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A new class of analytic functions involving a linear integral operator [1]

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Abstract

Using the linear operator $I_{\lambda,\ \mu}$ $(\lambda > -1, \mu > 0)$, we introduce and study a new class $Q(\lambda, \mu, \alpha, \varphi)$ of analytic functions. We derive inclusion relationship and integral representation. We also show that this class is closed under convolution with a convex function. Some applications of this theorem are also discussed.

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1 Introduction

Let A be the class of functions

(1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. We denote S^* and C be the subclasses of A, consisting of functions which are respectively starlike and convex univalent in E. A function $f \in A$ is subordinate to $g \in A$ (written as $f \prec g$), if and only if there exists a function $w(z)$, analytic in E, such that $w(0) = 0, |w(z)| < 1$ and $f(z) = g(w(z))$, $(z \in E)$.

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The class A is closed under the Hadamard product or convolution defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

We consider the following integral operator

$$I_{\lambda,\mu} : A \to A, \lambda > -1, \mu > 0; f \in A, \quad \text{defined by}$$

$$I_{\lambda,\mu} f(z) = (f_{\lambda,\mu} * f)(z), \qquad \text{see [2]} \tag{2}$$

where

$$\frac{z}{(1-z)^{\lambda+1}} * f_{\lambda,\mu}(z) = \frac{z}{(1-z)^{\mu}}.$$

Using (2) it can be easily verified that

$$(z(I_{\lambda+1,\mu} f(z)))' = (\lambda+1) I_{\lambda,\mu} f(z) - \lambda I_{\lambda+1,\mu} f(z) \tag{3}$$

and

$$(z(I_{\lambda,\mu} f(z)))' = \mu I_{\lambda,\mu+1} f(z) - (\mu - 1) I_{\lambda,\mu} f(z). \tag{4}$$

In particular, by taking $\lambda = n, \mu = 2, (n \in \mathbb{N}_0 = \{0, 1, 2, ...\})$ in (2), we obtain Noor integral operator introduced in [3].

Let

$$N = \{\varphi : z\varphi \in A, Re(\varphi(z)) > 0 \ for \ z \in E \text{ and } \ \varphi \text{ is convex univalent in } E\}.$$

It is known [6] that

$$S^*_{\lambda,\ \mu}(\varphi) = \left\{ f : f \in A \text{ and } \frac{z(I_{\lambda,\mu} f(z))'}{I_{\lambda,\mu} f(z)} \prec \varphi(z) \right\},$$

$$C_{\lambda,\ \mu}(\varphi) = \left\{ f : f \in A \text{ and } \frac{(z(I_{\lambda,\mu} f(z))')'}{(I_{\lambda,\mu} f(z))'} \prec \varphi(z) \right\}.$$

Clearly, $S^*_{1,2}(\varphi) = S^*(\varphi)$ and $C_{1,2}(\varphi) = C(\varphi)$.
For $0 \le \alpha \le 1$, and using the operator $I_{\lambda,\mu}$, we introduce the following class of analytic functions as

$$Q(\lambda,\ \mu, \alpha,\ \varphi) = \left\{ f : f \in A \text{ and } \frac{z(I_{\lambda,\mu} f(z))' + \alpha z^2 (I_{\lambda,\mu} f(z))''}{(1-\alpha)(I_{\lambda,\mu} f(z)) + \alpha z (I_{\lambda,\mu} f(z))'} \prec \varphi(z) \right\}.$$

Remark 1

$$f \in Q(\lambda, \mu, \alpha, \varphi) \text{ if and only if } \left\{(1-\alpha)(I_{\lambda,\mu}f(z)) + \alpha z(I_{\lambda,\mu}f(z))'\right\} \in S^*(\varphi).$$

2 Preliminary Results

Lemma 1 [6] *Let f_{λ,μ_i} and $f_{\lambda_i,\mu}$, $i = 1, 2$, be defined by* (2). *Then for $\lambda_i > -1$, $\mu_i > 0$, $i = 1, 2$,*

$$f_{\lambda,\ \mu_1} = f_{\lambda,\ \mu_2} * f_{\mu_2 - 1,\ \mu_1},$$

and

$$f_{\lambda_2,\ \mu} = f_{\lambda_1,\ \mu} * f_{\lambda_2,(\lambda_1+1)},$$

where

$$f_{\lambda,\mu}(z) = z + \sum_{n=1}^{\infty} \frac{(\mu)_n}{(\lambda+1)_n}\ a_n z^{n+1}, \tag{5}$$

and $f(z)$ is given by (1).

