**General Mathematics Vol. 19, No. 1 (2011), 45–57**

# Subordination and Superordination Results Associated with a Linear Operator [1]

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**Abstract**

In the present paper, we derive some subordination and superordination results associated with a linear operator. Several sandwich-type results involving this operator are also proved.

**2000 Mathematics Subject Classification:** 30C45.
**Key words and phrases:** Analytic functions; Hadamard product (or Convolution); Subordination and superordination between analytic functions; Dziok-Srivastava operator.

## 1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number $n$ and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a, n] := \{ \mathfrak{f} \in \mathcal{H}(\mathbb{U}) : \ \mathfrak{f}(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$

[1] *Received 1 April, 2009*
*Accepted for publication (in revised form) 23 September, 2009*

Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f)=\left\{\varepsilon\in\partial\mathbb{U}:\ \lim_{z\to\varepsilon}f(z)=\infty\right\},$$

and such that $f'(\varepsilon)\neq 0$ for $\varepsilon\in\partial\mathbb{U}\backslash E(f)$. The subclass of $Q$ for which $f(0)=a$ $(a\in\mathbb{C})$ is denoted by $Q(a)$.

Let $f,\ g\in\mathcal{A}$, where $f$ is given by (1.1) and $g$ is defined by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k.$$

Then the Hadamard product (or convolution) $f*g$ of the functions $f$ and $g$ is defined by

$$(f*g)(z):=z+\sum_{k=2}^{\infty}a_kb_kz^k=:(g*f)(z). \tag{1.2}$$

For two functions $f$ and $g$, analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{U}$, and write

$$f(z)\prec g(z)\qquad(z\in\mathbb{U}),$$

if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with

$$\omega(0)=0\quad\text{and}\quad|\omega(z)|<1\quad(z\in\mathbb{U})$$

such that

$$f(z)=g\big(\omega(z)\big)\qquad(z\in\mathbb{U}).$$

Indeed, it is known that

$$f(z)\prec g(z)\quad(z\in\mathbb{U})\Longrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z)\prec g(z)\quad(z\in\mathbb{U})\Longleftrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

For complex parameters

$$\alpha_j\in\mathbb{C}\ (j=1,\ldots,l),\ \beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\ (\mathbb{Z}_0^-:=\{0,-1,-2,\ldots\};j=1,\ldots,m),$$

the generalized hypergeometric function ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is given by

$$(1.3) \qquad {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leqq m+1;\ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, \ldots\};\ z \in \mathbb{U}),$$

where $(\upsilon)_k$ is the Pochhammer symbol defined by

$$(\upsilon)_0 = 1 \quad \text{and} \quad (\upsilon)_k = \upsilon(\upsilon+1)\cdots(\upsilon+k-1) \qquad (k \in \mathbb{N}).$$

Corresponding to a function $h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$, defined by

$$(1.4) \qquad h(\alpha_1, \cdots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\, {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

Dziok and Srivastava [1, 2, 3](see also [4, 5, 6]) considered a linear operator:

$$H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m):\ \mathcal{A} \longrightarrow \mathcal{A},$$

defined by the following Hadamard product:

$$(1.5) \qquad H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) := h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z)$$

We note that the linear operator $H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ includes other linear operators which were introduced and studied in [7, 8, 9] and so on.

Corresponding to the function $h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$, defined by (1.4), we introduce a function $h_\lambda(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)(\lambda > 0)$ given by

$$(1.6) \quad h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * h_\lambda(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \frac{z}{(1-z)^\lambda}.$$

Analogous to the Dziok-Srivastava operator $H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$, Kwon and Cho [10] defined a new linear operator $H_\lambda(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ on $\mathcal{A}$ as follows:

$$(1.7) \quad H_\lambda(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) := h_\lambda(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z).$$

For convenience, we write

$$(1.8) \quad H_{\lambda,l,m}(\alpha_j) := H_\lambda(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \quad (j \in \{1, 2, \ldots, l\}).$$

