General Mathematics Vol. 17, No. 4 (2009), 185–193

Growth Theorems for Perturbated Starlike Log-Harmonic Mappings of Complex Order

Yaşar Polatoğlu, Emel Yavuz Duman and H. Esra Özkan

Abstract

Let $\mathcal{H}(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation $\overline{f_{\bar{z}}} = wf_z\left(\overline{f}/f\right)$, where $w(z) \in \mathcal{H}(\mathbb{D})$ is the second dilatation of f such that $|w(z)| < 1$ for every $z \in \mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as $f = h(z)\overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$. If f vanishes at $z = 0$ but it is not identically zero, then f admits the representation $f = z|z|^{2\beta}h(z)\overline{g(z)}$, where $\mathrm{Re}\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$, $g(0) = 1$, $h(0) \neq 0$ (see [1], [2], [3]). Let $f = zh(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping of complex order b ($b \neq 0$, complex) if

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f} - 1\right)\right\} > 0, \quad z \in \mathbb{D}.$$

The class of all starlike log-harmonic mappings of complex order b is denoted by $\mathcal{S}^*_{\mathcal{LH}}(1-b)$. We also note that if $zh(z)$ is a starlike function of complex order b, then the starlike log-harmonic mapping $f = zh(z)\overline{g(z)}$ will be called a perturbated starlike log-harmonic mapping of complex order b, and the family of such mappings will be denoted by $\mathcal{S}^*_{\mathcal{LH}}(p)(1-b)$.

The aim of this paper is to obtain the growth theorems for the perturbated starlike log-harmonic mappings of complex order.

2000 Mathematical Subject Classification: 30C45, 30C55.

1 Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$, such that $p(z)$ in $\mathcal{P}$ if and only if

$$p(z) = \frac{1+\phi(z)}{1-\phi(z)} \tag{1}$$

for some functions $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. Therefore we have $p(0) = 1$ and $\mathrm{Re}p(z) > 0$ whenever $p(z) \in \mathcal{P}$.

Moreover, let $\mathcal{S}^*(1-b)$ denote the family of functions $s(z) = z + a_2 z^2 + \cdots$ regular in $\mathbb{D}$, and such that $s(z)$ in $\mathcal{S}^*(1-b)$ if and only if

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(z\frac{s'(z)}{s(z)} - 1\right)\right\} > 0 \quad \left(\text{or } 1 + \frac{1}{b}\left(z\frac{s'(z)}{s(z)} - 1\right) = p(z),\ p(z) \in \mathcal{P}\right). \tag{2}$$

Let $s_1(z)$ and $s_2(z)$ be analytic functions in $\mathbb{D}$ with $s_1(0) = s_2(0)$. We say that $s_1(z)$ subordinate to $s_2(z)$ and denote by $s_1(z) \prec s_2(z)$ if $s_1(z) = s_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. If $s_1(z) \prec s_2(z)$ then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$ (see [5]).

Finally, let $\mathcal{H}(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}$. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w\frac{\overline{f}}{f}f_z,$$

where $w(z) \in \mathcal{H}(\mathbb{D})$ is the second dilatation of f such that $|w(z)| < 1$ for every $z \in \mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f = h(z)\overline{g(z)},$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$, with $h(0) \neq 0$, and $g(0) = 1$.

On the other hand, if f vanishes at $z = 0$ and at no other point, then f admits the representation

$$f = z|z|^{2\beta}h(z)\overline{g(z)},$$

where $\mathrm{Re}\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and $h(0) \neq 0$, $g(0) = 1$. We note that the class of log-harmonic function is denoted by $\mathcal{S}_{\mathcal{LH}}$.

Let $f = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{LH}}$. We say that f is a starlike log-harmonic mapping of complex order b, if

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f} - 1\right)\right\} > 0, \quad z \in \mathbb{D}.$$

and denote by $\mathcal{S}^*_{\mathcal{LH}}(1-b)$ the set of all starlike log-harmonic mappings of complex order b. Also we denote by $\mathcal{S}^*_{\mathcal{LH}}(p)(1-b)$ the class of all functions in $\mathcal{S}_{\mathcal{LH}}$ for which $zh(z)\in\mathcal{S}^*(1-b)$ and $f(z)\in\mathcal{S}^*_{\mathcal{LH}}(1-b)$ for all $z\in\mathbb{D}$.

