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On a subclass of certain p-valent starlike functions with negative coefficients [1]

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Abstract

We introduce the subclass $T_\Omega(n, p, \lambda, \alpha, \beta)$ of analytic functions with negative coefficients defined by salagean operators D^n. In this paper we give some properties of functions in the class $T_\Omega(n, p, \lambda, \alpha, \beta)$ and obtain numerous sharp results including coefficient estimates, distortion theorems, closure theorems and modified Hadamard products of several functions belonging to the class $T_\Omega(n, p, \lambda, \alpha, \beta)$. We also obtain radii of close to convexity, starlikeness and convexity for the functions belonging to the class $T_\Omega(n, p, \lambda, \alpha, \beta)$ and consider integral operators associated with functions belonging to the class $T_\Omega(n, p, \lambda, \alpha, \beta)$.

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1 Introduction

Let A denote the class functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

For a function $f(z)$ in A, we define

$$D^0(z) = f(z),$$

$$D^1(z) = Df(z) = zf'(z)$$

and

$$D^n(z) = D(D^{n-1}f(z)) \quad n \in N = \{1, 2, 3, \cdots\}.$$

Note that

$$D^n(z) = D(D^{n-1}f(z)) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad n \in N_0 = \{0\} \smile N.$$

the differential operator D^n was introduced by Salagean [5].

Let $T(n, p)$ denote the class of functions $f(z)$ of the form:

$$(1) \qquad f(z) = z^p - \sum_{k=n}^{\infty} a_k z^{k+p}$$

$(a_{k+p} \geq 0; p \in N = \{1, 2, 3, \cdots\}; n \in N)$, which are analytic in the unit disc $U = \{z : |z| < 1\}$.

A function $f(z)$ belonging to $T(n, p)$ is in the class $T(n, p, \lambda, \alpha)$ if it satisfies

$$(2) \qquad Re\left\{\frac{zf'(z) + \lambda z^2 f(z)}{(1-\lambda)f(z) + \lambda zf(z)}\right\} > \alpha$$

for some α $(0 \leq \alpha < 1)$ and λ $(0 \leq \alpha \leq 1)$, and for all $z \in U$ [2].

We can write the following equalities for the functions $f(z)$ belong to the class $T(n,p)$

$$D^0(z) = f(z),$$

$$\begin{aligned} D^1(z) &= Df(z) = zf'(z) = z\left[pz^{p-1} - \sum_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}\right] \\ &= pz^p - \sum_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}, \end{aligned}$$

and

$$D^{\Omega}(z) = D(D^{\Omega-1}f(z)) \quad n \in N = \{1,2,3,\cdots\}.$$

Note that

$$D^{\Omega}(z) = D(D^{\Omega-1}f(z)) = p^{\Omega}z^p - \sum_{k=2}^{\infty}(k+p)^{\Omega}a_{k+p}z^{k+p}.$$

We define a new subclass as follows:

Definition 1 *A function $f(z)$ belonging to $T(n,p)$ is in the class $T_{\Omega}(n,p,\lambda,\alpha,\beta)$ if and only if*

$$\begin{aligned} &Re\left\{\frac{(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)}\right\} \\ &> \beta\left|\frac{(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)} - 1\right| + \alpha \end{aligned}$$

for some $\beta \geq 0, \alpha \quad (0 \leq \alpha < 1)$ and $\lambda \quad (0 \leq \lambda < 1)$ and for all $z \in U$.

We note that by specializing the parameters n, p, λ, α and β we obtain the following subclasses studied by various authors:

(i) $T_{\Omega}(n,p,\lambda,\alpha,0) = T_{\Omega}(n,p,\lambda,\alpha)$, (Kamali and Orhan [11])

(ii) $T_0(n,p,\lambda,\alpha) = T(n,p,\lambda,\alpha)$, (Altintas et. al. [2])

(iii) $T_0(n,1,0,\alpha) = T_0(n)$, (Srivastava et. al. [8])

(iv) $T_0(1, 1, 0, \alpha) = T^*(\alpha)$ (Silverman [7])

(v) $T_0(1, 1, 1, \alpha) = C(\alpha)$, (Silverman [7])

(vi) $T_0(n, 1, \lambda, \alpha) = P(n, \lambda, al)$ (Altintas [1])

(vii) $T_0(n, 1, \lambda, \alpha) = C(n, \lambda, \alpha)$. (Kamali and Akbulut [4]).

2 Coefficient Inequalities

Theorem 1 *A function $f(z) \in T(n, p)$ is in the class $T_\Omega(n, p, \lambda, \alpha, \beta)$ if and only if*

$$(3)\quad \sum_{k=n}^{\infty}(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)a_{k+p} \leq p^{n}(p-\alpha)(1+\lambda p-\lambda)$$

$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p \neq 1); p \in N; n \in N; \Omega \in N_0)$.

The result is sharp.