It is easily verified from the definition (1.7) that

$$(1.9) \; z\left(H_{\lambda,l,m}(\alpha_j+1)f\right)'(z) = \alpha_j H_{\lambda,l,m}(\alpha_j)f(z) - (\alpha_j - 1)H_{\lambda,l,m}(\alpha_j+1)f(z),$$

and

$$z\left(H_{\lambda,l,m}(\alpha_j)f\right)'(z) = \lambda H_{\lambda+1,l,m}(\alpha_j)f(z) - (\lambda-1)H_{\lambda,l,m}(\alpha_j)f(z). \tag{1.10}$$

We note that the operator $H_{\lambda,l,m}(\alpha_1)$ was introduced and investigated recently by Kwon and Cho [10], they defined several new classes of analytic functions by using this operator and investigated various inclusion properties of these classes. In the present paper, we derive some subordination and superordination results of this operator $H_{\lambda,l,m}(\alpha_j)$. Several sandwich-type results involving this operator are also proved.

## 2 A Set of Lemmas

The following lemmas will be required in our proposed investigation.

**Lemma 1.** (See [11]) *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ for all real $s$ and for all*

$$t \leqq -\frac{n\left(1+s^2\right)}{2} \qquad (n \in \mathbb{N})$$

*satisfies the condition $\Re(H(is,t)) \leqq 0$. If the function*

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$$

*is analytic in $\mathbb{U}$ and*

$$\Re\left(H(p(z), zp'(z))\right) > 0 \qquad (z \in \mathbb{U}),$$

*then*

$$\Re(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

**Lemma 2.** (See [12]) *Let $\kappa,\ \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If*

$$\Re(\kappa h(z) + \gamma) > 0 \qquad (z \in \mathbb{U}),$$

*then the solution of the following differential equation:*

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \qquad (z \in \mathbb{U};\ q(0) = c)$$

*is analytic in $\mathbb{U}$ and satisfies the inequality given by*

$$\Re(\kappa q(z) + \gamma) > 0 \qquad (z \in \mathbb{U}).$$

**Lemma 3.** (See [13]) *Let* $p \in Q(a)$ *and*

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (q \neq a;\ n \in \mathbb{N})$$

*be analytic in* $\mathbb{U}$. *If* $q$ *is not subordinate to* $p$, *then there exists two points*

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad \textit{and} \quad \xi_0 \in \partial\mathbb{U} \backslash E(f)$$

*such that*

$$q(\mathbb{U}_{r_0}) \subset p\,(\mathbb{U}), \quad q(z_0) = p\,(\xi_0) \quad \textit{and} \quad z_0 q'(z_0) = m\xi_0 p'(\xi_0) \quad (m \geqq n).$$

A function $P(z,t)$ $(z \in \mathbb{U};\ t \geqq 0)$ is said to be a subordination chain if $P(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqq 0$, $P(z,0)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $P(z,t_1) \prec P(z,t_2)$ for all $0 \leqq t_1 \leqq t_2$.

**Lemma 4.** (See [14]) *The function* $P(z,t):\ \mathbb{U} \times [0,\infty) \to \mathbb{C}$ *of the form*

$$P(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \qquad (a_1(t) \neq 0;\ t \geqq 0)\,,$$

*and* $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ *is a subordination chain if and only if*

$$\Re\left(\frac{z\,\partial P/\partial z}{\partial P/\partial t}\right) > 0 \qquad (z \in \mathbb{U};\ t \geqq 0).$$

**Lemma 5.** (See [15]) *Let* $q \in \mathcal{H}[a,1]$ *and* $\phi : \mathbb{C}^2 \to \mathbb{C}$. *Also set*

$$\phi\left(q(z), zq'(z)\right) \equiv h(z) \qquad (z \in \mathbb{U}).$$

*If* $P(z,t) := \phi\,(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap Q(a)$. *Then*

$$h(z) \prec \phi\left(p(z), zp'(z)\right) \qquad (z \in \mathbb{U})$$

*implies that*

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

*Furthermore, if* $\phi\,(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in Q(a)$, *then* $q$ *is the best subordination.*

# 3 Main Results

We first give the following subordination result.

