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A proof of Schur's Conjecture and an improvement

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Abstract

In the paper I proved first Schur's Conjecture by using the properties of Bessel's functions of the first species. The second main result is an identity verified by the product $\sin ax \sin a(1-x)$, containing Schur's Conjecture as a particular case ($a = \frac{\pi}{2}$).

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1 Introduction

Schur showed that $\sin \pi x$ has a development of the form

$$\sin \pi x = \sum_{k=1}^{\infty} c_k [x(1-x)]^k, \tag{1}$$

the series of the right-side hand being convergent for $|x| \leq 1$.

By computing the first coefficients, Schur observed that $c_1 \geq 0$, $c_2 > 0$ and conjuctered that all coefficients are positive.

In the first part of the paper we give a solution to this problem.

It is known that Bessel's first species function verify:

$$J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\cos z,\ J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin z,\ z\in\mathbb{R}, \tag{2}$$

$$J_{n+1}(z)+J_{n-1}(z)=\frac{2n}{z}J_n(z),\ n\in\mathbb{R}, \tag{3}$$

$$J_n(az)=z^n\sum_{k=0}^{\infty}\frac{(1-z^2)^k}{k!}\left(\frac{a}{2}\right)^k J_{n+k}(a),\ n\in\mathbb{R}, \tag{4}$$

$$J_n(z)=\frac{2(\frac{z}{2})^n}{\sqrt{\pi}\Gamma(n+\frac{1}{2})}\int_0^1(1-t^2)^{n-\frac{1}{2}}\cos xt\mathrm{d}t, n>-\frac{1}{2},\ z\in\mathbb{R}. \tag{5}$$

2 Main results

Propertie 1 *The coefficient c_k, $k=1,2,\ldots$ from (1) are strictly positive.*

Proof. Let $n=-\frac{1}{2}$ in (4). Then:

$$J_{-\frac{1}{2}}(az)=\frac{1}{\sqrt{z}}\sum_{k=0}^{\infty}\frac{(1-z^2)^k}{k!}\left(\frac{a}{2}\right)^k J_{k-\frac{1}{2}}(a)$$

Using (2), we obtain:

$$\sqrt{\frac{2}{\pi az}}\cos az=\frac{1}{\sqrt{z}}\sum_{k=0}^{\infty}\frac{(1-z^2)^k}{k!}\left(\frac{a}{2}\right)^k J_{k-\frac{1}{2}}(a)$$

$$\cos az=\sqrt{\frac{\pi a}{2}}\sum_{k=0}^{\infty}\frac{(1-z^2)^k}{k!}\left(\frac{a}{2}\right)^k J_{k-\frac{1}{2}}(a)$$

In the above equality, we consider $z = 2x - 1$, $a = \dfrac{\pi}{2}$. It follows:
$\cos\dfrac{\pi}{2}(2x-1) = \sin\pi x = \dfrac{\pi}{2}\displaystyle\sum_{k=1}^{\infty}\frac{[x(1-x)]^k}{k!}\pi^k J_{k-\frac{1}{2}}\left(\frac{\pi}{2}\right) + \frac{\pi}{2}J_{-\frac{1}{2}}\left(\frac{\pi}{2}\right)$

From (5), it follows:

$$\sin\pi x = 2\cdot\sum_{k=1}^{\infty}\frac{[x(1-x)]^k}{\Gamma(k)\Gamma(k+1)}\cdot\left(\frac{\pi}{2}\right)^{2k}\cdot\int_0^1(1-t^2)^{k-1}\cos\frac{\pi t}{2}dt$$

$$\sin\pi x = \sum_{k=1}^{\infty}c_k[x(1-x)]^k, \text{ where}$$

$$(5') \qquad c_k = \frac{2}{\Gamma(k)\Gamma(k+1)}\cdot\left(\frac{\pi}{2}\right)^{2k}\int_0^1(1-t^2)^{k-1}\cos\frac{\pi t}{2}dt > 0, (\forall)\, k\in\mathbb{N}^*.$$

