^b g(t)dt + g(x)\int_a^b f(t)dt\right], \end{aligned}$$

and

$$T_s(f,g) = \frac{1}{b-a}\int_a^b S_s(f,g)dx,$$

and define $\|.\|_\infty$ as the usual Lebesgue norm on $L_\infty[a,b]$, *i.e.*, $\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)|$ for $h \in L_\infty[a,b]$.

The following theorems deal with Ostrowski type inequalities involving two functions.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continous on $[a,b]$.*

1. If $|f''|, |g''|$ are convex on $[a,b]$ and $f'', g'' \in L_\infty[a,b]$, then

$$|S_s(f,g)| \leq [|g(x)|(2|f''(x)| + \|f''(t)\|_\infty) + |f(x)|(2|g''(x)| + \|g''(t)\|_\infty)] \times \left[\frac{1}{12} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\frac{(b-a)^2}{12}, \tag{5}$$

for all $x \in [a,b]$.

2. If $|f''|, |g''|$ are log-convex on $[a,b]$, then

$$|S_s(f,g)| \leq \frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|^2 |f''(x)| \frac{-\ln A + A - 1}{(\ln A)^2}dt + |f(x)|\int_a^b |x-t|^2 |g''(x)| \frac{-\ln B + B - 1}{(\ln B)^2}dt\right\}, \tag{6}$$

for all $x \in [a,b]$, where $A = \dfrac{|f''(t)|}{|f''(x)|} > 0$ and $B = \dfrac{|g''(t)|}{|g''(x)|} > 0$.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$.*

1. If $|f''|$, $|g''|$ are convex on $[a,x]$ and $[x,b]$, then

$$|S_s(f,g)| \leq \frac{1}{4}[|g(x)|M(x) + |f(x)|N(x)],$$

for $x \in [a,b]$, where

$$M(x) \leq \frac{(b-a)^2}{12}\left\{\left(\frac{x-a}{b-a}\right)^3|f''(a)| + \left(\frac{b-x}{b-a}\right)^3|f''(b)| + \left[\frac{3}{4} + \frac{9(x-\frac{a+b}{2})^2}{(b-a)^2}\right]|f''(x)|\right\},$$

and

$$(7)\qquad N(x) \leq \frac{(b-a)^2}{12}\left\{\left(\frac{x-a}{b-a}\right)^3 |g''(a)| + \left(\frac{b-x}{b-a}\right)^3 |g''(b)| + \left[\frac{3}{4}+\frac{9(x-\frac{a+b}{2})^2}{(b-a)^2}\right]|g''(x)|\right\},$$

for $x \in [a,b]$.

2. If $|f''|$, $|g''|$ *log-convex on* $[a,x]$ *and* $[x,b]$, *then*

$$|S_s(f,g)| \leq [|g(x)|\, H(x) + |f(x)|\, L(x)],$$

$x \in [a,b]$, *where*

$$H(x) = \frac{(b-a)^2}{4}\left\{\left(\frac{x-a}{b-a}\right)^3 \frac{A_1 (\ln A_1)^2 - 2A_1 \ln A_1 + 2A_1 - 2}{(\ln A_1)^3}|f''(a)| + \left(\frac{b-x}{b-a}\right)^3 \frac{B_1 (\ln B_1)^2 - 2B_1 \ln B_1 + 2B_1 - 2}{(\ln B_1)^3}|f''(b)|\right\},$$

and

$$(8)\ L(x) = \frac{(b-a)^2}{4}\left[\left(\frac{x-a}{b-a}\right)^3 \frac{A_2 (\ln A_2)^2 - 2A_2 \ln A_2 + 2A_2 - 2}{(\ln A_2)^3}|g''(a)| + \left(\frac{b-x}{b-a}\right)^3 \frac{B_2 (\ln B_2)^2 - 2B_2 \ln B_2 + 2B_2 - 2}{(\ln B_2)^3}|g''(b)|\right],$$

with $A_1 = \dfrac{|f''(x)|}{|f''(a)|} > 0,\ B_1 = \dfrac{|f''(x)|}{|f''(b)|} > 0,\ A_2 = \dfrac{|g''(x)|}{|g''(a)|} > 0$ *and* $B_2 = \dfrac{|g''(x)|}{|g''(b)|} > 0$.

The Grüss type inequalities are embodied in the following theorems.

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be absolutely continuous on* $[a,b]$.

1. If $|f''|$, $|g''|$ are convex on $[a,b]$ and f'', $g'' \in L_\infty[a,b]$, then

$$
\begin{aligned}
(9)\qquad |T_s(f,g)| &\le \frac{1}{12(b-a)^2}\int_a^b [|g(x)|\,(2|f''(x)| + \|f''(t)\|_\infty) \\
&+ |f(x)|\,(2|g''(x)| + \|g''(t)\|_\infty)]\,E(x)dx,
\end{aligned}
$$

for all $x \in [a,b]$, where $E(x) = \dfrac{(x-a)^3 + (b-x)^3}{3}$.