**Theorem 1.** *Let $f,\ g \in \mathcal{A}$ and $\lambda > 0$. If*

$$\Re\left(1+\frac{z\psi''(z)}{\psi'(z)}\right) > -\delta \qquad \left(z \in \mathbb{U};\ \psi(z) := \frac{H_{\lambda+1,l,m}(\alpha_j)g(z)}{z}\right), \tag{3.1}$$

*where*

$$\delta := \frac{1+\lambda^2-\left|1-\lambda^2\right|}{4\lambda}, \tag{3.2}$$

*then the following subordination relationship*

$$\frac{H_{\lambda+1,l,m}(\alpha_j)f(z)}{z} \prec \frac{H_{\lambda+1,l,m}(\alpha_j)g(z)}{z} \qquad (z \in \mathbb{U})$$

*implies that*

$$\frac{H_{\lambda,l,m}(\alpha_j)f(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z} \qquad (z \in \mathbb{U}).$$

*Furthermore, the function $\frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z}$ is the best dominant.*

*Proof.* Let us define the functions $\mathcal{F}$ and $\mathcal{G}$ by

$$\mathcal{F}(z) := \frac{H_{\lambda,l,m}(\alpha_j)f(z)}{z},\quad \mathcal{G}(z) := \frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z}. \tag{3.3}$$

We here assume, without loss of generality, that $\mathcal{G}$ is analytic and univalent on $\overline{\mathbb{U}}$ and

$$\mathcal{G}'(\zeta) \neq 0 \qquad (|\zeta| = 1).$$

If not, then we replace $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}(\rho z)$ and $\mathcal{G}(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, and we can use them in the proof of our result. Therefore, our results would follow by letting $\rho \to 1$.

We first show that if the function $\mathcal{Q}$ be defined by

$$\mathcal{Q}(z) := 1+\frac{z\mathcal{G}''(z)}{\mathcal{G}'(z)} \qquad (z \in \mathbb{U}), \tag{3.4}$$

then

$$\Re(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}).$$

In view of (1.10) and the definitions of $\mathcal{G}$ and $\psi$, we know that

$$\psi(z) = \mathcal{G}(z) + \frac{1}{\lambda}\, z\mathcal{G}'(z). \tag{3.5}$$

Differentiating both sides of (3.5) with respect to $z$, we get

$$\psi'(z) = \left(1 + \frac{1}{\lambda}\right)\mathcal{G}'(z) + \frac{1}{\lambda} z\mathcal{G}''(z). \tag{3.6}$$

After some simple calculations, in conjunction with (3.4) and (3.6), we easily get the following relationship:

$$1 + \frac{z\psi''(z)}{\psi'(z)} = \mathcal{Q}(z) + \frac{z\mathcal{Q}'(z)}{\mathcal{Q}(z) + \lambda} := \quad (z) \qquad (z \in \mathbb{U}). \tag{3.7}$$

We also deduce from (3.1) and (3.7) that

$$\Re\left( \quad (z) + \lambda\right) > 0 \qquad (z \in \mathbb{U}). \tag{3.8}$$

Furthermore, by Lemma 2, we conclude that the differential equation (3.7) has a solution $\mathcal{Q} \in \mathcal{H}(\mathbb{U})$ with

$$\quad (0) = \mathcal{Q}(0) = 1.$$

Let us put

$$H(u, v) := u + \frac{v}{u + \lambda} + \delta, \tag{3.9}$$

where $\delta$ is given by (3.2). From (3.1), (3.7) and (3.9), we obtain

$$\Re\left(H(\mathcal{Q}(z), z\mathcal{Q}'(z))\right) > 0 \qquad (z \in \mathbb{U}).$$

Now we proceed to show that

$$\Re(H(is, t)) \leqq 0 \qquad \left(s \in \mathbb{R};\ t \leqq -\frac{1 + s^2}{2}\right), \tag{3.10}$$