2 Main Results

Lemma 1 *Let $f=zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{LH}}$. Then*

$$f\in\mathcal{S}^*_{\mathcal{LH}}\ iff\ \left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right)\prec\frac{2z}{1-z}.$$

Proof. If $f\in\mathcal{S}^*_{\mathcal{LH}}$, then we have

$$0<\mathrm{Re}\left(\frac{zf_z-\bar{z}f_{\bar{z}}}{f}\right)=\mathrm{Re}\left(1+z\frac{h'(z)}{h(z)}-\bar{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\right)=\mathrm{Re}\left(1+z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right)$$

iff

$$1+z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}=p(z)=\frac{1+\phi(z)}{1-\phi(z)}\ iff\ z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}=\frac{2\phi(z)}{1-\phi(z)}\ iff$$

$$z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\prec\frac{2z}{1-z}.$$

Theorem 1 *Let $f=zh(z)\overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{LH}}(1-b)$. Then*

$$\frac{(1-r)^{|b|-Reb}}{(1+r)^{|b|+Reb}}\le\left|\frac{h(z)}{g(z)}\right|\le\frac{(1+r)^{|b|-Reb}}{(1-r)^{|b|+Reb}}\quad(|z|=r<1).\tag{6}$$

Proof. The function $\frac{2z}{1-z}$ maps $|z|=r$ onto the circle with center $C(r)=\left(\frac{2r^2}{1-r^2},0\right)$ and radius $\rho(r)=\frac{2r}{1-r^2}$. Therefore using the definition of the subordination and Lemma 1, we get

$$\left|\left(z\frac{h'(z)}{h(z)}-\bar{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\right)-\frac{2\{(\mathrm{Re}b)+(\mathrm{Im}b)\}r^2}{1-r^2}\right|\le\frac{2|b|r}{1-r^2}.\tag{7}$$

The inequality (7) can be written in the form

$$\frac{2\{(\mathrm{Re}b)r-|b|\}r}{1-r^2} \leq \mathrm{Re}\left\{z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right\} \leq \frac{2\{(\mathrm{Re}b)r+|b|\}r}{1-r^2}. \tag{8}$$

On the other hand, we have

$$\mathrm{Re}\left\{z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right\} = r\frac{\partial}{\partial r}\left(\log|h(z)| - \log|g(z)|\right).$$

Thus the inequality (8) can be written in the form

$$\frac{2\{(\mathrm{Re}b)r-|b|\}}{(1-r)(1+r)} \leq \frac{\partial}{\partial r}\left(\log|h(z)| - \log|g(z)|\right) \leq \frac{2\{(\mathrm{Re}b)r+|b|\}}{(1-r)(1+r)}. \tag{9}$$

Integrating both sides of (9) from 0 to r we get (6).

Theorem 2 *Let $h(z) = 1 + a_1 z + a_2 z^2 + \cdots$ be an analytic function in the open unit disc $\mathbb{D}$. If $zh(z)$ is starlike of complex order b, then*

$$\frac{(1-r)^{|b|-Reb}}{(1+r)^{|b|+Reb}} \leq |h(z)| \leq \frac{(1+r)^{|b|-Reb}}{(1-r)^{|b|+Reb}} \quad (|z| = r < 1). \tag{10}$$

Proof. If $zh(z)$ is a starlike function of complex order b, then we have

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(z\frac{(zh(z))'}{zh(z)} - 1\right)\right\} > 0 \text{ implies } 1 + \frac{1}{b}\left(z\frac{(zh(z))'}{zh(z)} - 1\right) = p(z),$$

where $p(z) \in \mathcal{P}$. Then we have

$$\left|\left\{1 + \frac{1}{b}\left(z\frac{(zh(z))'}{zh(z)} - 1\right)\right\} - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}$$

implies

$$\frac{1-2|b|r+(2(\mathrm{Re}b)-1)r^2}{1-r^2} \leq \mathrm{Re}\left(z\frac{(zh(z))'}{zh(z)}\right) \leq \frac{1+2|b|r+(2(\mathrm{Re}b)-1)r^2}{1-r^2}. \tag{11}$$

On the other hand we have

$$\mathrm{Re}\left(z\frac{(zh(z))'}{zh(z)}\right) = r\frac{\partial}{\partial r}\log|zh(z)|.$$

Therefore the inequality (11) can be written in the form

(12)

$$\frac{1-2|b|r+(2(\mathrm{Re}b)-1)r^2}{r(1-r)(1+r)} \leq \frac{\partial}{\partial r}\log|zh(z)| \leq \frac{1+2|b|r+(2(\mathrm{Re}b)-1)r^2}{r(1-r)(1+r)},$$

and upon integration of both sides of (14) from 0 to r, we get (10) (see [4]).

Corollary 1 *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{\mathcal{LH}}(p)(1-b)$. *Then*

$$F(|b|, Reb, -r) \leq |h(z)+zh'(z)| \leq F(|b|, Reb, r) \quad (|z| = r < 1),$$

where

$$F(|b|, Reb, r) = \frac{(1+r)^{|b|-Reb}}{(1-r)^{|b|+Reb}}\frac{|1+(2b-1)r^2|+2|b|r}{1-r^2}.$$

Proof. Since $\varphi(z) \in \mathcal{S}^*(1-b)$, then using the definition of the subordination we can write

$$\left|\left\{1+\frac{1}{b}\left(z\frac{\varphi'(z)}{\varphi(z)}-1\right)\right\} - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}$$

implies

$$\left|z\frac{\varphi'(z)}{\varphi(z)} - \frac{1+(2b-1)r^2}{1-r^2}\right| \leq \frac{2|b|r}{1-r^2}.$$

After simple calculations from above inequality, then we have

$$\frac{|1+(2b-1)r^2|-2|b|r}{1-r^2} \leq \left|z\frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{|1+(2b-1)r^2|+2|b|r}{1-r^2}.$$

The last inequality can be written in the form

$$|h(z)|\frac{|1+(2b-1)r^2|-2|b|r}{1-r^2} \leq |(zh(z))'| \leq |h(z)|\frac{|1+(2b-1)r^2|+2|b|r}{1-r^2}.$$

Using the Theorem 2 we obtain the result.