2. If $|f''|$, $|g''|$ are log-convex on $[a,b]$, then

$$
\begin{aligned}
(10)\, |T_s(f,g)| &\le \frac{1}{2(b-a)^2}\int_a^b \Bigg\{ |g(x)| \int_a^b |x-t|^2 \, |f''(x)| \frac{-\ln A + A - 1}{(\ln A)^2} dt \\
&+ |f(x)| \int_a^b |x-t|^2 \, |g''(x)| \frac{-\ln B + B - 1}{(\ln B)^2} dt \Bigg\} dx,
\end{aligned}
$$

for all $x \in [a,b]$, where $A = \dfrac{|f''(t)|}{|f''(x)|} > 0$ and $B = \dfrac{|g''(t)|}{|g''(x)|} > 0$.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$.*

1. If $|f''|, |g''|$ are convex on $[a,b]$ and $f'', g'' \in L_\infty[a,b]$, then

$$
\begin{aligned}
(11)\, |T_s(f,g)| &\le \frac{b-a}{48}\int_a^b \Bigg\{ \left(\frac{x-a}{b-a}\right)^3 [|g(x)|\,(|f''(a)| + 3|f''(x)|) \\
&+ |f(x)|\,(|g''(a)| + 3|g''(x)|)] + [|g(x)|\,(|f''(b)| + 3|f''(x)|) \\
&+ |f(x)|\,(|g''(b)| + 3|g''(x)|)] \left(\frac{b-x}{b-a}\right)^3 \Bigg\} dx.
\end{aligned}
$$

2. If $|f''|$, $|g''|$ are log-convex on $[a,b]$, then

$$(12)\quad |T_s(f,g)| \leq \frac{b-a}{4}\int_a^b \left\{ \left(\frac{x-a}{b-a}\right)^3 \left[\frac{A_1(\ln A_1)^2 - 2A_1\ln A_1 + 2A_1 - 2}{(\ln A_1)^3} \times |g(x)|\,|f''(a)| + \frac{A_2(\ln A_2)^2 - 2A_2\ln A_2 + 2A_2 - 2}{(\ln A_2)^3}|f(x)|\,|g''(a)| \right] + \left[\frac{B_1(\ln B_1)^2 - 2B_1\ln B_1 + 2B_1 - 2}{(\ln B_1)^3}|g(x)|\,|f''(b)| + \frac{B_2(\ln B_2)^2 - 2B_2\ln B_2 + 2B_2 - 2}{(\ln B_2)^3}|f(x)|\,|g''(b)| \right] \left(\frac{b-x}{b-a}\right)^3 \right\} dx.$$

The next theorem contains Čebyšev type inequalities.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$.*

1. If $|f''|, |g''|$ are convex on $[a,b]$ and $f'', g'' \in L_\infty[a,b]$, then

$$(13)\quad \left|\widetilde{T}_s(f,g)\right| \leq \frac{1}{36(b-a)^3}\int_a^b [2|f''(x)| + \|f''\|_\infty + 2|g''(x)| + \|g''\|_\infty] E^2(x)dx,$$

for all $x\in[a,b]$, where $E(x) = \dfrac{(x-a)^3 + (b-x)^3}{3}$.

2. If $|f''|$, $|g''|$ are log-convex on $[a,b]$, then

$$(14)\quad \left|\widetilde{T}_s(f,g)\right| \leq \frac{1}{(b-a)^3}\int_a^b \left\{ |f''(x)|\int_a^b |x-t|^2 \frac{-\ln A + A - 1}{(\ln A)^2}dt + |g''(x)|\int_a^b |x-t|^2\frac{-\ln B + B - 1}{(\ln B)^2}dt \right\} dx,$$

for all $x\in[a,b]$, where $A = \dfrac{|f''(t)|}{|f''(x)|} > 0$ and $B = \dfrac{|g''(t)|}{|g''(x)|} > 0$.

3 Proofs of Theorems

Proof of Theorem 2.

From the hypothesis of Theorem 2, it is easy to verify that the following identity holds:

$$
\begin{aligned}
(15)\ f(x) &= \frac{1}{b-a}\int_a^b f(t)dt + (x-\frac{a+b}{2})f'(x) \\
&- \frac{1}{b-a}\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda) f''((1-\lambda)x+\lambda t)\,d\lambda\right] dt.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(16)\ g(x) &= \frac{1}{b-a}\int_a^b g(t)dt + (x-\frac{a+b}{2})g'(x) \\
&- \frac{1}{b-a}\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda) g''((1-\lambda)x+\lambda t)\,d\lambda\right] dt,
\end{aligned}
$$

for $x \in [a,b]$. Multiplying both sides of (15) and (16) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting, we have:

$$
\begin{aligned}
f(x)g(x) &= \frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)dt + f(x)\int_a^b g(t)dt\right] \\
&+ \frac{1}{2}(x-\frac{a+b}{2})\left[f'(x)g(x)+f(x)g'(x)\right] \\
&- \frac{1}{2(b-a)}\left\{g(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda) f''((1-\lambda)x+\lambda t)\,d\lambda\right] dt\right. \\
&+ \left. f(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda) g''((1-\lambda)x+\lambda t)\,d\lambda\right] dt\right\}.
\end{aligned}
$$