Proof. Assume that the inequality (3) holds true. Note the fact that

$$Re(w) > \beta|w-1| + \alpha \Leftrightarrow Re\{w(1+\beta e^{i\theta}) - \beta e^{i\theta}\} > \alpha, \quad -\overline{\Lambda} \leq \theta \leq \overline{\Lambda}.$$

or equivalently

$$Re\left\{\frac{(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)}(1+\beta e^{i\theta}) - \beta e^{i\theta}\right\} > \alpha.$$

Then for $0 < |z| = r < 1$,

$$(4)\quad Re\left\{\frac{[(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'](1-\beta e^{i\theta}) - \beta e^{i\theta}(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)}{(1-\lambda)^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)}\right\} > \alpha$$

Let

$$\begin{aligned} A(z) &= [(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'](1+\beta e^{i\theta}) \\ &- \beta e^{i\theta}[(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)] \end{aligned}$$

and

$$B(z) = (1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z).$$

Then (4) is equivalent to

$$|A + (1-\alpha)B| > |A - (1+\alpha)|, \quad 0 \le \alpha < 1.$$

For $A(z)$ and $B(z)$ as above, we have

$$|A + (1-\alpha)B| = |[p(1+\beta) - (\alpha+\beta-1)](1+\lambda p-\lambda)p^n z^p$$
$$-\sum_{k=n}^{\infty}(\lambda k + \lambda p + 1 - \lambda)(k+p)^n[(k+p)(1+\beta) - \alpha\beta]a_{k+p}z^{k+p}|$$
$$\ge (p-\alpha)(1+\lambda p-\lambda)p^n|z|^p - \sum_{k=n}^{\infty}(\lambda k + \lambda p + 1 - \lambda)(k+p)^n \times$$
$$[(k+p)(1+\beta) - \alpha\beta]a_{k+p}|z|^{k+p}$$
$$\ge (p-\alpha)(1+\lambda p-\lambda)p^n|z|^p - \sum_{k=n}^{\infty}(\lambda k + \lambda p + 1 - \lambda)(k+p)^n \times$$
$$[k+p-\alpha+\beta(k+p-1)|a_{k+p}|z|^{k+p}$$

and similarly

$$|A - (1-\alpha)B| = |(p+\beta p - (\beta-1-\alpha)(1+\lambda p-\lambda)p^n z^p$$
$$-\sum_{k=n}^{\infty}(\lambda k + \lambda p + 1 - \lambda)(k+p)^n[(k+p)(1+\beta) - \alpha\beta]a_{k+p}z^{k+p}|$$
$$< (\alpha-p)(1+\lambda p-\lambda)p^n|z|^p + \sum_{k=n}^{\infty}(\lambda k + \lambda p + 1 - \lambda)(k+p)^n \times$$
$$[(k+p-\alpha+\beta(k+p-1)a_{k+p}|z|^{k+p}.$$

Therefore,

$$|A + (1-\alpha)B| - |A - (1+\alpha)B|$$
$$\geq 2(p-\alpha)(1+\lambda p-\lambda)p^n|z|^p - 2\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^n \times$$
$$[k+p-\alpha+\beta(k+p-1)]a_{k+p}|z|^{k+p} \geq 0.$$

Letting $r \to 1$, we obtain

$$\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^n[k+p-\alpha+\beta(k+p-1)]a_{k+p} \leq (p-\alpha)(1+\lambda p-\lambda)p^n$$

which yields (3).

On the other hand we must have

$$Re\left\{\frac{(1-\lambda)z(D^{\Omega}f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z) + \lambda D^{\Omega+1}f(z)}(1+\beta e^{i\theta}) - \beta e^{i\theta}\right\} > \alpha.$$

Upon choosing the values of z on the positive real axis where $0 < |z| = r < 1$, the above inequality reduces to

$$Re\left\{\frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega+1} - \sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega+1}[k+p-\alpha+\beta(k+p-1)]a_{k+p}r^{k+p}}{(1+\lambda p-\lambda)p^{\Omega} - \sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}[(k+p-\alpha+\beta(k+p-1)]a_{k+p}r^{k+p}}\right\} > 0.$$

Letting $r \to 1$, we get the desired result.

Finally, we note that the assertion (3) is sharp, the extremal function being

$$f(z) = z^p - \frac{p^n(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}. \tag{5}$$

Corollary 1 *Let the function $f(z)$ defined by (1) be in the class $T_{\Omega}(n,p,\lambda,\alpha,\beta)$. Then*

$$a_{k+p} \leq \frac{(p-\alpha)(1+\lambda p-\lambda)p^n}{(\lambda k+\lambda p+1-\lambda)(k+p)^n[k+p-\alpha+\beta(k+p-1)]},\ (k \geq j+1). \tag{6}$$

The equality in (6) is attained from the function $f(z)$ given by (5).

