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Hyers-Ulam-Rassias Stability of additive type Functional Equation

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Abstract

In this paper, the Hyers-Ulam-Rassias stability of additive type functional equation

$$f(rx+sy)=\frac{r+s}{2}f(x+y)+\frac{r-s}{2}f(x-y)$$

$r, s \in \mathbb{R}$ and $r \neq \pm s$ over a unital C^*-algebra will be investigate.

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1 Introduction and preliminaries

One of the interesting questions in the theory of functional equations concerning the problem of the stability of functional equations is as follows:

when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The first stability problem was raised by Ulam during his talk at the University of Wisconsin in 1940 [11].

Given a group G_1, a metric group (G_2, d), and a positive number ε, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$?

Ulam's problem was partially solved by Hyers in 1941 in the context of Banach spaces with $\varepsilon = \delta$ as shown below [3].

Theorem 1.1 *(D. H. Hyers (1941)). Let E_1 be a normed vector space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all x in E_1 where $\varepsilon > 0$ is a constant. Then the limit

$$g(x) = \lim_n 2^{-n} f(2^n x)$$

exists for each $x \in E_1$ and g is the unique additive mapping satisfying

$$\|f(x) - g(x)\| \leq \varepsilon$$

for all $x \in E_1$. Also, if for each x the function $t \to f(tx)$ from $\mathbb{R}$ to E_2 is continuous for each fixed x, then g is linear. If f is continuous at a single point of E_1, then g is continuous in E_1.

Aoki [1] and Th.M. Rassias [10] provided a generalization of the Hyers theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.2 *(Th.M. Rassias). Let $f : E \to E_0$ be a mapping from a normed vector space E into a Banach space E_0 subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E_0$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \to f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The above inequality has provided a lot of influence in the development of what is now known as a generalized Hyers-Ulam-Rassias stability of functional equations. P. Gavruta [2] provided a further generalization of the Th.M. Rassias theorem. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam-Rassias stability to a number of functional equations and mappings (see [6, 7, 8, 10]). We also refer the readers to the books [11, 5].

Th. M. Rassias (1990) during the 27'th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda (1991) gave an affirmative solution to this question for $p > 1$. It is shown that there is no analogue of Rassias result for $p = 1$, [5].

In this paper, we introduce the following additive functional equation

$$f(rx + sy) = \frac{r+s}{2} f(x+y) + \frac{r-s}{2} f(x-y) \tag{1.3}$$

$r, s \in \mathbb{R}$ and $r \neq s$, We investigate the Hyers-Ulam-Rassias stability of the functional equation (1.3) in Banach modules over a unital C^*-algebra. These results are applied to investigate homomorphisms between unital C^*-algebras.

2 Hyers-Ulam-Rassias stability of the functional equation (1.3) in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with norm $|.|$, unit 1. Also we assume that X and Y are (unit linked) normed left A-module and Banach left A-module with norms $\|.\|_X$ and $\|.\|_Y$, respectively. Let $U(A)$ be the set of unitary elements in A and let $r, s \in \mathbb{R}$ and $r \neq s$. For a given mapping $f : X \to Y, u \in U(A)$ and a given $\mu \in \mathbb{C}$, we define $D_u f$, $D_\mu f : X^2 \to Y$ by

$$D_u f(x, y) := f(rux + suy) - \frac{r+s}{2} uf(x+y) - \frac{r-s}{2} uf(x-y),$$

$$D_\mu f(x,y) := f(r\mu x + s\mu y) - \frac{r+s}{2}\mu f(x+y) - \frac{r-s}{2}\mu f(x-y)$$

for all $x, y \in X$. An additive mapping $f : X \to Y$ is called A-linear if $f(ax) = af(x)$ for all $x \in X$ and all $a \in A$.

Proposition 1 *[9] Let $L : X \to Y$ be a mapping with $L(0) = 0$ such that*

$$D_u L(x,y) = 0 \quad \forall x, y \in X, \forall u \in U(A).$$

Then L is A-linear.

Corollary 1 *Let $L : X \to Y$ be a mapping with $L(0) = 0$ such that $D_1 L(x,y) = 0$ for all $x, y \in X$. Then L is additive.*

Corollary 2 *A mapping $L : X \to Y$ with $L(0) = 0$ satisfies $D_\mu L(x,y) = 0$ for all $x, y \in X$ and all $\mu \in T := \{\mu \in C : |\mu| = 1\}$, if and only if L is $\mathbb{C}$-linear.*

Now, we investigate the Hyers-Ulam-Rassias stability of the functional equation (1.3) in Banach modules.

We recall that throughout this paper $r, s \in \mathbb{R}$ with $r+s, r-s \neq 0$.