This gives

$$S_s(f,g) = \frac{-1}{2(b-a)}\left\{g(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda) f''((1-\lambda)x+\lambda t)\,d\lambda\right]dt\right.$$

$$\left. + f(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda) g''((1-\lambda)x+\lambda t)\,d\lambda\right]dt\right\}, \quad (17)$$

where

$$S_s(f,g) = f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)dt + f(x)\int_a^b g(t)dt\right]$$
$$- \frac{1}{2}(x - \frac{a+b}{2})\left[f'(x)g(x) + f(x)g'(x)\right].$$

1. Since $|f''|, |g''|$ are convex on $[a,b]$, from (17), we observe that

$$|S_s(f,g)| \le \frac{1}{2(b-a)}$$
$$\times\left\{|g(x)|\int_a^b |x-t|^2\left[|f''(x)|\int_0^1 (1-\lambda)^2 d\lambda + |f''(t)|\int_0^1 \lambda(1-\lambda)\,d\lambda\right]dt\right.$$
$$\left. + |f(x)|\int_a^b |x-t|^2\left[|g''(x)|\int_0^1 (1-\lambda)^2 d\lambda + |g''(t)|\int_0^1 \lambda(1-\lambda)\,d\lambda\right]dt\right\}.$$

Now $\int_0^1 (1-\lambda)^2 d\lambda = \frac{1}{3}$ and $\int_0^1 \lambda(1-\lambda)\,d\lambda = \frac{1}{6}$. We thus have:

$$|S_s(f,g)| \le \frac{1}{12(b-a)}\left[|g(x)|\int_a^b |x-t|^2 (2|f''(x)| + |f''(t)|)\,dt\right.$$
$$\left. + |f(x)|\int_a^b |x-t|^2 (2|g''(x)| + |g''(t)|)\,dt\right]$$
$$\le \frac{1}{12(b-a)}\left[\sup_{t\in[a,b]} (2|f''(x)| + |f''(t)|)\,|g(x)|\right.$$

$$
\begin{aligned}
&+ \left.\sup_{t\in[a,b]} \left(2\left|g''(x)\right| + \left|g''\left(t\right)\right|\right)\left|f(x)\right|\right] \int_a^b \left|x-t\right|^2 dt \\
&= \frac{1}{12\left(b-a\right)}\left\{\left|g(x)\right| \sup_{t\in[a,b]}\left[2\left|f''(x)\right| + \left|f''\left(t\right)\right|\right]\right. \\
&+ \left.\left|f(x)\right| \sup_{t\in[a,b]}\left[2\left|g''(x)\right| + \left|g''\left(t\right)\right|\right]\right\}\left[\frac{\left(x-a\right)^3+\left(b-x\right)^3}{3}\right].
\end{aligned}
$$

Now $\frac{\left(x-a\right)^3+\left(b-x\right)^3}{3} = \left[\frac{\left(b-a\right)^2}{12} + (x-\frac{a+b}{2})^2\right]\left(b-a\right)$. We therefore have:

$$
\begin{aligned}
\left|S_s(f,g)\right| &\le \frac{\left(b-a\right)^2}{12}\left[\frac{1}{12} + \frac{(x-\frac{a+b}{2})^2}{\left(b-a\right)^2}\right] \\
&\times \left[\left|g(x)\right|\left[2\left|f''x\right| + \left\|\left|f''\left(t\right)\right|\right\|_\infty\right] + \left|f(x)\right|\left[2\left|g''(x)\right| + \left\|\left|g''\left(t\right)\right|\right\|_\infty\right]\right]
\end{aligned}
$$

2. Since $\left|f''\right|, \left|g''\right|$ are log-convex on $[a,x]$ and $[x,b]$, we have from (17):

$$
\begin{aligned}
\left|S_s(f,g)\right| &\le \frac{1}{2\left(b-a\right)}\left\{\left|g(x)\right|\int_a^b \left|x-t\right|^2\left[\int_0^1 \left(1-\lambda\right)\left|f''\left(x\right)\right|^{1-\lambda}\left|f''(t)\right|^\lambda d\lambda\right] dt\right. \\
&+ \left.\left|f(x)\right|\int_a^b \left|x-t\right|^2\left[\int_0^1 \left(1-\lambda\right)\left|g''\left(x\right)\right|^{1-\lambda}\left|g''(t)\right|^\lambda d\lambda\right]\right\} dt \\
&= \frac{1}{2\left(b-a\right)}\left\{\left|g(x)\right|\int_a^b \left|x-t\right|^2\left[\left|f''\left(x\right)\right|\int_0^1 \left(1-\lambda\right)\left|\frac{f''(t)}{f''\left(x\right)}\right|^\lambda d\lambda\right] dt\right. \\
&+ \left.\left|f(x)\right|\int_a^b \left|x-t\right|^2\left[\left|g''\left(x\right)\right|\int_0^1 \left(1-\lambda\right)\left|\frac{g''(t)}{g''\left(x\right)}\right|^\lambda d\lambda\right] dt\right\} \\
&= \frac{1}{2\left(b-a\right)}\left\{\left|g(x)\right|\int_a^b \left|x-t\right|^2\left|f''\left(x\right)\right|\frac{-\ln A + A - 1}{\left(\ln A\right)^2}dt\right.
\end{aligned}
$$

$$+ \quad |f(x)| \int_a^b |x-t|^2 |g''(x)| \frac{-\ln B + B - 1}{(\ln B)^2} dt \Bigg\},$$

where $A = \left| \frac{f''(t)}{f''(x)} \right|$ and $B = \left| \frac{g''(t)}{g''(x)} \right|$.

