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Application of Differential subordination on p-valent Functions with a fixed point

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Abstract

Making Use of the familiar differential subordination Structure in this paper, we investigate a new class of p-valent functions with a fixed point w. Some results connected to sharp coefficient bounds, Distortion Theorem and other important properties are obtained.

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1 Introduction

Let $S_w(P)$ for a fixed point w denote the class of functions $f(z)$ of the form

$$f(z) = \frac{1}{(z-w)^p} + \sum_{n=1}^{+\infty} a_{n+p}(z-w)^{n+p} \qquad a_{n+p} \geq 0 \quad (1.1)$$

For the function $f(z)$ in the class $S_w(P)$, Frasin and Darus [3] defined the differential operator D^k as follow:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = (z-w)\acute{f}(z) + \frac{1+p}{(z-w)^p}$$

$$D^2 f(z) = (z-w)(D^1 f(z))' + \frac{1+p}{(z-w)^p}$$

and for $k = 1, 2, 3...$

$$D^k f(z) = (z-w)(D^{k-1} f(z))' + \frac{p+1}{(z-w)^p}$$

$$= \frac{1}{(z-w)^p} + \sum_{n=1}^{+\infty} (n+p)^k a_{n+p} (z-w)^{n+p} \quad (1.2)$$

See also [4].

Now we define the class $\Omega^w p(A, B)$ consisting the functions $f(z) \epsilon S_w(p)$ such that

$$-\frac{(z-w)(D^k f(z))''}{[D^k f(z)]'} \prec H(z) \quad (1.3)$$

where $H(z) = (p+1)\frac{1+Az}{1+Bz}$, $A = B + (C-B)(1-\lambda)$, $-1 \leq B < C \leq 1$, $0 \leq \lambda < 1$ and " $\prec$ " denotes the subrdination Symbol. See [2],[5] and [1].

2 Main Results

In this section we find sharp Coefficient estimates and Integral representation for the class $\Omega_p^w(A, B)$.

Theorem 2.1 *Let $f(z) \in S_w(p)$, then $f(z) \in \Omega_p^w(A,B)$ if and only if*

$$\sum_{n=1}^{+\infty}[(n+2p)(B+1)+(p+1)(C-B)(1-\lambda)]a_{n+p} < P(p+1)(C-B)(1-\lambda). \quad (2.1)$$

The result is sharp for the function $h(z)$ given by

$$h(z) = \frac{1}{(z-w)^p} + \frac{P(p+1)(C+B)(1-\lambda)}{(n+2p)(B+1)+(p+1)(C+B)(1\lambda)}(z-w)^n, n = 1, 2, ... \quad (2.2)$$

proof. Let $f(z) \in \Omega_p^w(A,B)$, then the inequality (1.2) or inequality

$$|\frac{(z-w)(D^k f(z))'' + (p+1)'(D^k f(z))'}{B(z-w)(D^k f(z))'' + (p+1)[B+(C-B)(1-\lambda)](D^k f(z))'}| < 1 \quad (2.3)$$

holds true, therefore by using (1.1) we have

$$|\frac{\sum_{n=1}^{+\infty}(n+2p)(n+p)^{k+1}a_{n+p}(z-w)^{n+p-1}}{-p(p+1)(C-B)(1-\lambda)+A}| < 1$$

where

$$A = \sum_{n=1}^{+\infty}[(p+1)(B+(C-B)(1-\lambda))+B(n+p+1)](n+p)^k a_{n+p}(z-w)^{n+p-1}.$$

Since $Re(z) \leq (z)$ for all z, therefore

$$Re\{\frac{\sum_{n=1}^{+\infty}(n+2p)(n+p)^{k+1}a_{n+p}(z-w)^{n+p-1}}{p(p+1)(C-B)(1-\lambda)-A}\} < 1.$$

where

$$A = \sum_{n=1}^{+\infty}[(p+1)(B+(C-B)(1-\lambda))+B(n+p+1)](n+p)^k a_{n+p}(z-w)^{n+p-1}$$

By letting $(z-w) \to \bar{1}$ through real valves, we have

$$\sum_{n=1}^{+\infty}[(n+2p)(B+1)+(p+1)(C-B)(1-\lambda)]a_{n+p} < p(p+1)(C-B)(1-\lambda).$$