By taking $\beta = 0$ in above Theorem 1, we get

Corollary 2 *Let the function $f(z)$ be defined by (1) then $f(z) \in T_\Omega(n,p,\lambda,\alpha,0)$ if and only if*

$$(7)\quad \sum_{k=n}^{\infty}(\lambda k + \lambda p + 1 - \lambda)(k+p)^n(k+p-\alpha)a_{k+p} \le (p-\alpha)(1+\lambda p-\lambda)p^n.$$

3 Distortion Theorems

Theorem 2 *Let the function $f(z)$ be defined by (1) be in the class $T_\Omega(n,p,\lambda,\alpha,\beta)$. Then for $|z| = r < 1$,*

$$(8)\qquad |f(z)| \ge |z|^p - \frac{(p-\alpha)(1+\lambda p-\lambda)p^\Omega}{(n+p)^\Omega(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}|z|^{p+n}$$

and

$$(9)\qquad |f(z)| \le |z|^p + \frac{(p-\alpha)(1+\lambda p-\lambda)p^\Omega}{(n+p)^\Omega(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}|z|^{p+n}$$

for $z \in U$. The inequalities in (8) and (9) are attained for the function $f(z)$ given by

$$(10)\quad f(z) = z^p - \frac{(p-\alpha)(1+\lambda p-\lambda)p^\Omega}{(n+p)^\Omega(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}z^{p+n}.$$

Proof. Note that

$$\begin{aligned}(11)\qquad & (n+p)^\Omega(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]\sum_{k=n}^{\infty}a_{k+p} \\ & \le \sum_{k=n}^{\infty}(k+p)^\Omega(\lambda k+\lambda p+1-\lambda)[k+p-\alpha+\beta(k+p-1)]a_{k+p} \\ & \le (p-\alpha)(1+\lambda p-\lambda)p^\Omega,\end{aligned}$$

this last inequality following from Theorem 1. Thus

$$\begin{aligned}|f(z)| &\le |z|^p + \sum_{k=n}^{\infty}|a_{k+p}||z|^{k+p} \le |z|^p + |z|^{n+p}\sum_{k=n}^{\infty}|a_{k+p}| \\ &\le |z|^p + \frac{(p-\alpha)(1+\lambda p-\lambda)p^\Omega}{(n+p)^\Omega(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}|z|^{n+p}.\end{aligned}$$

Similarly,

$$|f(z)| \geq |z|^p - \sum_{k=n}^{\infty} |a_{k+p}||z|^{k+p} \geq |z|^p - |z|^{n+p} \sum_{k=n}^{\infty} |a_{k+p}|$$
$$\geq |z|^p - \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega}(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}|z|^{n+p}.$$

This completes the proof of Theorem 2.

Corollary 3 *Let the function $f(z)$ be defined by (1) then $f(z) \in T_{\Omega}(n,p,\lambda,\alpha)$. Then for $|z| = r_1$,*

$$|f(z)| \geq |z|^p - \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega}(\lambda n+\lambda p+1-\lambda)(n+p-\alpha)}|z|^{p+n} \tag{12}$$

and

$$|f(z)| \leq |z|^p + \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega}(\lambda n+\lambda p+1-\lambda)(n+p-\alpha)}|z|^{p+n} \tag{13}$$

for $z \in U$. The inequalities in (8) and (9) are attained for the function $f(z)$ given by

$$|f(z)| = |z|^p - \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega-1}(\lambda n+\lambda p+1-\lambda)(n+p-\alpha)}|z|^{p+n} \tag{14}$$

Proof. Taking $\beta = 0$ in Theorem 2, we immediately obtain (12) and (13).

Theorem 3 *Let the function $f(z) \in T_{\Omega}(n,p,\lambda,\alpha,\beta)$ be defined by (1). Then for $|z| = r < 1$,*

$$|f'(z)| \geq p|z|^{p-1} - \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega-1}(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}|z|^{p+n-1} \tag{15}$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega-1}(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}|z|^{p+n-1}. \tag{16}$$

Proof. We have

$$|f'(z)| \leq p|z|^{p-1} + \sum_{k=n}^{\infty}(k+p)a_{k+p}|z|^{k+p-1} \tag{17}$$
$$\leq p|z|^{p-1} + |z|^{n+p-1}\sum_{k=n}^{\infty}(k+p)a_{k+p}.$$

In view of Theorem 1, we have

$$\sum_{k=n}^{\infty}(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)a_{k+p} \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)$$

and then

$$(n+p)^{\Omega-1}[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)\sum_{k=n}^{\infty}(k+p)a_{k+p}$$
$$\leq \sum_{k=n}^{\infty}(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)a_{k+p}$$
$$\leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)$$

or

$$\sum_{k=n}^{\infty}(k+p)a_{k+p} \leq \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)}. \tag{18}$$

A substitution of (18) into (17) yields

$$|f'(z)| \leq p|z|^{p-1} + \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega-1}(\lambda n+\lambda p+1-\lambda)[n+p-\alpha+\beta(n+p-1)]}|z|^{p+n-1}.$$

On the other hand,

$$|f'(z)| \geq p|z|^{p-1} - \sum_{k=n}^{\infty}(k+p)a_{k+p}|z|^{k+p-1}$$
$$\geq p|z|^{p-1} - |z|^{n+p-1}\sum_{k=n}^{\infty}(k+p)a_{k+p}$$
$$\geq p|z|^{p-1} - \frac{(p-\alpha)(1+\lambda p-\lambda)p^{\Omega}}{(n+p)^{\Omega-1}(\lambda n+\lambda p+1-\lambda[(n+p-\alpha+\beta(n+p-1)]}|z|^{p+n-1}.$$