Proof of Theorem 3.

From the hypothesis of the Theorem 3, the following identity [5] holds:

$$\begin{aligned}
(18) \qquad f(x) &= \frac{1}{b-a} \int_a^b f(t)dt + (x - \frac{a+b}{2}) f'(x) \\
&- \frac{1}{2(b-a)} \Bigg\{ (x-a)^3 \int_0^1 \lambda^2 |f''((1-\lambda)a + \lambda x)| d\lambda \\
&+ (b-x)^3 \int_0^1 \lambda^2 |f''((1-\lambda)b + \lambda x)| d\lambda \Bigg\}.
\end{aligned}$$

similarly,

$$\begin{aligned}
(19) \qquad g(x) &= \frac{1}{b-a} \int_a^b g(t)dt + (x - \frac{a+b}{2}) g'(x) \\
&- \frac{1}{2(b-a)} \Bigg\{ (x-a)^3 \int_0^1 \lambda^2 |g''((1-\lambda)a + \lambda x)| d\lambda \\
&+ (b-x)^3 \int_0^1 \lambda^2 |g''((1-\lambda)b + \lambda x)| d\lambda \Bigg\}.
\end{aligned}$$

Multiplying both sides of (18) and (19) by $g(x)$ and $f(x)$, adding the resulting identities and rewriting, we have:

$$(20)\ S_s(f,g) = \frac{-1}{4}\left\{g(x)\left[\left(\frac{x-a}{b-a}\right)^3\int_0^1 \lambda^2\left|f''((1-\lambda)a+\lambda x)\right|d\lambda + \left(\frac{b-x}{b-a}\right)^3\int_0^1 \lambda^2\left|f''((1-\lambda)b+\lambda x)\right|d\lambda\right] + f(x)\left[\left(\frac{x-a}{b-a}\right)^3\int_0^1 \lambda^2\left|g''((1-\lambda)a+\lambda x)\right|d\lambda + \left(\frac{b-x}{b-a}\right)^3\int_0^1 \lambda^2\left|g''((1-\lambda)b+\lambda x)\right|d\lambda\right]\right\}(b-a)^2.$$

1. Since $|f''|, |g''|$ are convex on $[a,b]$, from (20) we observe that

$$(21)\qquad |S_s(f,g)| \leq \frac{1}{4}\left[|g(x)|\,M(x) + |f(x)|\,N(x)\right],$$

where

$$M(x) = \frac{(x-a)^3}{b-a}\int_0^1 \lambda^2\left|f''((1-\lambda)a+\lambda x)\right|d\lambda + \frac{(b-x)^3}{b-a}\int_0^1 \lambda^2\left|f''((1-\lambda)b+\lambda x)\right|d\lambda,$$

and

$$N(x) = \frac{(x-a)^3}{b-a}\int_0^1 \lambda^2\left|g''((1-\lambda)a+\lambda x)\right|d\lambda + \frac{(b-x)^3}{b-a}\int_0^1 \lambda^2\left|g''((1-\lambda)b+\lambda x)\right|d\lambda.$$

Next, using the property of functions whose modulus of second derivatives are convex, we observe that

$$\begin{aligned}\int_0^1 \lambda^2 \left|f''((1-\lambda)a+\lambda x)\right| d\lambda &\leq \left|f''(a)\right| \int_0^1 \lambda^2(1-\lambda) + \left|f''(x)\right| \int_0^1 \lambda^3 d\lambda \\ &= \frac{1}{12}\left|f''(a)\right| + \frac{1}{4}\left|f''(x)\right|.\end{aligned}$$

Similarly, $\int_0^1 \lambda^2 \left|f''((1-\lambda)b+\lambda x)\right| d\lambda \leq \frac{1}{12}\left|f''(b)\right| + \frac{1}{4}\left|f''(x)\right|$, also
$\int_0^1 \lambda^2 \left|g''((1-\lambda)a+\lambda x)\right| d\lambda \leq \frac{1}{12}\left|g''(a)\right| + \frac{1}{4}\left|g''(x)\right|$, and
$\int_0^1 \lambda^2 \left|g''((1-\lambda)b+\lambda x)\right| d\lambda \leq \frac{1}{12}\left|g''(b)\right| + \frac{1}{4}\left|g''(x)\right|$.
We thus have

$$\begin{aligned}M(x) &\leq \frac{(b-a)^2}{12}\Bigg\{\left(\frac{x-a}{b-a}\right)^3\left|f''(a)\right| + \left(\frac{b-x}{b-a}\right)^3\left|f''(b)\right| \\ &+ 3\left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right]\left|f''(x)\right|\Bigg\}.\end{aligned}$$

Now

$$3\left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right] = \frac{3}{4} + 9\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}.$$

We, therefore have:

$$\begin{aligned}M(x) &\leq \frac{(b-a)^2}{12}\Bigg\{\left(\frac{x-a}{b-a}\right)^3\left|f''(a)\right| + \left(\frac{b-x}{b-a}\right)^3\left|f''(b)\right| \\ &+ \left[\frac{3}{4} + 9\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\left|f''(x)\right|\Bigg\}.\end{aligned} \tag{22}$$

Similarly,

$$N(x) \leq \frac{(b-a)^2}{12}\left\{\left(\frac{x-a}{b-a}\right)^3 |g''(a)| + \left(\frac{b-x}{b-a}\right)^3 |g''(b)| + \left[\frac{3}{4} + 9\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]|g''(x)|\right\}. \tag{23}$$

The inequalities (21), (22) and (23) establish the required inequality.