Conversely, Let (2.1) holds true, if we Let $(z-w)\partial\partial\triangle\triangle^*$ where $\partial\triangle^*$ denotes the boundary of $\triangle^*$, then we have

$$|\frac{(z-w)(D^kf(z))''+(p+1)(D^kf(z))'}{B(z-w)(D^kf(z))''+(p+1)[B+(C-B)(1-\lambda)](D^kf(z))'}| \leq$$

$$\frac{\sum_{n=1}^{+\infty}(n+2p)(n+p)^{k+1}|a_{n+p}|}{P(p+1)(C-B)(1-\lambda)-\sum_{n=1}^{+\infty}[B(n+2p)+(p+1)(C+B)(1+\lambda)]|a_{n+p}|}$$

$(by(2.1))$

< 1

Thus by the Maximum modolus Theorem, we conclude $f(z) \in \Omega_p^w(A,B)$.

Theorem 2.2 *If $f(z) \in \Omega_p^w(A,B)$, then*

$$D^kf(z) = \int_0^z [exp\int_0^z \frac{(p+1)(\delta M(t)-1)}{(t-w)[1-BM(t)}dt]ds \quad (2.4)$$

where $\delta = B+(c-B)(1-\lambda)$ and $|M(z)| < 1$.

proof. Since $f(z) \in \Omega_p^w(A,B)$, so (1.2) or equivalently (2.3) holds true. Hence

$$\frac{(z-w)(D^kf(z))''+(p+1)(D^kf(z))'}{B(z-w)(D^kf(z))''+(p+1)\delta(D^kf(z))'} = M(z),$$

where $|M(z)| < 1, z\epsilon\triangle^*$ and $\delta = B+(C-B)(1-\lambda)$.

This yields

$$\frac{[D^kf(z)]''}{[D^kf(z)]'} = \frac{(p+1)[\delta M(z)-1]}{(z-w)(1-BM(z))}.$$

After integration we obtain the required result.

Remark. Theorem 2.1 shows that if $f(z)\epsilon\Omega_p^w(A,B)$, then

$$|a_{n+p}| \leq \frac{p(p+1)(C-B)(1-\lambda)}{(2p+1)(B+1)+(p+1)(C-B)(1-\lambda)}, n=1,2,3,... \quad (2.5)$$

3 Distortion Bounds and Extreme points

In this section we investigate about Distortion theorem and extreme points of the class $\Omega_p^w(A, B)$.

Theorem 3.1 *Let* $f(z)\epsilon\Omega_p^w(A, B)$*, then*

$$r^{-p} - \frac{p(p+1)(c-B)(1-\lambda)}{(2p+1)(B+1)+(p+1)(C-B)(1-\lambda)}r^{p+1} < |D^k f(z)|$$

$$< r^{-p} + \frac{p(p+1)(C-B)(1-\lambda)}{(2p+1)(B+1)+(p+1)(C-B)(1-\lambda)}r^{p+1} \quad (3.1)$$

where $0 < |z-w| = r < 1$.

proof. By theorem 2.1 and (2.5) we have

$$|D^k f(z)| = |\frac{1}{(z-w)^p} + \sum_{n=1}^{+\infty}(n+p)^k a_{n+p}(z-w)^{n+p}|$$

$$\leq r^{-p} + \sum_{n=1}^{+\infty}(n+p)^k |a_{n+p}| r^{n+p}$$

$$< r^{-p} + \frac{p(p+1)(C-B)(1-\lambda)}{(2p+1)(B+1)+(p+1)(C-B)(1-\lambda)}r^{p+1}.$$

similarly we obtain

$$|D^k f(z)| \geq r^{-p} - \frac{p(p+1)(C-B)(1-\lambda)}{(2p+1)(B+1)+(p+1)(C-B)(1-\lambda)}r^{p+1}.$$

So the proof is complete.