In what follows we are concerned with the determination of the coefficients $d_k(a)$ from the development:

$$(6) \qquad \sin ax\sin a(1-x) = \sum_{k=1}^{\infty}x^k(1-x)^k d_k(a), a\in\mathbb{R}, |x|\le 1.$$

Choosing in (4) $n = -\frac{1}{2}$, we can write

$$J_{-\frac{1}{2}}(az) = \frac{1}{\sqrt{z}}\sum_{k=0}^{\infty}\frac{(1-z^2)^k}{k!}\left(\frac{a}{2}\right)J_{k-\frac{1}{2}}(a),$$

$$(7) \qquad J_{-\frac{1}{2}}(az) - \frac{1}{\sqrt{z}}J_{-\frac{1}{2}}(a) = \frac{1}{\sqrt{z}}\sum_{k=1}^{\infty}\frac{(1-z^2)^k}{k!}\left(\frac{a}{2}\right)^k J_{k-\frac{1}{2}}(a).$$

From(2), equality (7) can be written as follows:

$$(8) \qquad \cos az - \cos a = \sqrt{\frac{\pi a}{2}}\sum_{k=1}^{\infty}\frac{(1-z^2)^k}{k!}\left(\frac{a}{2}\right)^k J_{k-\frac{1}{2}}(a),$$

and for $z = 2x-1$, we have:

$$\sin(ax)\sin a(1-x) = \frac{1}{2}\sqrt{\frac{\pi a}{2}}\sum_{k=1}^{\infty}\frac{[x(1-x)]^k}{k!}(2a)^k J_{k-\frac{1}{2}}(a).$$

Thus,

$$\sin(ax)\sin a(1-x) = \sum_{k=1}^{\infty} d_k(a)[x(1-x)]^k, \text{ where}$$

$$(8') \qquad d_k(a) = \frac{1}{2}\sqrt{\frac{\pi a}{2}} \cdot \frac{(2a)^k}{k!} J_{k-\frac{1}{2}}(a),\ k = 1, 2, \ldots$$

By using (3) for $n = k - \frac{1}{2}$, we obtain for $z = a$:

$$J_{k+\frac{1}{2}}(a) = \frac{2k-1}{a} J_{k-\frac{1}{2}}(a) - J_{k-1-\frac{1}{2}}(a),$$

$$(9) \qquad d_{k+1}(a) = \frac{2(2k-1)}{k+1} d_k(a) - \frac{4a^2}{k(k+1)} d_{k-1}(a),\ k = 2, 3, \ldots$$

The coefficients $d_1(a)$ and $d_2(a)$ are:

$$d_1(a) = \frac{1}{2}\sqrt{\frac{\pi a}{2}} \cdot \frac{2a}{1!} J_{1/2}(a) = a \sin a \ \text{(see (2))}$$

$$d_2(a) = \frac{1}{2}\sqrt{\frac{\pi a}{2}} \cdot \frac{4a^2}{2} J_{\frac{3}{2}}(a) = \frac{1}{a}\sqrt{\frac{2}{\pi a}}(\sin a - a\cos a),\ a > 0. \text{(see (3) and (2))}$$

From (5) and (8′) we have:

$$(10) \qquad d_k(a) = \frac{a^{2k}}{\Gamma(k+1)\Gamma(k)} \int_0^1 (1-t^2)^{k-1} \cos at\, dt$$