2. Since $|f''|$, $|g''|$ are log-convex on $[a,b]$, we have from (20)

$$\begin{aligned}
|S_s(f,g)| &\leq \frac{1}{4}(b-a)^2\left\{|g(x)|\left[\left(\frac{x-a}{b-a}\right)^3\int_0^1 \lambda^2 |f''(a)|^{1-\lambda}|f''(x)|^{\lambda}d\lambda\right.\right.\\
&+ \left.\left(\frac{b-x}{b-a}\right)^3\int_0^1 \lambda^2 |f''(b)|^{1-\lambda}|f''(x)|^{\lambda}d\lambda\right]\\
&+ |f(x)|\left[\left(\frac{x-a}{b-a}\right)^3\int_0^1 \lambda^2 |g''(a)|^{1-\lambda}|g''(x)|^{\lambda}d\lambda\right.\\
&+ \left.\left.\left(\frac{b-x}{b-a}\right)^3\int_0^1 \lambda^2 |g''(b)|^{1-\lambda}|g''(x)|^{\lambda}d\lambda\right]\right\}\\
&= \frac{1}{4}(b-a)^2\left\{|g(x)|\left[\left(\frac{x-a}{b-a}\right)^3|f''(a)|\int_0^1 \lambda^2\left(\frac{|f''(x)|}{|f''(a)|}\right)^{\lambda}d\lambda\right.\right.\\
&+ \left.\left(\frac{b-x}{b-a}\right)^3|f''(b)|\int_0^1 \lambda^2\left(\frac{|f''(x)|}{|f''(b)|}\right)^{\lambda}d\lambda\right]\\
&+ |f(x)|\left[\left(\frac{x-a}{b-a}\right)^3|g''(a)|\int_0^1 \lambda^2\left(\frac{|g''(x)|}{|g''(a)|}\right)^{\lambda}d\lambda\right.\\
&+ \left.\left.\left(\frac{b-x}{b-a}\right)^3|g''(b)|\int_0^1 \lambda^2\left(\frac{|g''(x)|}{|g''(b)|}\right)^{\lambda}d\lambda\right]\right\}.
\end{aligned} \tag{24}$$

For any $C > 0$, we have:

$$\int_0^1 \lambda^2 C^\lambda d\lambda = \frac{C(\ln C)^2 - 2C\ln C + 2C - 2}{(\ln C)^3}.$$

Also, let $A_1 = \frac{|f''(x)|}{|f''(a)|}$, $B_1 = \frac{|f''(x)|}{|f''(b)|}$, $A_2 = \frac{|g''(x)|}{|g''(a)|}$ and $B_2 = \frac{|g''(x)|}{|g''(b)|}$.

We therefore have from (24)

$$|S_s(f,g)| \le [|g(x)| H(x) + |f(x)| L(x)],$$

for $x \in [a,b]$, where

$$\begin{aligned} H(x) &= \frac{1}{4}(b-a)^2 \left[\left(\frac{x-a}{b-a} \right)^3 \frac{A_1(\ln A_1)^2 - 2A_1 \ln A_1 + 2A_1 - 2}{(\ln A_1)^3} |f''(a)| \right. \\ &+ \left. \left(\frac{b-x}{b-a} \right)^3 \frac{B_1(\ln B_1)^2 - 2B_1 \ln B_1 + 2B_1 - 2}{(\ln B_1)^3} |f''(b)| \right], \end{aligned}$$

and

$$\begin{aligned} L(x) &= \frac{1}{4}(b-a)^2 \left[\left(\frac{x-a}{b-a} \right)^3 \frac{A_2(\ln A_2)^2 - 2A_2 \ln A_2 + 2A_2 - 2}{(\ln A_2)^3} |g''(a)| \right. \\ &+ \left. \left(\frac{b-x}{b-a} \right)^3 \frac{B_2(\ln B_2)^2 - 2B_2 \ln B_2 + 2B_2 - 2}{(\ln B_2)^3} |g''(b)| \right]. \end{aligned} \tag{25}$$

Proof of Theorem 4.