Lemma 2 [4] *If $f \in C$, $g \in S^*$, then for each function h analytic in E,*

$$\frac{(f * hg)(E)}{(f * g)(E)} \subset \overline{Co}h(E),$$

where $\overline{Co}h(E)$ denotes the closed convex hull of $h(E)$.

Lemma 3 *Let $0 < \alpha \leq \beta$. If $\beta \geq 2$ or $\alpha + \beta \geq 3$, then the function*

$$f_{\beta-1,\alpha}(z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} z^{n+1}, \qquad z \in E$$

belong to class C of convex functions.

Lemma 3 is a special case of Theorem 2.13 contained in [5].

3 Main Results

Theorem 1 *Let $\lambda > -1,\ \mu > 0,\ 0 \leq \alpha \leq 1$. If $f \in Q(\lambda, \mu, \alpha, \varphi)$, then*

$$f = \left[\sum_{n=0}^{\infty} \frac{(\lambda+1)_n}{(\mu)_n (1+\alpha)_n} z^{n+1} \right] * \exp \int_0^z \frac{\varphi(w(t))}{t} dt, \tag{6}$$

where $w(z)$ is analytic in E, with $w(0) = 0,\ |w(z)| < 1$ for $z \in E$.

Proof. Let $f \in Q(\lambda, \mu, \alpha, \varphi)$. Then there exists a function $w(z)$ analytic in E, with $w(0) = 0,\ |w(z)| < 1$ such that

$$\frac{z(I_{\lambda,\mu} f(z))' + \alpha z^2 (I_{\lambda,\mu} f(z))''}{(1-\alpha)(I_{\lambda,\mu} f(z)) + \alpha z (I_{\lambda,\mu} f(z))'} = \varphi(w(z)). \tag{7}$$

From (7) and after some simplifications, we have

$$I_{\lambda,\mu} \left[(1-\alpha)(f(z)) + \alpha z f'(z) \right] = \exp \int_0^z \frac{\varphi(w(t))}{t} dt. \tag{8}$$

Let

$$\phi(z) = (1-\alpha)\frac{z}{1-z} + \alpha \frac{z}{(1-z)^2}.$$

Then

$$(\phi * f)(z) = (1-\alpha) f(z) + \alpha z f'(z). \tag{9}$$

From (2), (8) and (9), we have

$$(I_{\lambda,\mu}\phi) * f = \exp \int_0^z \frac{\varphi(w(t))}{t} dt. \tag{10}$$

From (5) and (10), we obtain the required result.

Theorem 2 *Let $0 < \mu_1 \leq \mu_2, \lambda > -1, 0 \leq \alpha \leq 1$ and $\varphi \in N$. If $\mu_2 \geq 2$ or $\mu_1 + \mu_2 \geq 3$, then*

$$Q(\lambda, \mu_2, \alpha, \varphi) \subset Q(\lambda, \mu_1, \alpha, \varphi).$$

Proof. Let $f \in Q(\lambda, \mu_2, \alpha, \varphi)$. Then there exists a function $w(z)$ analytic in E, with $w(0) = 0,\ |w(z)| < 1$ such that

$$\frac{z(I_{\lambda,\mu_2} f(z))' + \alpha z^2 (I_{\lambda,\mu_2} f(z))''}{(1-\alpha)(I_{\lambda,\mu_2} f(z)) + \alpha z (I_{\lambda,\mu_2} f(z))'} = \varphi(w(z)). \tag{11}$$

Let

$$p(z) = \frac{z(I_{\lambda,\mu_1} f(z))' + \alpha z^2 (I_{\lambda,\mu_1} f(z))''}{(1-\alpha)(I_{\lambda,\mu_1} f(z)) + \alpha z (I_{\lambda,\mu_1} f(z))'}. \tag{12}$$

From (2), (12) and using Lemma 1, we have

$$p(z) = \frac{z(f_{\lambda,\mu_2} * f_{\mu_2-1,\lambda} * f)' + \alpha z^2 (f_{\lambda,\mu_2} * f_{\mu_2-1,\lambda} * f)''}{(1-\alpha)(f_{\lambda,\mu_2} * f_{\mu_2-1,\lambda} * f) + \alpha z (f_{\lambda,\mu_2} * f_{\mu_2-1,\lambda} * f)'}. \tag{13}$$

