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Kalecki's model of business cycle. Data dependence[1]

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Abstract

In this paper we study date dependence for a delay equation which models a business cycle. The study is made using weakly Picard operators.

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1 Introduction

Let (X, d) be a metric space and $A : X \longrightarrow X$ an operator. We shall use the following notations:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\},$$

the set of valid parts of X;

$$F_A := \{x \in X \mid A(x) = x\},$$

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the fixed point set of A;

$$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\},$$

the family of the nonempty invariant subset of A.

$$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in \mathbb{N}.$$

Definition 1.1. *([2],[3]) An operator A is weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$ and the limit (which depend on x) is a fixed point of A.*

Definition 1.2. *([2],[3]) If the operator A is weakly Picard operator and $F_A = \{x^*\}$ then by definition A is Picard operator.*

Definition 1.3. *([2],[3]) If A is weakly Picard operator, then we consider the operator*

$$A^\infty : X \longrightarrow X, A^\infty(x) = \lim_{n\longrightarrow\infty} A^n(x).$$

Observation 1.1. $A^\infty(x) = F_A$.

Definition 1.4. *([2],[3]) Let be A an WPO and $c > 0$. The operator A is c-weakly Picard operator if*

$$d(x, A^\infty(x) \leq c \cdot d(x, A(X)).$$

for all $x \in X$.

Observation 1.2. *([4]) If (X, d) is a metric space and $A : X \longrightarrow X$ an operator is a a-contraction, then A is c-weakly Picard operator with $c = \frac{1}{1-a}$.*

The next result it is a characteristic of weakly Picard operator, respectively c-weakly Picard.

Theorem 1.1. *([2],[3]) Let (X, d) be a metric space and $A : X \longrightarrow X$ an operator. The operator A is weakly Picard operator (c-weakly Picard operator) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

such that:

(a) $X_\lambda \in I(A)$;

(b) $A \mid X_\lambda : X_\lambda \longrightarrow X_\lambda$ *is a Picard (c-Picard)operator, for all* $\lambda \in A$.

2 Main result

We consider the equation

$$(1) \qquad I'(t) = \frac{m}{t} I(t) - (\frac{m}{\tau} + n) I(t-\tau) + nu, t \in [0,T]$$

where $\tau > 0, m = \frac{p}{1-\alpha}, p > 0, n > 0, \alpha \in (0,1), u > 0$, with the initial condition

$$(2) \qquad I(t) = \varphi(t), t \in [-\tau, 0]$$

with $\varphi : [-\tau, 0] \longrightarrow \mathbb{R}$.

The Cauchy problem $(1)+(2)$ is equivalent with the next integral equation

$$(3) \quad I(t) = \begin{cases} \varphi(0) + \int\limits_0^t [\frac{m}{\tau} I(s) - (\frac{m}{\tau} + \eta) I(s-\tau) + nu] ds & , \quad t \in [0,T] \\ \varphi(t) & , \quad t \in [-\tau, 0] \end{cases}$$

We search the solutions for the integral equation (3) in the continuous functions space $(C[-\tau,T] \mid \cdot \mid_r)$ endowed with Bielecki norm,

$$|x|_r = \sup_{t \in [-\tau, T]} |x(t)| e^{-rt}.$$

Next, we consider the operator $A : C[-\tau, T] \longrightarrow C[-\tau, T]$, defined by

$$A(I)(t) = \begin{cases} \varphi(0) + \int\limits_0^t [\frac{m}{\tau} I(s) - (\frac{m}{\tau} + \eta) I(s-\tau) + nu] ds & , \quad t \in [0,T] \\ \varphi(t) & , \quad t \in [-\tau, 0] \end{cases}$$

Then for any $I_1, I_2 \in C[-\tau, T]$ we have

$$|A(I_1)(t)-A(I_2)(t)| \leq \int_0^t [\frac{m}{\tau}|I_1(s)-I_2(s)|+(\frac{m}{\tau}+n)|I_1(s-\tau)-I_2(s-\tau)|]ds \leq$$

$$\leq |I_1 - I_2|_r \int_0^t \frac{m}{\tau}e^{rs} + (\frac{m}{\tau} + n)e^{r(s-\tau)} ds \leq$$

$$\leq I_1 - I_2|_r e^{rt} \frac{\frac{2m}{\tau} + n}{r} e^{rt}$$

It follows that

$$|A(I_1) - A(I_2)|_r \leq \frac{\frac{2m}{\tau} + n}{r} \cdot |I_1 - I_2|_r$$

Using the Banach principle of fixed point it results that equation (3) has, in $C[-\tau, T]$, a unique solution $I^\star(\cdot, \varphi)$.
In the following lines, using the characterization theorem of the weakly Picard operator we show that equation (1) has a infinity of solutions. Indeed, for $\varphi \in C[-\tau, 0]$, we consider

$$X_\varphi = \{x \in C[-\tau, T] | x_{|[-\tau,0]} = \varphi\}.$$

We choose $r > 0$ thus

$$\frac{2m}{\tau} + n < 1$$

it results

$$|A(I_1) - A(I_2)|_r \leq \frac{(\frac{2m}{\tau} + n)}{r} |I_1 + I_2|.$$

According to the above, the operator $A|_{X_\varphi} : X_\varphi \longrightarrow X_\varphi$ is a Picard operator. Then A is a weakly Picard operator and as consequence the equation (1) has a infinity of solutions.

Next we assume that it exists $\eta > 0$ thus

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta.$$

Let $I^\star(\cdot,\varphi_1), I^\star(\cdot,\varphi_2)$ be the solution of the equation (3) with data φ_1, φ_2. Then

$$|I^\star(t,\varphi_1) - I^\star(t,\varphi_2)| \leq$$

$$\leq \eta \int_0^t [\frac{m}{\tau}|I^\star(s,\varphi_1) - I^\star(s,\varphi_2)| + (\frac{m}{\tau} + n)|I^\star(s-\tau,\varphi_1) - I^\star(t,\varphi_2)|]ds.$$

According with Theorem 14.6 (see [1] pp 145) we obtain

$$|I_1(t) - I_2(t)| \leq \eta k(m,n,\tau)e^{\int_0^t (\frac{2m}{\tau}+n)ds} \leq \eta k(m,n,\tau)e^{(\frac{2m}{\tau}+n)T}$$

So, from above we obtain the following result

Theorem 2.1. *We consider the equation (1). Then:*

(a) the equation (1) has, in $(C([-\tau,T]), |\cdot|_r)$ *a infity of solutions;*

(b) the problem (1)+(2) has, in $C([-\tau,T]), |\cdot|_r$*, a unique solution* $I^\star(\cdot,\varphi)$*;*

(c) if there exists $\eta > 0$ *such that*

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta,$$

for all $t \in [-\tau,T]$*, then there exists* $k(m.n.\tau) > 0$ *such that*

$$|I^\star(t,\varphi_1) - I^\star(t,\varphi_2)| \leq \eta k(m,n,\tau)e^{(\frac{2m}{\tau}+n)T}.$$

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