Corollary 4 *Let the function $f(z) \in T_\Omega(n,p,\lambda,\alpha)$ be defined by (1). Then for $|z| = r < 1$,*

$$|f'(z)| \geq p|z|^{p-1} - \frac{(p-\alpha)(1+\lambda p-\lambda)p^\Omega}{(n+p)^{\Omega-1}(\lambda n+\lambda p+1-\lambda)(n+p-\alpha)}|z|^{p+n-1} \tag{19}$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{(p-\alpha)(1+\lambda p-\lambda)p^\Omega}{(n+p)^{\Omega-1}(\lambda n+\lambda p+1-\lambda)(n+p-\alpha)}|z|^{p+n-1}. \tag{20}$$

Proof. Taking $\beta = 0$ in Theorem 3, we obtain (19) and (20).

4 Closure Theorems and Extreme Points

Let the function $f_j(z)$ be defined, for $j = 1, 2, \cdots, m$ by

$$f_j(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,j} z^{k+p} \quad (a_{k+p,j} \geq 0) \tag{21}$$

for $z \in U$.

We shall prove that the following results for the closure of functions in the class $T_\Omega(n,p,\lambda,\alpha,\beta)$.

Theorem 4 *Let the functions $f_j(z)$ defined by (21) be in the class $T_\Omega(n,p,\lambda,\alpha,\beta)$ for every $j = 1, 2, \cdots, m$. Then the functions $h(z)$ defined by*

$$h(z) = \sum_{j=1}^{m} c_j f_j(z),\ (c_j \geq 0)$$

is also in the class $T_\Omega(n,p,\lambda,\alpha,\beta)$, where

$$\sum_{j=1}^{m} c_j = 1.$$

Proof. According to the definition $h(z)$, we can write

$$\begin{aligned} h(z) &= \sum_{j=1}^{m} c_j \left[z^p - \sum_{k=n}^{\infty} a_{k+p,j} z^{k+p} \right] \\ &= \left(\sum_{j=1}^{m} c_j \right) z^p - \sum_{k=n}^{\infty} \left(\sum_{j=1}^{m} c_j a_{k+p,j} \right) z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \left(\sum_{j=1}^{m} c_j a_{k+p,j} \right) z^{k+p}. \end{aligned}$$

Further, since $f_j(z)$ are in $T_\Omega(n,p,\lambda,\alpha,\beta)$ for every $j = 1, 2, \cdots, m$ we get

$$\sum_{k=n}^{\infty} (k+p)^\Omega (\lambda k+\lambda p+1-\lambda)[k+p-\alpha+\beta(k+p-1)] a_{k+p,j} \leq (p-\alpha)(1+\lambda p-\lambda)p^\Omega$$

for every $j = 1, 2, \cdots, m$. Hence we can see that

$$\begin{aligned} &\sum_{k=n}^{\infty} (k+p)^\Omega (\lambda k + \lambda p + 1 - \lambda)[k + p - \alpha + \beta(k + p - 1)] \left(\sum_{j=1}^{m} c_j a_{k+p,j} \right) \\ &= \sum_{k=n}^{\infty} (k+p)^\Omega (\lambda k + \lambda p + 1 - \lambda)[k + p - \alpha + \beta(k + p - 1)] \\ &\times (c_1 a_{k+p,1} + c_2 a_{k+p,2} + \cdots + c_m a_{k+p,m}) \\ &= c_1 \sum_{k=n}^{\infty} (k+p)^\Omega (\lambda k + \lambda p + 1 - \lambda)[k + p - \alpha + \beta(k + p - 1)] a_{k+p,1} \\ &+ c_2 \sum_{k=n}^{\infty} (k+p)^\Omega (\lambda k + \lambda p + 1 - \lambda)[k + p - \alpha + \beta(k + p - 1)] a_{k+p,2} \\ &+ \cdots c_m \sum_{k=n}^{\infty} (k+p)^\Omega (\lambda k + \lambda p + 1 - \lambda)[k + p - \alpha + \beta(k + p - 1)] a_{k+p,m} \\ &\leq c_1 (p - \alpha)(1 + \lambda p - \lambda) p^\Omega + c_2 (p - \alpha)(1 + \lambda p - \lambda) p^\Omega + \cdots \\ &+ c_m (p - \alpha)(1 + \lambda p - \lambda) p^\Omega = \left(\sum_{j=1}^{m} c_j \right) (p - \alpha)(1 + \lambda p - \lambda) p^\Omega \\ &= (p - \alpha)(1 + \lambda p - \lambda) p^\Omega \end{aligned}$$

which implies that $h(z) \in T_\Omega(n,p,\lambda,\alpha,\beta)$. Thus we have the Theorem.