Theorem 2.1 *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{2^k}\varphi(2^k x, 2^k y) = 0, \tag{2.1}$$

$$\begin{aligned} \widetilde{\varphi}(x) := \sum_{k=0}^{\infty} \frac{1}{2^k}\Big\{\varphi\big(\frac{2^{k+1} r x}{r^2 - s^2}, \frac{-2^{k+1} s x}{r^2 - s^2}\big) \\ + \varphi\big(\frac{2^k x}{r+s}, \frac{2^k x}{r+s}\big) + \varphi\big(\frac{2^k x}{r-s}, \frac{-2^k x}{r-s}\big)\Big\} < \infty, \end{aligned} \tag{2.2}$$

$$\|D_1 f(x,y)\|_Y \le \varphi(x,y) \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{1}{2}\widetilde{\varphi}(x) \tag{2.4}$$

for all $x \in X$.

Proof. It follows from (2.3)that

$$\begin{aligned}&\Big\|D_1 f(x,y) - D_1 f\big(\frac{x+y}{2}, \frac{x+y}{2}\big) - D_1 f\big(\frac{x-y}{2}, \frac{y-x}{2}\big)\Big\|_Y\\ &\le \varphi(x,y) + \varphi\big(\frac{x+y}{2}, \frac{x+y}{2}\big) + \varphi\big(\frac{x-y}{2}, \frac{y-x}{2}\big)\end{aligned}$$

for all $x, y \in X$. Therefore

$$\begin{aligned}&\Big\|f(rx+sy) - f\big(\frac{r+s}{2}(x+y)\big) - f\big(\frac{r-s}{2}(x-y)\big)\Big\|_Y\\ &\le \varphi(x,y) + \varphi\big(\frac{x+y}{2}, \frac{x+y}{2}\big) + \varphi\big(\frac{x-y}{2}, \frac{y-x}{2}\big)\end{aligned} \tag{2.5}$$

for all $x, y \in X$. Replacing x by $\frac{1}{r+s}x + \frac{1}{r-s}y$ and y by $\frac{1}{r+s}x - \frac{1}{r-s}y$ in (2.5), we get

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\|_Y \le \varphi\big(&\frac{x}{r+s} + \frac{y}{r-s}, \frac{x}{r+s} - \frac{y}{r-s}\big)\\ &+ \varphi\big(\frac{x}{r+s}, \frac{x}{r+s}\big) + \varphi\big(\frac{y}{r-s}, \frac{-y}{r-s}\big)\end{aligned} \tag{2.6}$$

for all $x, y \in X$. Letting $y = x$ in (2.6), we get

$$\begin{aligned}\|f(2x) - 2f(x)\|_Y \le \varphi\big(&\frac{2rx}{r^2-s^2}, \frac{-2sx}{r^2-s^2}\big)\\ &+ \varphi\big(\frac{x}{r+s}, \frac{x}{r+s}\big) + \varphi\big(\frac{x}{r-s}, \frac{-x}{r-s}\big)\end{aligned} \tag{2.7}$$

for all $x \in X$. For convenience, set

$$\psi(x) := \varphi\big(\frac{2rx}{r^2-s^2}, \frac{-2sx}{r^2-s^2}\big) + \varphi\big(\frac{x}{r+s}, \frac{x}{r+s}\big) + \varphi\big(\frac{x}{r-s}, \frac{-x}{r-s}\big)$$

for all $x \in X$. It follows from (2.2) that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \psi(2^k x) = \widetilde{\varphi}(x) < \infty \tag{2.8}$$

for all $x \in X$. Replacing x by $2^k x$ in (2.7) and dividing both sides of (2.7) by 2^{k+1}, we get

$$\left\| \frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^k} f(2^k x) \right\|_Y \leq \frac{1}{2^{k+1}} \psi(2^k x)$$

for all $x \in X$ and all $k \in \mathbb{N}$. Therefore we have

$$\begin{aligned} \left\| \frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^m} f(2^m x) \right\|_Y &\leq \sum_{l=m}^{k} \left\| \frac{1}{2^{l+1}} f(2^{l+1}x) - \frac{1}{2^l} f(2^l x) \right\|_Y \\ &\leq \frac{1}{2} \sum_{l=m}^{k} \frac{1}{2^l} \psi(2^l x) \end{aligned} \tag{2.9}$$

for all $x \in X$ and all integers $k \geq m \geq 0$. It follows from (2.8) and (2.9) that the sequence $\{\frac{f(2^k x)}{2^k}\}$ is a Cauchy sequence in Y for all $x \in X$, and thus converges by the completeness of Y. So we can define the mapping $L : X \to Y$ by

$$L(x) = \lim_{k \to \infty} \frac{f(2^k x)}{2^k}$$

for all $x \in X$. Letting $m = 0$ in (2.9) and taking the limit as $k \to \infty$ in (2.9), we obtain the desired inequality (2.4). It follows from the definition of L, (2.1) and (2.3) that

$$\begin{aligned} \|D_1 L(x,y)\|_Y &= \lim_{k \to \infty} \frac{1}{2^k} \|D_1 f(2^k x, 2^k y)\|_Y \\ &\leq \lim_{k \to \infty} \frac{1}{2^k} \varphi(2^k x, 2^k y) = 0 \end{aligned}$$

for all $x, y \in X$. Therefore the mapping $L : X \to Y$ satisfies the equation (1.3) and $L(0) = 0$. Hence by Proposition 1, L is a additive mapping.