Remark 1 $2d_k\left(\frac{\pi}{2}\right) = c_k$ *(see (5′) and (10)):*
if $a \in \left(0, \frac{\pi}{2}\right)$ *the coefficients* $d_k(0)$ *are positive.*
By using (10), for $a \in \left(0, \frac{\pi}{2}\right]$ *we obtain:*

$$d_{k+1} < \frac{a^2}{k(k+1)} \cdot \frac{a^{2k}}{\Gamma(k)\Gamma(k+1)} \int_0^1 (1-t^2)^k \cos at\, dt,$$

$$d_{k+1}(a) < \frac{a^2}{k(k+1)} d_k(a),\ \frac{d_{k+1}(a)}{d_k(a)} < \frac{a^2}{k(k+1)} < \frac{\pi^2}{24}$$

Taking $k = 2, 3, \ldots, n-1$ *and multiplying the inequalities:*

$$(11) \qquad d_n(a) < \left(\frac{\pi^2}{24}\right)^{n-2} d_2(a)$$

Propertie 2 *If* $a \in \left(0, \frac{\pi}{2}\right]$, *the coefficients* $d_k(a)$ *from the development (6) verify:*

i) $2d_k\left(\frac{\pi}{2}\right) = c_k$, *where* c_k *are the coefficients from the development (1)*

ii) $(d_n(a))_{n\geq 2}$ *is decreasing.*

iii) $d_n(a) < \left(\frac{\pi^2}{24}\right)^{n-2} d_2(a), (\forall) n \geq 2.$

iv) $d_{k+1}(a) = \frac{1}{k(k+1)}[2k(2k-1)d_k(a) - 4a^2 d_{k-1}(a)],\ k \in \mathbb{N}^*,$
$d_1(a) = a \sin a, d_2(a) = a(\sin a - a\cos a)$

Propertie 3 *The coefficients* $d_k(a)$ *verify:*

$$\frac{1}{2} \cdot \frac{(2a)^{2k}}{(2k)!}\left[1 - \frac{a^2}{2(2k+1)} + \frac{a^4}{8(2k+1)(2k+3)} - \frac{a^6}{48(2k+1)(2k+3)(2k+5)}\right] <$$
$$< d_k(a) < \frac{1}{2} \cdot \frac{(2a)^{2k}}{(2k)!} \cdot \left[1 - \frac{a^2}{2(2k+1)} + \frac{a^4}{8(2k+1)(2k+3)} - \right.$$
$$\left. - \frac{a^6}{48(2k+1)(2k+3)(2k+5)} + \frac{a^8}{24 \cdot 16(2k+1)(2k+3)(2k+5)(2k+7)}\right]$$

Proof. From

$$J_{k-\frac{1}{2}}(a) = \frac{(2a)^{k-\frac{1}{2}}}{\sqrt{\pi}} \sum_{j\geq 0} \frac{(-1)^j (k+j-1)!}{j!(2k+2j-1)!} a^{2j},\ k \in \mathbb{N}^*,$$
$$J_{k-\frac{1}{2}}(a) = \frac{(2a)^{k-\frac{1}{2}}}{\sqrt{\pi}} \cdot \frac{(k-1)!}{(2k-1)!} \cdot \left[1 - \frac{a^2}{2(2k+1)} + \frac{a^4}{8(2k+1)(2k+3)} - \right.$$
$$\left. - \frac{a^6}{48(2k+1)(2k+3)(2k+5)} + \frac{a^8}{24 \cdot 16(2k+1)(2k+3)(2k+5)(2k+7)} - \dots\right]$$

we obtain

$$\frac{2}{\sqrt{\pi}}(2a)^{k-\frac{1}{2}} \frac{k!}{(2k)!}\left[1 - \frac{a^2}{2(2k+1)} + \frac{a^4}{8(2k+1)(2k+3)} - \right.$$
$$\left. - \frac{a^6}{48(2k+1)(2k+3)(2k+5)}\right] < J_{k-\frac{1}{2}(a)} < \frac{2}{\sqrt{\pi}}(2a)^{k-\frac{1}{2}} \frac{k!}{(2k)!}\ [1-$$