Indeed, from (3.9), we have

$$\Re(H(is, t)) = \Re\left(is + \frac{t}{is + \lambda} + \delta\right) = \frac{t\lambda}{|\lambda + is|^2} + \delta \leqq -\frac{\Psi(\lambda, s)}{2\,|\lambda + is|^2},$$

where

$$\Psi(\lambda, s) := (\lambda - 2\delta)s^2 - 4\delta\lambda s - 2\delta\lambda^2 + \lambda. \tag{3.11}$$

For $\delta$ given by (3.2), the coefficient of $s^2$ in the quadratic expression $\Psi(\lambda, s)$ given by (3.11) is positive or equal to zero. Furthermore, we observe that the quadratic expression $\Psi(\lambda, s)$ by $s$ in (3.11) is a perfect square, which implies that (3.10) holds. Thus, by Lemma 1, we conclude that

$$\Re(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}),$$

that is, $\mathcal{G}(z)$ by (3.3) is convex.

To prove $\mathcal{F} \prec \mathcal{G}$, we let the function $P(z,t)$ be defined by

$$P(z,t) := \mathcal{G}(z) + \left(\frac{1+t}{\lambda}\right) z\mathcal{G}'(z) \qquad (z \in \mathbb{U};\ 0 \leqq t < \infty), \tag{3.12}$$

since $\mathcal{G}$ is convex and $\lambda > 0$, we have

$$\frac{\partial P(z,t)}{\partial z}|_{z=0} = \mathcal{G}'(0)\left(1 + \frac{1+t}{\lambda}\right) \neq 0 \qquad (z \in \mathbb{U};\ 0 \leqq t < \infty),$$

and

$$\Re\left(\frac{z\,\partial P(z,t)/\partial z}{\partial P(z,t)/\partial t}\right) = \Re\left(\lambda + (1+t)\mathcal{Q}(z)\right) > 0 \qquad (z \in \mathbb{U}).$$

Therefore, by Lemma 4, we obtain that $P(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\psi(z) = \mathcal{G}(z) + \frac{1}{\lambda} z\mathcal{G}'(z) = P(z,0),$$

and

$$P(z,0) \prec P(z,t) \qquad (z \in \mathbb{U};\ 0 \leqq t < \infty),$$

which implies that

$$P(\zeta,t) \notin P(\mathbb{U},0) = \psi(\mathbb{U}) \qquad (\zeta \in \partial\mathbb{U};\ 0 \leqq t < \infty). \tag{3.13}$$

If $\mathcal{F}$ is not subordinate to $\mathcal{G}$, by Lemma 3, we know that there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$\mathcal{F}(z_0) = \mathcal{G}(\zeta_0) \quad \text{and} \quad z_0\mathcal{F}'(z_0) = (1+t)\zeta_0\mathcal{G}'(\zeta_0) \quad (0 \leqq t < \infty). \tag{3.14}$$

Hence, by virtue of (1.10) and (3.14), we have

$$P(\zeta_0,t) = \mathcal{G}(\zeta_0) + \frac{1+t}{\lambda}\zeta_0\mathcal{G}'(\zeta_0) = \mathcal{F}(z_0) + \frac{1}{\lambda} z_0\mathcal{F}'(z_0) =$$

$$= \frac{H_{\lambda+1,l,m}(\alpha_j)f(z_0)}{z_0} \in \psi(\mathbb{U}).$$

But this contradicts to (3.13). Thus, we deduce that $\mathcal{F} \prec \mathcal{G}$. Considering $\mathcal{F} = \mathcal{G}$, we see that the function $\mathcal{G}$ is the best dominant. The proof of Theorem 1 is evidently completed. □

By similarly applying the method of proof of Theorem 1 and using (1.9), we easily get the following result.