To prove the uniqueness of L, let $L' : X \to Y$ be another additive mapping satisfying (2.4). Therefore it follows from (2.4) and (2.8) that

$$\begin{aligned}\|L(x) - L'(x)\|_Y &= \lim_{k\to\infty} \frac{1}{2^k} \left\| f(2^k x) - L'(2^k x) \right\|_Y \\ &\leq \frac{1}{2} \lim_{k\to\infty} \frac{1}{2^k} \sum_{l=0}^{\infty} \frac{1}{2^l} \psi(2^{l+k} x) \\ &= \frac{1}{2} \lim_{k\to\infty} \sum_{l=k}^{\infty} \frac{1}{2^l} \psi(2^l x) = 0\end{aligned}$$

for all $x \in X$. So $L(x) = L'(x)$ for all $x \in X$. It completes the proof.

Theorem 2.2 *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, 1)$ satisfying (2.1), (2.2) and*

$$\|D_u f(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in X$ and all $u \in U(A)$. Then there exists a unique A-linear mapping $L : X \to Y$ satisfying (2.4) for all $x \in X$.

Proof. The proof follows by letting $u = 1$ in (2.1) and using Proposition 1.

Corollary 3 *Let δ, ε, p and q be non-negative real numbers such that $0 < p, q < 1$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_1 f(x, y)\|_Y \leq \delta + \varepsilon(\|x\|_X^p + \|y\|_X^q)$$

$$(\|D_u f(x, y)\|_Y \leq \delta + \varepsilon(\|x\|_X^p + \|y\|_X^q))$$

for all $x, y \in X$ *(and all* $u \in U(A)$*). Then there exists a unique additive (A-linear) mapping* $L : X \to Y$ *such that*

$$\|f(x) - L(x)\|_Y$$

$$\leq 3\delta + \frac{2|r|^p + |r+s|^p + |r-s|^p}{(2-2^p)|r^2 - s^2|^p}\varepsilon\|x\|_X^p + \frac{2|s|^q + |r+s|^q + |r-s|^q}{(2-2^q)|r^2 - s^2|^q}\varepsilon\|x\|_X^q$$

for all $x \in X$*.*

Proof. Define $\varphi(x, y) := \delta + \varepsilon(\|x\|_X^p + \|y\|_X^q)$, and apply Theorem 2.1 (Theorem 2.2).

Corollary 4 *Let* δ*,* ε*,* p *and* q *be non-negative real numbers such that* $\lambda := p + q \neq 1$ *and* $|r| \neq |r|^\lambda$*. Assume that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|D_1 f(x, y)\|_Y \varepsilon \|x\|_X^p \|y\|_X^q$$

$$(\|D_u f(x, y)\|_Y \leq \varepsilon\|x\|_X^p + \|y\|_X^q)$$

for all $x, y \in X$ *(and all* $u \in U(A)$*). Then* f *is additive (A-linear).*

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan 2 (1950), 64-66.

[2] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431-436.

[3] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222-224.

[4] D.H. Hyers, G. Isac and Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhauser, Basel, 1998.

[5] D. Hyers, G. Isac and Th. M. Rassias, Stability of functional equations in sevral variables, Birkhauser, (1998).

[6] A. Najati, On the stability of a quartic functional equation, J. Math. Anal. Appl. 340 (2008), 569-574.

[7] A. Najati and M.B. Moghimi, Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces, J. Math. Anal. Appl. 337 (2008), 399-415.

[8] A. Najati and C. Park, The Pexiderized Apollonius-Jensen type additive mapping and isomorphisms between C^*-algebras, J. Difference Equat. Appl. 14 (2008), 459-479.

[9] A. Najati and A. Rahimi, Homomorphisms between C^*-algebras and their stabilities, Acta Univ. Apulensis, No 19, (2009).

[10] Th. M. Rassias, Functional Equations, Inequalities and Applications, Proc. Amer. Math. Soc., 72 (1978), 297-300.

[11] S.M. Ulam, A Collection of the Mathematical Problems, Interscience Publ. New York, 1960.

[12] S. M. Ulam, On the stability of the linear functional equations, Proc. Nat. Acad. Sci. U.S.A., 27 (1941). 222-224.

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