Theorem 5 *$T_\Omega(n,p,\lambda,\alpha,\beta)$ is a convex set.*

Proof. Let the function

$$f_v(z) = z - \sum_{k=n}^{\infty} a_{k+p,v} z^k \quad (a_{k+p,v} \geq 0; v = 1,2) \tag{22}$$

be in the class $T_\Omega(n,p,\lambda,\alpha,\beta)$. It is sufficient to show that the function $h(z)$ defined by

$$H(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \leq \mu \leq 1) \tag{23}$$

is also in the class $T_\Omega(n,p,\lambda,\alpha,\beta)$. Since, for $0 \leq \mu \leq 1$,

$$h(z) = z - \sum_{k=n}^{\infty} [\mu a_{k+p,1} + (1-\mu) a_{k+p,2}] z^k \tag{24}$$

with the aid of Theorem 1, we have

$$\begin{aligned}(25)\ &\sum_{k=n}^{\infty}(k+p)^\Omega(\lambda k+\lambda p+1-\lambda)[k+p-\alpha+\beta(k+p-1)][\mu a_{k+p,1}+(1-\mu)a_{k+p,2}]\\ &= \mu \sum_{k=n}^{\infty}(k+p)^\Omega(\lambda k+\lambda p+1-\lambda)[k+p-\alpha+\beta(k+p-1)]a_{k+p,1}\\ &+(1-\mu)\sum_{k=n}^{\infty}(k+p)^\Omega(\lambda k+\lambda p+1-\lambda)[k+p-\alpha+\beta(k+p-1)]a_{k+p,2}\\ &\leq \mu(p-\alpha)(1+\lambda p-\lambda)p^\Omega + (1-\mu)(p-\alpha)(1+\lambda p-\lambda)p^\Omega\\ &= (p-\alpha)(1+\lambda p-\lambda)p^\Omega\end{aligned}$$

which implies that $h(z) \in T_\Omega(n,p,\alpha,\beta)$. Hence, $T_\Omega(n,p,\lambda,\alpha,\beta)$ is a convex set.

As a consequence of Theorem 6 there exists the extreme points of the $T_\Omega(n,p,\lambda,\alpha,\beta)$.

Theorem 6 *Let*

$$f_{n-1}(z) = z^p \tag{26}$$

and

$$f_k(z) = z^p - \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)} z^k, \tag{27}$$

$(k \geq n)$, for $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$ and $n \in N$. Then $f(z)$ is in the class $T_{\Omega}(n,p,\lambda,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=n-1}^{\infty} \eta_k f_k(z), \tag{28}$$

where

$$\eta_k \geq 0 \quad (k \geq n-1) \quad \text{and} \quad \sum_{k=n-1}^{\infty} \eta_k = 1. \tag{29}$$

Proof. Assume that

$$f(z) = \sum_{k=n-1}^{\infty} \eta_k f_k(z). \tag{30}$$

Then

$$\begin{aligned} f(z) &= \sum_{k=n-1}^{\infty} \eta_k f_k(z) = \eta_{n-1} f_{n-1}(z) + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \eta_{n-1} z^p + \sum_{k=n}^{\infty} \eta_k \left[z^p - \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)} \right] z^{k+p} \\ &= \left(\sum_{k=n-1}^{\infty} \eta_k \right) z^p - \sum_{k=n}^{\infty} \eta_k \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)} z^{k+p}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=n}^{\infty} \eta_k \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega}[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)} \\ &\times \frac{(n+p)^{\Omega}[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)} \\ &= \sum_{k=n}^{\infty} \eta_k = \sum_{k=n-1}^{\infty} \eta_k - \eta_{k-1} = 1 - \eta_{n-1} \leq 1, \end{aligned}$$

so, by Theorem 1, $f(z) \in T_\Omega(n,p,\lambda,\alpha,\beta)$.

Conversely, assume that the function $f(z)$ defined by (1) belongs to the class $T_\Omega(n,p,\lambda,\alpha,\beta)$. Then

$$a_{k+p} \leq \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^\Omega[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)}, \quad (k \geq n). \tag{31}$$

Setting

$$\eta_k = \frac{(n+p)^\Omega[(n+p-\alpha)+\beta(n+p-1)](\lambda n+\lambda p-\lambda+1)}{p^\Omega(p-\alpha)(1+\lambda p-\lambda)} \tag{32}$$

and

$$\eta_{n-1} = 1 - \sum_{k=n}^{\infty} \eta_k. \tag{33}$$

Then

$$\begin{aligned} f(z) &= z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)} \quad \eta_k z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \eta_k[z^p - f_k(z)] = z^p - \sum_{k=n}^{\infty} \eta_k z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \eta_{n-1} z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) = \eta_{n-1} f_{n-1}(z) + \sum_{k=n}^{\infty} \eta_k f_k(z) = \sum_{k=n-1}^{\infty} \eta_k f_k(z). \end{aligned}$$

This completes the proof.