$$-\frac{a^2}{2(2k+1)}+\frac{a^4}{8(2k+1)(2k+3)}\ \frac{a^6}{48(2k+1)(2k+3)(2k+5)}+$$
$$\left.\frac{a^8}{24\cdot 16(2k+1)(2k+3)(2k+5)(2k+7)}\right],$$
$$\frac{1}{2}\cdot\frac{(2a)^{2k}}{(2k)!}\left[1-\frac{a^2}{2(2k+1)}+\frac{a^4}{8(2k+1)(2k+3)}-\frac{a^6}{48(2k+1)(2k+3)(2k+5)}\right]<$$
$$< d_k(a) < \frac{1}{2}\cdot\frac{(2a)^{2k}}{(2k)!}\cdot\left[1-\frac{a^2}{2(2k+1)}+\frac{a^4}{8(2k+1)(2k+3)}-\right.$$
$$\left.-\frac{a^6}{48(2k+1)(2k+3)(2k+5)}+\frac{a^8}{24\cdot 16(2k+1)(2k+3)(2k+5)(2k+7)}\right]$$

Propertie 4 *Let* $p \in \mathbb{N}^*$, $p \geq 2$. *Exists* $\theta = \theta(x) \in (1, \frac{4}{3})$ *such that*

$$\sin ax \sin a(1-x) = \sum_{k=1}^{p-1} d_k(a)[x(1-x)]^k + \theta(x) d_p(a)[x)(1-x)]^p.$$

Proof. From

$$\sin ax \sin a(1-x) = \sum_{k=1}^{\infty} d_k(a)[x(1-x)]^k$$

and Property 2 we have

$$\sum_{k=1}^{p} d_k(a)[x(1-x)]^k < \sin ax \sin a(1-x) < \sum_{k=1}^{p} d_k(a)[x(1-x)]^k + d_p(a)\sum_{k=p+1}^{\infty}$$
$$[x(1-x)]^k = \sum_{k=1}^{p} d_k(a)[x(1-x)]^k + d_p(a)[x(1-x)]^p\cdot[x(1-x)]\sum_{k=0}^{\infty}[x(1-x)]^k,$$
$$\sum_{k=1}^{p} d_k(a)[x(1-x)]^k < \sin ax \sin a(1-x) < \sum_{k=1}^{p} d_k(a)[x(1-x)]^k + d_p(a)\cdot$$
$$\cdot[x(1-x)]^p\tfrac{x(1-x)}{1-x+x^2}$$
$$\sum_{k=1}^{p} d_k(a)[x(1-x)]^k < \sin ax \sin a(1-x) < \sum_{k=1}^{p} d_k(a)[x(1-x)]^k + \frac{1}{3}d_p(a)\cdot$$
$$\cdot[x(1-x)]^p$$
$$\sum_{k=1}^{p-1} d_k(a)[x(1-x)]^k + d_p(a)[x(1-x)]^p < \sin ax \sin a(1-x) < \sum_{k=1}^{p-1} d_k(a)\cdot$$

$\cdot[x(1-x)]^k + \frac{4}{3}d_p(a)[x(1-x)]^p.$

We obtain:

$\sin ax \sin a(1-x) = \sum_{k=1}^{p-1} d_k(a)[x(1-x)]^k + d_p(a)[x(1-x)]^p \cdot \theta(x)$ where $\theta(x) \in (1, \frac{4}{3})$.

Remark 2 *For $a = \frac{\pi}{2}$ and $p = 5$, we obtain*

$$\sin \pi x = \pi x(1-x) + \pi[x(1-x)]^2 + \pi\left(2 - \frac{\pi^2}{6}\right)[x(1-x)]^3 + \pi\left(5 - \frac{\pi^2}{2}\right)$$
$$[x(1-x)]^4 + \pi\left(\frac{\pi^4}{120} - \frac{3\pi^2}{2} + 14\right)[x(1-x)]^5 \cdot \theta(x),\ \theta(x) \in (1, \tfrac{4}{3})$$

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