Theorem 3.2 *The function* $f(z)$ *of the from* (1.1) *belongs to* $\Omega_p^w(A, B)$ *if and only if it can be expressed by*

$$f(z) = \sum_{n=p}^{+\infty} \gamma_n f_n(z), \gamma_n \geq 0, \quad n = p, p+1, ... \quad (3.2)$$

where $f_p(z)=\frac{1}{(z-w)^p}$ (3.2),

$$f_n(z)=\frac{1}{(z-w)^p}+\frac{p(p+1)(C-B)(1-\lambda)}{(n+2p)+(B+1)+(p+1)(C-B)(1-\lambda)}(z-w)^n, n=p+1,p+2,...$$

$$\sum_{n=p}^{+\infty}\gamma_n=1.$$

proof. Let

$$f(z)=\sum_{n=p}^{+\infty}\gamma_n f_n(z)=\gamma_p f_p(z)+$$

$$\sum_{n=p+1}^{+\infty}\gamma_n[\frac{1}{(z-w)^p}+\frac{p(p+1)(C-B)(1-\lambda)}{(n+2p)(B+1)+(p+1)(C-B)(1-\lambda)}(z-w)^n]$$

$$=\frac{1}{(z-w)^p}+\sum_{n=p}^{+\infty}\frac{p(p+1)(C-B)(1-\lambda)}{(n+2p)(B+1)+(p+1)(C-B)(1-\lambda)}\gamma_n(z-w)^n.$$

Now by using Theorem 2.1 we conclude that $f(z)\in\Omega_p^w(A,B)$. Conversely, if $f(z)$ given by (1.1) belongs to $\Omega_p^w(A,B)$. by letting $\gamma_p=1-\sum_{n=p+1}^{+\infty}\gamma_n$ where

$$\gamma_n=\frac{(n+2p)(B+1)+(p+1)(C-B)(1-\lambda)}{p(p+1)(C-B)(1-\lambda)}a_{n+p}, n=1,2,...$$

we conclude the required result.

4 Radii of starlikeness and convexity

In the last section we introduce the Radii of Starlikeness and convexity for functions in the class $\Omega_p^w(A,B)$.

Theorem 4.1 *If $f(z) \in \Omega_p^w(A,B)$, then f is starlike of order $\xi(0 \leq \xi < p)$ in disk $|z-w| < R$, and it is convex of order ξ in disk $|zw| < R2$ where*

$$R_1 = inf_n \frac{[(2p+1)(B+1)+(p+1)(C-B)?(1-\lambda)](p-\xi)}{(n+3p+\xi)p(p+1)(C-B)(1-\lambda)} \frac{1}{n+p}$$

$$R_2 = inf_n \frac{(n+p)[(2p+1)(B+1)+(p+1)(C-B)(1-\lambda)]}{(n+3p+\xi)(p+1)(C-B)(1-\lambda)} \frac{1}{n+p}$$

proof. For Starlikeness it is enough to show that

$$|\frac{(z-w)f'(z)}{f(z)} + p| < p_\xi,$$

but

$$|\frac{(z-w)f'(z)}{f(z)} + p| = |\frac{\sum_{n=1}^{+\infty}(n+2p)a_{n+p}(z-w)^{n+p}{}^{n+p}}{\frac{1}{(z-w)^p} + \sum_{n=1}^{+\infty} a_{n+p}(z-w}| \leq$$

$$\frac{\sum_{n=1}^{+\infty}(n+2p)a_{n+p}|z-w|^{n+p}}{1-\sum_{n=1}^{+\infty} a_{n+p}|z-w|^{n+p}} \geq p-\xi$$

or

$$\sum_{n=1}^{+\infty}(n+2p)a_{n+p}|z-w|^{n+p} \leq p-\xi-(p-\xi)\sum_{n=1}^{+\infty} a_{n+p}|z-w|^{n+p}$$

$$\sum_{n=1}^{+\infty} \frac{(n+3p+\xi)}{p-\xi} a_{n+p}|z-w|^{n+p} \leq 1$$

by using (2.5) we obtain

$$\sum_{n=1}^{+\infty} \frac{(n+3p+\xi)}{p-\xi} a_{n+p}|z-w|^{n+p} \leq$$

$$\sum_{n=1}^{+\infty} \frac{(n+3p+\xi)P(P+1)(C-B)(1-\lambda)}{[(2P+1)(B+1)+(P+1)(C-B)(1-\lambda)](p-\xi)}|z-w|^{n+p} \leq 1$$

So it is enough to suppose

$$|z-w|^{n+p} \leq \frac{[(2P+1)(B+1)+(P+1)(C-B)(1-\lambda)](p-\xi)}{(n+3p+\xi)P(P+1)(C-B)(1-\lambda)}.$$

For convexity by using the fact that " $f(z)$ is convex if and only if $zf'(z)$ is starlike " and by an easy calculation we conclude the required result.

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