Theorem 7 *Let the function $f(z)$ defined by (1) be in the class $T_\Omega(n,p,\lambda,\alpha,\beta)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r$, where*

$$\begin{aligned} r &= r(n,p,\lambda,\alpha,\beta,\rho) \\ &= \inf_k \left[\left(\frac{k+p}{p}\right)^{\Omega-1} \left(\frac{p-\rho}{p-\alpha}\right) \left[\frac{(k+p-\alpha)+\beta(k+p-1)}{(k+p-\rho)+\beta(k+p-1)}\right] \right. \\ &\left. \left(\frac{\lambda k+\lambda p-\lambda+1}{\lambda p-\lambda+1}\right)\right]^{\frac{1}{k}}, \ (k \geq j+1) \end{aligned} \tag{34}$$

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}+1-p\right|\leq p-\rho,$$

$(0\leq\rho<1)$ for $|z|<r$. We have

$$\left|\frac{zf'(z)}{f(z)}+1-p\right|=\left|\frac{zf'(z)+(1-p)f(z)}{f(z)}\right|$$

$$=\left|\frac{p(p-1)z^{p-1}-\sum\limits_{k=n}^{\infty}(k+p)(k+p-1)a_{k+p}z^{k+p-1}}{pz^{p-1}-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}}\right.$$

$$\left.-\frac{p(p-1)z^{p-1}+\sum\limits_{k=n}^{\infty}(p-1)(k+p)a_{k+p}z^{k+p-1}}{pz^{p-1}-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}}\right|$$

$$=\left|\frac{-\sum\limits_{k=n}^{\infty}(k+p)ka_{k+p}z^{k+p-1}}{pz^{p-1}-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}}\right|\leq\frac{\sum\limits_{k=n}^{\infty}k(k+p)a_{k+p}|z|^{k}}{p-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^{k}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}+1-p\right|\leq p-\rho,$$

if

$$\sum_{k=n}^{\infty}\frac{(k+p-\rho)(k+p)}{p(p-\rho)}\,a_{k+p}|z|^{k}\leq 1. \tag{35}$$

But, by Theorem 1 confirms that

$$\sum_{k=n}^{\infty}\frac{(k+p^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{n}(p-\alpha)(1+\lambda p-\lambda)}\,a_{k+p}\leq 1.$$

Hence (34) will be true if

$$\frac{(k+p-\rho)(k+p)}{p(p-\rho)}|z|^{k}\leq\frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{n}(p-\alpha)(1+\lambda p-\lambda)}$$

that is, if

(36)
$$|z| \leq \left[\frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)p(p-\rho)}{p^n(p-\alpha)(1+\lambda p-\lambda)(k+p-\rho)(k+p)}\right]^{\frac{1}{k}}, \ (k \geq n).$$

Theorem 7 follows easily from (36).

Theorem 8 *Let the function $f(z)$ defined by (1) be in the class $T_j(n, m, \lambda, \alpha, \beta)$. Then $f(z)$ starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

(37)
$$r_2 = r_2(n, m, \lambda, \alpha, \beta, \rho) = \inf_k \left[\frac{(1-\rho)k^{n-1}[(k-\alpha)+\beta(k-1)](1+(k^m-1)\lambda)}{1-\alpha}\right]^{\frac{1}{k-1}}, \ (k \geq j+1).$$

The result is sharp, with the extremal function $f(z)$ given by (7).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$$

for $|z| < r_2(n, m, \lambda, \alpha, \beta, \rho)$, where $r_2(n, m, \lambda, \alpha, \beta, \rho)$ is given by (37). Indeed we find again from the Definition 1, that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=j+1}^{\infty}(k-1)a_k|z|^{k-1}}{1 - \sum_{k=j+1}^{\infty} a_k|z|^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \rho$$

if

(38)
$$\sum_{k=j+1}^{\infty}\left(\frac{k-\rho}{1-\rho}\right) a_k|z|^{k-1} \leq 1.$$

But, by Theorem 1, (38) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1} \leq \frac{k^n[(k-\alpha)+\beta(k-1)](1+(k^m-1)\lambda)}{1-\alpha}$$

that is, if

(39)
$$|z| \leq \left[\frac{(1-\rho)k^{n-1}[(k-\alpha)+\beta(k-1)](1+(k^m-1)\lambda)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}}, \ (k \geq j+1).$$

Theorem 8 follows easily from (39).

Corollary 5 *Let the function $f(z)$ defined by (1) be in the class $T_j(n,m,\lambda,\alpha,\beta)$. Then $f(z)$ is convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_3$, where*

(40)
$$r_3 = r_3(n,m,\lambda,\alpha,\beta,\rho) = \inf_k \left[\frac{(1-\rho)k^{n-1}[(k-\alpha)+\beta(k-1)](1+(k^m-1)\lambda)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}}, \ (k \geq j+1).$$

The result is sharp, with the extremal function $f(z)$ given by (5).

5 Modified Hadmard Products

Let the function $f(z)$ be defined by (1) and function $g(z)$ be defined by

$$g(z) = z^p - \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (b_{k+p} \geq 0; p \in N, n \in N)$$

be in the same class $T_\Omega(n,p,\alpha,\lambda,\beta)$. We define the modified Hadamard product of the functions $f(z)$ and $g(z)$ is defined by

$$f * g(z) = z^p - \sum_{k=j+1}^{\infty} a_{k+p} b_{k+p} z^{k+p}. \tag{41}$$

Theorem 9 *Let each of the functions $f(z)$ and $g(z)$ be in the class $T_\Omega(n,p,\alpha,\lambda,\beta)$. Then*

$$f * g(z) \in T_\Omega(n,p,\delta,\lambda,\beta)$$

where

(42)
$$\delta \leq \frac{p(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)]^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)[k+p+\beta(k+p-1)]}{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)]^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman, we need to find the largest δ such that