Corollary 2 *Let $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{\mathcal{LH}}(p)(1-b)$. Then*

$$\left(\frac{1-r}{1+r}\right)^{2|b|} \leq |g(z)| \leq \left(\frac{1+r}{1-r}\right)^{2|b|}. \tag{13}$$

Proof. The result is a consequence of Theorem 1 and Theorem 2.

Theorem 3 *Let $f = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{LH}}(p)(1-b)$. Then*

$$F(|b|, -r) \leq |g'(z)| \leq F(|b|, r), \tag{14}$$

where

$$F(|b|, r) = \left(\frac{1+r}{1-r}\right)^{2|b|} \frac{|1 + (2b-1)r^2| + 2|b|r}{(1-r)^2}$$

for all $|z| = r < 1$.

Proof. Let $f = z|z|^{2\beta}h(z)\overline{g(z)}$ be an element of $\mathcal{S}_{\mathcal{LH}}$. Then f is the solution of the non-linear elliptic partial differential equation

$$w(z) = \frac{\overline{f_{\bar{z}}}}{\overline{f}} \frac{f}{f_z},$$

from which it follows that

$$w(z) = \frac{\overline{f_{\bar{z}}}}{\overline{f}} \frac{f}{f_z} = \frac{\overline{\beta} + z\frac{g'(z)}{g(z)}}{\beta + z\frac{\varphi'(z)}{\varphi(z)}}, \; w(0) = \frac{\overline{\beta}}{\beta + 1},$$

where $w(z)$ is the second dilatation of f and $\varphi(z) \in \mathcal{S}^*(1-b)$, and we are studying on the Riemann branch which is $1^{2\beta} = 1$, $\text{Re}\beta > -1/2$. If we take $\beta = 0$, then $w(z)$ satisfy the conditions of Schwarz lemma. Therefore we have

$$-r \leq |w(z)| = \left|\frac{g'(z)/g(z)}{\varphi'(z)/\varphi(z)}\right| \leq r.$$

Using Theorem 2, Corollary 1 and Corollary 2 we obtain (14).

Corollary 3 *If* $f = zh(z)\overline{g(z)}$ *is an element of* $\mathcal{S}^*_{\mathcal{LH}}(p)(1-b)$*, then*

$$r\frac{(1-r)^{3|b|-Reb}}{(1+r)^{3|b|+Reb}} \leq |f| \leq r\frac{(1+r)^{3|b|-Reb}}{(1-r)^{3|b|+Reb}}.$$

Proof. Follows immediately from Theorem 2 and Corollary 2.

Corollary 4 *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{\mathcal{LH}}(p)(1-b)$*. Then*

$$G(|b|,-r) - r^2G(|b|,r) \leq J_f \leq G(|b|,-r) \quad (|z| = r < 1), \tag{16}$$

where

$$G(|b|,-r) = \frac{(1+r)^{6|b|-2Reb}}{(1-r)^{6|b|+2Reb}}\frac{|1+(2b-1)r^2|+2|b|r}{|1+(2b-1)r^2|-2|b|r}.$$

Proof. Using Theorem 3 and

$$J_f = |f|^2\left(\left|\frac{\varphi'(z)}{\varphi(z)}\right|^2 - \left|\frac{g'(z)}{g(z)}\right|^2\right)$$

we obtain (16) after simple calculations.

References

[1] Z. Abdulhadi and D. Bshouty, *Univalent functions in* $H\overline{H}(\mathbb{D})$, Trans. Amer. Math. Soc., 305 (1998), 841–849.

[2] Z. Abdulhadi, W. Hengartner, *On pointed univalent log-harmonic mappings*, J. Math. Anal. Appl., 203 (2) (1996), 333–351.

[3] Z. Abdulhadi, Y. Abu Muhanna, *Starlike log-harmonic mappings of order* α, JIPAM, Vol. 7, Issue 4, Article 123, (2006).

[4] I.I. Bavrin, *Functions star- and convex-univalent of order α with weight*, Doklady Math., Vol. 76, Issue 3 (2007), 848–850.

[5] A.W. Goodman, *Univalent Functions*, Vol. I., Mariner Pub. Comp. Inc., New Jersey, 1983.

Department of Mathematics and Computer Science,
Faculty of Science and Letters,
İstanbul Kültür University,
34156 İstanbul, Turkey
E-mail: y.polatoglu@iku.edu.tr, e.yavuz@iku.edu.tr, e.ozkan@iku.edu.tr