From the proof of Theorem 2, we have

$$
(26)\quad f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)dt+f(x)\int_a^b g(t)dt\right]
$$
$$
-\frac{1}{2}(x-\frac{a+b}{2})\left[f'(x)g(x)+f(x)g'(x)\right]
$$
$$
=-\frac{1}{2(b-a)}\left\{g(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)f''\left[(1-\lambda)x+\lambda t\right]d\lambda\right]dt\right.
$$
$$
\left.+f(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)g''\left[(1-\lambda)x+\lambda t\right]d\lambda\right]dt\right\}.
$$

Integrating w.r.t x from a to b, we get

$$
(27)\quad T_s(f,g)=\frac{-1}{2(b-a)^2}
$$
$$
\times\int_a^b\left\{g(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)f''\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt\right.
$$
$$
\left.+f(x)\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)g''\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt\right\}dx
$$

1. Since $|f''|$, $|g''|$ are convex on $[a,b]$, we have from (27)

$$
|T_s(f,g)|\le\frac{1}{2(b-a)^2}
$$
$$
\times\int_a^b\left\{|g(x)|\int_a^b |x-t|^2\left[|f''(x)|\int_0^1 (1-\lambda)^2d\lambda+|f''(t)|\int_0^1 \lambda(1-\lambda)d\lambda\right]dt\right.
$$
$$
\left.+|f(x)|\int_a^b |x-t|^2\left[|g''x|\int_0^1 (1-\lambda)^2d\lambda+|g''(t)|\int_0^1 \lambda(1-\lambda)d\lambda\right]dt\right\}dx
$$

$$= \frac{1}{2(b-a)^2}\int_a^b \left\{ |g(x)| \int_a^b |x-t|^2 \left[\frac{|f''(x)|}{3}+\frac{|f''(t)|}{6}\right] dt \right.$$

$$\left. + |f(x)| \int_a^b |x-t|^2 \left[\frac{|g''(x)|}{3}+\frac{|g''(t)|}{6}\right] dt \right\} dx$$

$$\leq \frac{1}{2(b-a)^2}\int_a^b \left\{ |g(x)| \int_a^b |x-t|^2 \ ess\sup \left[\frac{|f''(x)|}{3}+\frac{|f''(t)|}{6}\right] dt \right.$$

$$\left. + |f(x)| \int_a^b |x-t|^2 \ ess\sup \left[\frac{|g''(x)|}{3}+\frac{|g''(t)|}{6}\right] dt \right\} dx$$

$$= \frac{1}{12(b-a)^2}\int_a^b \{ |g(x)| \left[2|f''(x)| + \||f''(t)|\|_\infty\right]$$

$$+ |f(x)| \left[2|g''(x)| + \||g''(t)|\|_\infty\right] \} E(x) dx,$$

where

$$\int_a^b |x-t|^2 dt = E(x) = \frac{(x-a)^3+(b-x)^3}{3}.$$

2. Since $|f''|$, $|g''|$ are log-convex on $[a,b]$, we have from (26)

$$|T_s(f,g)| \leq \frac{1}{2(b-a)^2}$$

$$\times \int_a^b \left\{ |g(x)| \int_a^b |x-t|^2 \left[\int_0^1 (1-\lambda)|f''(x)|^{1-\lambda}|f''(t)|^{\lambda} d\lambda\right] dt \right.$$

$$\left. + \ |f(x)| \int_a^b |x-t|^2 \left[\int_0^1 (1-\lambda)|g''(x)|^{1-\lambda}|g''(t)|^{\lambda} d\lambda\right] dt \right\} dx$$

$$= \frac{1}{2(b-a)^2}\int_a^b \left\{ |g(x)| \int_a^b |x-t|^2 \left[|f''(x)| \int_0^1 (1-\lambda)\left|\frac{f''(t)}{f''(x)}\right|^{\lambda} d\lambda\right] dt \right.$$

$$
\begin{aligned}
&+ \quad |f(x)| \int_a^b |x-t|^2 \left[|g''(x)| \int_0^1 (1-\lambda) \left| \frac{g''(t)}{g''(x)} \right|^{\lambda} d\lambda \right] dt\, dx \\
&= \quad \frac{1}{2(b-a)^2} \int_a^b \left\{ |g(x)| \int_a^b |x-t|^2 |f''(x)| \frac{-\ln A + A - 1}{(\ln A)^2} dt \right. \\
&+ \quad \left. |f(x)| \int_a^b |x-t|^2 |g''(x)| \frac{-\ln B + B - 1}{(\ln B)^2} dt \right\} dx,
\end{aligned}
$$

where $A = \left| \frac{f''(t)}{f''(x)} \right|$, $B = \left| \frac{g''(t)}{g''(x)} \right|$.

Proof of Theorem 5

From the proof of Theorem 3, we have

$$
\begin{aligned}
S_s(f,g) \quad &= \quad \frac{-1}{4} \left\{ g(x) \left[\left(\frac{x-a}{b-a} \right)^3 \int_0^1 \lambda^2 f''((1-\lambda)a + \lambda x) d\lambda \right. \right. \\
&+ \quad \left. \left(\frac{b-x}{b-a} \right)^3 \int_0^1 \lambda^2 f''((1-\lambda)b + \lambda x) d\lambda \right] \\
&+ \quad f(x) \left[\left(\frac{x-a}{b-a} \right)^3 \int_0^1 \lambda^2 g''((1-\lambda)a + \lambda x) d\lambda \right. \\
&+ \quad \left. \left. \left(\frac{b-x}{b-a} \right)^3 \int_0^1 \lambda^2 g''((1-\lambda)b + \lambda x) d\lambda \right] \right\} (b-a)^2 .
\end{aligned}
$$