From (13) and using some properties of convolution, we obtain

$$p(z) = \frac{f_{\mu_2-1,\lambda} * \left[z(I_{\lambda,\mu_2} f(z))' + \alpha z^2 (I_{\lambda,\mu_2} f(z))''\right]}{f_{\mu_2-1,\lambda} * [(1-\alpha)(I_{\lambda,\mu_2} f(z)) + \alpha z (I_{\lambda,\mu_2} f(z))']}.$$

Using (11) and after some simplifications, we have

$$p(z) = \frac{f_{\mu_2-1,\lambda} * \varphi(w(z)) \, [(1-\alpha)(I_{\lambda,\mu_2} f(z)) + \alpha z (I_{\lambda,\mu_2} f(z))']}{f_{\mu_2-1,\lambda} * [(1-\alpha)(I_{\lambda,\mu_2} f(z)) + \alpha z (I_{\lambda,\mu_2} f(z))']}. \tag{14}$$

It follows from Remark 1, that

$$\left\{(1-\alpha)(I_{\lambda,\mu_2} f(z)) + \alpha z (I_{\lambda,\mu_2} f(z))'\right\} \in S^*(\varphi),$$

since $f \in Q(\lambda, \mu, \alpha, \varphi)$. Also by Lemma 3, $f_{\mu_2-1,\lambda} \in C$. Therefore, by (14), we have

$$p(E) \subset \overline{Co}\varphi(w(t)) \subset \varphi(E),$$

$\varphi \in N$ in E. Hence $p(z) \prec \varphi(z)$ and consequently $f \in Q(\lambda, \mu_1, \alpha, \varphi)$. $\square$

Special Cases. For $\alpha = 0, 1$, we obtain the result proved in [6] as special cases.

Theorem 3 *Let* $\varphi \in N,\ \lambda > -1,\ \mu > 0$ *and* $\psi \in C$. *If* $f \in Q(\lambda, \mu, \alpha, \varphi)$, *then* $f * \psi \in Q(\lambda, \mu, \alpha, \varphi)$.

Proof. Let $F = f * \psi$ and set

$$p(z) = \frac{z(I_{\lambda,\mu}F(z))' + \alpha z^2 (I_{\lambda,\mu}F(z))''}{(1-\alpha)(I_{\lambda,\mu}F(z)) + \alpha z (I_{\lambda,\mu}F(z))'}. \tag{15}$$

From (2), (15) and after some simplifications, we have

$$p(z) = \frac{\psi * \left[z(I_{\lambda,\mu}f(z))' + \alpha z^2 (I_{\lambda,\mu}f(z))''\right]}{\psi * \left[(1-\alpha)(I_{\lambda,\mu}f(z)) + \alpha z (I_{\lambda,\mu}f(z))'\right]}.$$

Now proceeding in a similar way as in Theorem 2, we obtain the required result.

Applications of Theorem 3

The class $Q(\lambda, \mu, \alpha, \varphi)$ is invariant under the following integral operators

$$\text{(i)} \quad f_1(z) = \int_0^z \frac{f(t)}{t} dt,$$

$$\text{(ii)} \quad f_2(z) = \frac{2}{t} \int_0^z f(t) dt,$$

$$\text{(iii)} \quad f_3(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt, \ |x| \le 1, \ x \ne 1,$$

$$\text{(iv)} \quad f_4(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt, \ \Re(c) > -1.$$

The proof immediately follows from Theorem 3, since we can write, see [1], $f_i = f * \varphi_i$, for $i = 1, 2, 3, 4$ with

$$\begin{aligned}
\varphi_1(z) &= -\log(1-z), \\
\varphi_2(z) &= -2\left[\frac{z + \log(1-z)}{z}\right], \\
\varphi_3(z) &= \frac{1}{1-x} \log\left(\frac{1-xz}{1-z}\right), \ |x| \le 1, \ x \ne 1, \\
\varphi_4(z) &= \sum_{m=1}^{\infty} \frac{1+c}{m+c} z^m, \Re(c) > -1
\end{aligned}$$

and each φ_i is convex for $i = 1, 2, 3, 4$.

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