(43)
$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}[(k+p-\delta)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\delta)(1+\lambda p-\lambda)}\ a_{k+p}b_{k+p} \leq 1.$$

Since

(44)
$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}\ a_{k+p} \leq 1$$

and

(45)
$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}\ b_{k+p} \leq 1$$

by the Cauchy-Schwarz inequality, we have

(46)
$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}\ \sqrt{a_{k+p}b_{k+p}} \leq 1.$$

Thus it is sufficient to show that

$$\frac{(k+p)^{\Omega}[(kp-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}\ a_{k+p}b_{k+p}$$
$$\leq \frac{(k+p)^{\Omega}[(kp-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}\ \sqrt{a_{k+p}b_{k+p}}.$$

That is that

(47)
$$\sqrt{a_{k+p}b_{k+p}} \leq \frac{[(k+p-\alpha)+\beta(k+p-1)](p-\delta)}{[(k+p-\delta)+\beta(k+p-1)](p-\alpha)}, (k \geq n).$$

Note that

$$(48)\quad \sqrt{a_{k+p}b_{k+p}} \le \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}, (k \ge n).$$

Consequently, we need only to prove that

$$\frac{p^{\Omega}(p-\alpha)(1-\lambda p-\lambda)}{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-1)} \le \frac{[(k+p-\alpha)+\beta(k+p-1)](p-\delta)}{[(k+p-\delta)+\beta(k+p-1)](p-\alpha)}\ (k \ge n),$$

or, equivalently, that

$$(49)\quad \delta \le \frac{p(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)]^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)[k+p+\beta(k+p-1)]}{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)]^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}.$$

Since

$$(50)\quad \Phi(k) = \frac{p(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)]^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)[k+p+\beta(k+p-1)]}{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)]^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}$$

is an increasing function of k $(k \ge n)$, letting $k = n$ in (50) we obtain

$$(51)\quad \delta \le \frac{p(n+p)^{\Omega}[(n+p-\alpha)+\beta(n+p-1)]^2(\lambda n+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)[nv+p+\beta(n+p-1)]}{(n+p)^{\Omega}[(n+p-\alpha)+\beta(n+p-1)]^2(\lambda n+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}$$

which proves the pain assertion of Theorem.

6 Integral Operators

Theorem 10 *Let the function $f(z)$ defined by (1) be in the class $T_{\Omega}(n,p,\lambda,\alpha,\beta)$, and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by*

$$(52)\qquad F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt,\ (c > -p)$$

also belongs to the class $T_{\Omega}(n,p,\lambda,\alpha,\beta)$.

Proof. From the representation (52) of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=n}^{\infty} b_{k+p} z^k,$$

where

$$b_k = \left(\frac{c+p}{c+p+k}\right) a_{k+p}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{k=n}^{\infty} (k+p)^{\Omega}[(k+p-\alpha) + \beta(k+p-1)](\lambda k + \lambda p - \lambda + 1) b_{k+p} \\
&\sum_{k=n}^{\infty} (k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1) \left(\frac{c+p}{c+p+k}\right) a_{k+p} \\
&\leq \sum_{k=n}^{\infty} (k+p)^{\Omega}[(k+p-\alpha) + \beta(k+p-1)](\lambda k + \lambda p - \lambda + 1) a_{k+p} \\
&\leq p^{\Omega}(p-\alpha)(1+\lambda p - \lambda),
\end{aligned}$$

since $f(z) \in T_{\Omega}(n, p, \lambda, \alpha, \beta)$. Hence, by Theorem 1, $F(z) \in T_{\Omega}(n, p, \lambda, \alpha, \beta)$, since $F(z) \in T_{\Omega}(n, p, \lambda, \alpha\beta$.

Hence by Theorem 1, $F(z) \in T_{\Omega}(n, p, \lambda, \alpha, \beta)$.

Theorem 11 *Let the function $f(z)$ be in the class $T_{\Omega}(n, p, \lambda, \alpha, \beta)$ and let c be a real number such that $c > -p$. Then the function $f(z)$ given by (52) is univalent in $|z| < R_p^*$, where*

(53)
$$R_p^* = \inf_k \left\{ \left(\frac{k+p}{p}\right)^{\Omega-1} \left(\frac{c+p}{c+p+k}\right) \left[\frac{(k+p-\alpha) + \beta(k+p-1)}{p-\alpha}\right] \left(\frac{\lambda k + \lambda p - \lambda + 1}{\lambda p - \lambda + 1}\right) \right\}^{\frac{1}{k}}, (k \geq n).$$

The result is sharp.