Integrating the above w.r.t.x from a to b, we have

$$T_s(f,g) =$$

$$
\begin{aligned}
&= \frac{-(b-a)}{4}\int_a^b \left\{\left(\frac{x-a}{b-a}\right)^3\left[g(x)\int_0^1 \lambda^2 \left|f''((1-\lambda)a+\lambda x)\right| d\lambda\right.\right. \\
&+ \left. f(x)\int_0^1 \lambda^2 \left|g''((1-\lambda)a+\lambda x)\right| d\lambda\right] \\
&\qquad + \left(\frac{b-x}{b-a}\right)^3\left[g(x)\int_0^1 \lambda^2 \left|f''((1-\lambda)b+\lambda x)\right| d\lambda\right. \\
&\qquad + \left.\left. f(x)\int_0^1 \lambda^2 \left|g''((1-\lambda)b+\lambda x)\right| d\lambda\right]\right\} dx.
\end{aligned}
$$

1. Since $|f''|$, $|g''|$ are convex on $[a,b]$, we have

$$
\begin{aligned}
|T_s(f,g)| &\le \frac{b-a}{4}\int_a^b \left\{\left(\frac{x-a}{b-a}\right)^3\left[|g(x)|\int_0^1 \left[\lambda^2(1-\lambda)|f''(a)|+\lambda^3|f''(x)|\right] d\lambda\right.\right. \\
&\quad + \left.|f(x)|\int_0^1 \left[\lambda^2(1-\lambda)|g''(a)|+\lambda^3|g''(x)|\right] d\lambda\right] \\
&\quad + \left(\frac{b-x}{b-a}\right)^3\left[|g(x)|\int_0^1 \left[\lambda^2(1-\lambda)|f''(b)|+\lambda^3|f''(x)|\right] d\lambda\right. \\
&\quad + \left.\left.|f(x)|\int_0^1 \left[\lambda^2(1-\lambda)|g''(b)|+\lambda^3|g''(x)|\right] d\lambda\right]\right\} dx \\
&= \frac{b-a}{48}\int_a^b \left\{\left(\frac{x-a}{b-a}\right)^3 \left[|g(x)|\left(|f''(a)|+3|f''(x)|\right)\right.\right. \\
&\quad + \left.|f(x)|\left(|g''(a)|+3|g''(x)|\right)\right] \\
&\quad + \left[|g(x)|\left(|f''(b)|+3|f''(x)|\right)\right. \\
&\quad + \left.\left.|f(x)|\left(|g''(b)|+3|g''(x)|\right)\right]\left(\frac{b-x}{b-a}\right)^3\right\} dx.
\end{aligned}
$$

2. Since $|f''|$, $|g''|$ are log-convex on $[a, x]$, $[x, b]$, we have from (20)

$$
\begin{aligned}
|T_s(f,g)| &= \frac{b-a}{4}\int_a^b \left\{ \left(\frac{x-a}{b-a}\right)^3 \left[|g(x)|\,|f''(a)| \int_0^1 \lambda^2 \left(\frac{|f''(x)|}{|f''(a)|}\right)^{\lambda} d\lambda \right.\right. \\
&+ \left. |f(x)|\,|g''(b)| \int_0^1 \lambda^2 \left(\frac{|g''(x)|}{|g''(a)|}\right)^{\lambda} d\lambda \right] \\
&+ \left(\frac{b-x}{b-a}\right)^3 \left[|g(x)|\,|f''(b)| \int_0^1 \lambda^2 \left(\frac{|f''(x)|}{|f''(b)|}\right)^{\lambda} d\lambda \right. \\
&+ \left.\left. |f(x)|\,|g''(b)| \int_0^1 \lambda^2 \left(\frac{|g''(x)|}{|g''(b)|}\right)^{\lambda} d\lambda \right] \right\} dx.
\end{aligned}
$$

Using the results of Theorem 3, we have

$$
\begin{aligned}
|T_s(f,g)| &= \frac{b-a}{4} \\
&\times \int_a^b \left\{ \left(\frac{x-a}{b-a}\right)^3 \left[|g(x)|\,|f''(a)| \frac{A_1(\ln A_1)^2 - 2A_1 \ln A_1 + 2A_1 - 2}{(\ln A_1)^3} \right.\right. \\
&+ \left. |f(x)|\,|g''(a)| \frac{A_2(\ln A_2)^2 - 2A_2 \ln A_2 + 2A_2 - 2}{(\ln A_2)^3} \right] \\
&+ \left(\frac{b-x}{b-a}\right)^3 \left[|g(x)|\,|f''(b)| \frac{B_1(\ln B_1)^2 - 2B_1 \ln B_1 + 2B_1 - 2}{(\ln B_1)^3} \right. \\
&+ \left.\left. |f(x)|\,|g''(b)| \frac{B_2(\ln B_2)^2 - 2B_2 \ln B_2 + 2B_2 - 2}{(\ln B_2)^3} \right] \right\} dx.
\end{aligned}
$$