Proof. From (52) we have

$$\begin{aligned} f(z) &= \frac{z^{1-c}(z^c F(z))'}{c+1} \\ &= z - \sum_{k=n}^{\infty} \left(\frac{c+p+k}{c+p} \right) \quad a_{k+p} z^{k+p}, \ (c > -p) \end{aligned}$$

in order to obtain the required result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p$$

whenever $|z| < R_p^*$, where R_p^* is given by (53). Now

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| \frac{pz^{p-1} - \sum\limits_{k=n}^{\infty} (k+p) \left(\frac{c+p+k}{c+p} \right) a_{k+p} z^{k+p-1}}{z^{p-1}} - p \right|$$

$$- \sum_{k=n}^{\infty} (k+p) \left(\frac{c+p+k}{c+p} \right) a_{k+p} z^k \leq \sum_{k=n}^{\infty} (k+p) \left(\frac{c+p+k}{c+p} \right) a_{k+p} |z|^k.$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p$$

if

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p} \right) \left(\frac{c+p+k}{c+p} \right) a_{k+p} |z|^k \leq 1. \tag{54}$$

But Theorem 1 confirms that

$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}[(k+p-\alpha) + \beta(k+p-1)](\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p-\alpha)(1+\lambda p - \lambda)} a_{k+p} \leq 1. \tag{55}$$

Hence, (54) will be satisfied if

$$\begin{aligned} &\left(\frac{k+p}{p} \right) \left(\frac{c+p+k}{c+p} \right) |z|^k \\ &\leq \left(\frac{k+p}{p} \right)^{\Omega-1} \left[\frac{(k+p-\alpha) + \beta(k+p-1)}{p-\alpha} \right] \left(\frac{\lambda k + \lambda p - \lambda + 1}{\partial p - \lambda + 1} \right) \quad (k \geq n) \end{aligned}$$

that is if

$$(56)\qquad |z| < \left[\left(\frac{k+p}{p}\right)^{\Omega-1}\left(\frac{c+p}{c+p+k}\right)\right.$$

$$\left.\left[\frac{(k+p-\alpha)+\beta(k+p-1)}{p-\alpha}\right]\left(\frac{\lambda k+\lambda p-\lambda+1}{\lambda p-\lambda+1}\right)\right]^{\frac{1}{k}},\ (k\geq n).$$

The required result follows from (56). Sharpness of the result follows if we take

$$(57)\qquad f(z) = z^p - \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}[(k+p-\alpha+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}\cdot$$

$$\left(\frac{c+p+k}{c+p}\right)z^{k+p}(k\geq n).$$

Theorem 12 *Let the function $f(z)$ defined by (1) be in the class $T_{\Omega}(n,p,\lambda,\alpha,\beta)$. Then $f(z)$ is close-to-convex of order ρ $(0\leq\rho<1)$ in $|z|<r$, where*

$$\begin{aligned} r &= r(n,p,\lambda,\alpha,\beta,\rho) \\ &= \inf_k \left[\left(\frac{k+p}{p}\right)^{\Omega-1}\left(\frac{p-\rho}{p-\alpha}\right)\left[\frac{(k+p-\alpha)+\beta(k+p-1)}{(k+p-\rho)+\beta(k+p-1)}\right]\right. \\ &\quad \left.\left(\frac{\lambda k+\lambda p-\lambda+1}{\lambda p-\lambda+1}\right)\right]^{\frac{1}{k}},\ (k\geq +1). \end{aligned}$$

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}+1-p\right| \leq p-\rho$$

$(0\leq\rho<1)$ for $|z|<r$. We have

$$\left|\frac{zf'(z)}{f(z)}+1-p\right| = \left|\frac{zf'(z)+(1-p)f'(z)}{f(z)}\right|$$

$$= \left|\frac{p(p-1)z^{p-1}-\sum_{k=n}^{\infty}(k+p)(k+p-1)a_{k+p}z^{k+p-1}}{pz^{p-1}-\sum_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}}\right.$$

$$\left.\frac{p(p-1)z^{p-1}+\sum\limits_{k=n}^{\infty}(p-1)(k+p)a_{k+p}z^{k+p-1}}{pz^{p-1}-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}}\right|$$

$$=\left|\frac{-\sum\limits_{k=n}^{\infty}(k+p)ka_{k+p}z^{k+p-1}}{pz^{p-1}-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}}\right|\leq\frac{\sum\limits_{k=n}^{\infty}k(k+p)a_{k+p}|z|^{k}}{p-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^{k}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}+1-p\right|\leq p-\rho,$$

if

$$\sum_{k=n}^{\infty}\frac{(k+p-\rho)(k+p)}{p(p-\rho)}\ a_{k+p}|z|^{k}\leq 1. \tag{58}$$

But, by Theorem 1 confirms that

$$\sum_{k=n}^{\infty}\frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{n}(p-\alpha)(1+\lambda p-\lambda)}\ a_{k+p}\leq 1.$$

Hence (58) will be true if

$$\frac{(k+p-\rho)(k+p)}{p(p-\rho)}|z|^{k}\leq\frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)}{p^{n}(p-\alpha(1+\lambda p-\lambda)}$$

that is, if

(59)

$$|z|\leq\left[\frac{(k+p)^{\Omega}[(k+p-\alpha)+\beta(k+p-1)](\lambda k+\lambda p-\lambda+1)p(p-\rho)}{p^{n}(p-\alpha)(1+\lambda p-\lambda)(k+p-\rho)(k+p)}\right]^{\frac{1}{k}},\ (k\geq n).$$

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