Proof of Theorem 6

From the hypothesis of Theorem 6, and using the identities (15) and (16), we have:

$$\left[f\left(x\right)-\frac{1}{b-a}\int\limits_{a}^{b}f(t)dt-(x-\frac{a+b}{2})f'(x)\right] \times\left[g\left(x\right)-\frac{1}{b-a}\int\limits_{a}^{b}g(t)dt-(x-\frac{a+b}{2})g'(x)\right] \tag{28}$$

$$=\left\{\frac{1}{b-a}\int\limits_{a}^{b}\left(x-t\right)^{2}\left[\int\limits_{0}^{1}\left(1-\lambda\right)f''\left(\left(1-\lambda\right)x+\lambda t\right)d\lambda\right]dt\right\} \times\left\{\frac{1}{b-a}\int\limits_{a}^{b}\left(x-t\right)^{2}\left[\int\limits_{0}^{1}\left(1-\lambda\right)g''\left(\left(1-\lambda\right)x+\lambda t\right)d\lambda\right]dt\right\}.$$

Integrating both sides of (28) w.r.t. x from a to b, we have

$$\widetilde{T}_{s}(f,g)=\frac{1}{(b-a)^{3}}\int\limits_{a}^{b}\left\{\left[\int\limits_{a}^{b}(x-t)^{2}\left(\int\limits_{0}^{1}(1-\lambda)\,f''\left((1-\lambda)\,x+\lambda t\right)d\lambda\right)dt\right] \times\int\limits_{a}^{b}(x-t)^{2}\left(\int\limits_{0}^{1}(1-\lambda)\,g''\left((1-\lambda)\,x+\lambda t\right)d\lambda\right)dt\right\}dx, \tag{29}$$

where $\widetilde{T}_{s}(f,g)=\frac{1}{b-a}\int\limits_{a}^{b}\widetilde{S}_{s}(f,g)dx$ and

$$\widetilde{S}_{s}\left(f,g\right)=f(x)g(x)-(x-\tfrac{a+b}{2})\left[f(x)g'(x)+g(x)f'(x)\right]$$

$$- \frac{1}{b-a}\left[f(x)\int_a^b g(t)dt + g(x)\int_a^b f(t)dt\right] + (x-\frac{a+b}{2})^2 f'(x)g'(x)$$
$$+ (x-\frac{a+b}{2})\left[\frac{1}{b-a}f'(x)\int_a^b g(t)dt + \frac{1}{b-a}g'(x)\int_a^b f(t)dt\right]$$
$$+ \left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right).$$

1. Since $|f''|$, $|g''|$ are convex on $[a,b]$, we have

$$|T_s(f,g)|$$
$$\leq \frac{1}{(b-a)^3}\int_a^b \left\{\int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)^2 |f''(x)| + \lambda(1-\lambda)|f''(t)|\, d\lambda\right] dt\right.$$
$$\left.\times \int_a^b (x-t)^2\left[\int_0^1 (1-\lambda)^2 |g''(x)| + \lambda(1-\lambda)|g''(t)|\, d\lambda\right] dt\right\} dx$$
$$= \frac{1}{(b-a)^3}\int_a^b \left[\int_a^b (x-t)^2\left(\frac{|f''(x)|}{3} + \frac{|f''(t)|}{6}\right) dt\right.$$
$$\left.\times \int_a^b (x-t)^2\left(\frac{|g''(x)|}{3} + \frac{|g''(t)|}{6}\right) dt\right] dx$$
$$\leq \frac{1}{(b-a)^3}\int_a^b \left\{ ess\sup_{t\in[a,b]}\left(\frac{|f''(x)|}{3} + \frac{|f''(t)|}{6}\right)\right.$$
$$\left.\times\, ess\sup_{t\in[a,b]}\left(\frac{|g''(x)|}{3} + \frac{|g''(t)|}{6}\right)\left[\int_a^b (x-t)^2\, dt\right]^2\right\} dx$$
$$\leq \frac{1}{36(b-a)^3}\int_a^b (2|f''(x)| + \|f''\|_\infty)(2|g''(x)| + \|g''\|_\infty) E^2(x) dx,$$

where $E(x) = \dfrac{(x-a)^3 + (b-x)^3}{3}$.

2. Since $|f''|$, $|g''|$ are log-convex on [a,b], from (29) we observe that

$$\begin{aligned}
\left|\widetilde{T}_s(f,g)\right| &\leq \frac{1}{(b-a)^3} \int_a^b \left\{ \int_a^b |x-t|^2 |f''(x)| \left[\int_0^1 (1-\lambda) \left| \frac{f''(t)}{f''(x)} \right|^\lambda d\lambda \right] dt \right. \\
&\times \left. \int_a^b |x-t|^2 |g''(x)| \left[\int_0^1 (1-\lambda) \left| \frac{g''(t)}{g''(x)} \right|^\lambda d\lambda \right] dt \right\} dx \\
&= \frac{1}{(b-a)^3} \int_a^b \left\{ \int_a^b |x-t|^2 |f''(x)| \frac{-\ln A + A - 1}{(\ln A)^2} dt \right. \\
&\times \left. \int_a^b |x-t|^2 |g''(x)| \frac{-\ln B + B - 1}{(\ln B)^2} dt \right\} dx,
\end{aligned}$$

where $A = \left| \dfrac{f''(t)}{f''(x)} \right|$ and $B = \left| \dfrac{g''(t)}{g''(x)} \right|$.

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