**Corollary 1.** *Let* $f,\ g \in \mathcal{A}$, $\lambda > 0$ *and* $\Re(\alpha_j + 1) > 0$. *If*

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\eta \qquad \left(z \in \mathbb{U};\ \varphi(z) := \frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z}\right),$$

*where*

$$\eta := \frac{1 + |\alpha_j + 1|^2 - \left|1 - (\alpha_j + 1)^2\right|}{4\Re(\alpha_j + 1)}, \tag{3.15}$$

*then the following subordination relationship*

$$\frac{H_{\lambda,l,m}(\alpha_j)f(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z} \qquad (z \in \mathbb{U})$$

*implies that*

$$\frac{H_{\lambda,l,m}(\alpha_j + 1)f(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j + 1)g(z)}{z} \qquad (z \in \mathbb{U}).$$

*Furthermore, the function* $\frac{H_{\lambda,l,m}(\alpha_j+1)g(z)}{z}$ *is the best dominant.*

If $f$ is subordinate to $F$, then $F$ is superordinate to $f$. We now derive the following superordination result.

**Theorem 2.** *Let* $f,\ g \in \mathcal{A}$ *and* $\lambda > 0$. *If*

$$\Re\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right) > -\delta \qquad \left(z \in \mathbb{U};\ \psi(z) := \frac{H_{\lambda+1,l,m}(\alpha_j)g(z)}{z}\right),$$

*where* $\delta$ *is given by (3.2), also let the function* $\frac{H_{\lambda+1,l,m}(\alpha_j)f}{z}$ *is univalent in* $\mathbb{U}$ *and* $\frac{H_{\lambda,l,m}(\alpha_j)f}{z} \in Q$, *then the following subordination relationship*

$$\frac{H_{\lambda+1,l,m}(\alpha_j)g(z)}{z} \prec \frac{H_{\lambda+1,l,m}(\alpha_j)f(z)}{z} \qquad (z \in \mathbb{U})$$

*implies that*

$$\frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)f(z)}{z} \qquad (z \in \mathbb{U}).$$

*Furthermore, the function* $\frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z}$ *is the best subdominant.*

*Proof.* Suppose that the functions $\mathcal{F}$ and $\mathcal{G}$ are defined by (3.3), $\mathcal{Q}$ is defined by (3.4). By applying the similar method as in the proof of Theorem 1, we get

$$\Re(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}).$$

Next, to arrive at our desired result, we show that $\mathcal{G} \prec \mathcal{F}$. For this, we suppose that the function $P(z,t)$ be defined by (3.12). Since $\lambda > 0$ and $\mathcal{G}$ is convex, by applying the similar method as in Theorem 1, we deduce that $P(z,t)$ is subordination chain. Therefore, by Lemma 5, we conclude that $\mathcal{G} \prec \mathcal{F}$. Furthermore, since the differential equation

$$\psi(z) = \mathcal{G}(z) + \frac{1}{\lambda} z\mathcal{G}'(z) := \phi\left(\mathcal{G}(z), z\mathcal{G}'(z)\right)$$

has a univalent solution $\mathcal{G}$, it is the best subordination. We thus complete the proof of Theorem 2. □

By similarly applying the method of proof of Theorem 2 and using (1.9), we easily get the following result.

**Corollary 2.** *Let $f,\ g \in \mathcal{A}$, $\lambda > 0$ and $\Re(\alpha_j + 1) > 0$. If*

$$\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\eta \qquad \left(z \in \mathbb{U};\ \varphi(z) := \frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z}\right),$$

*where $\eta$ is given by (3.15), also let the function $\frac{H_{\lambda,l,m}(\alpha_j)f}{z}$ is univalent in $\mathbb{U}$ and $\frac{H_{\lambda,l,m}(\alpha_j+1)f}{z} \in Q$, then the following subordination relationship*

$$\frac{H_{\lambda,l,m}(\alpha_j)g(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)f(z)}{z} \qquad (z \in \mathbb{U})$$

*implies that*

$$\frac{H_{\lambda,l,m}(\alpha_j+1)g(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j+1)f(z)}{z} \qquad (z \in \mathbb{U}).$$

*Furthermore, the function $\frac{H_{\lambda,l,m}(\alpha_j+1)g(z)}{z}$ is the best subdominant.*

Combining the above mentioned subordination and superordination results involving the operator $H_{\lambda,l,m}(\alpha_j)$, we get the following "sandwich-type result".

**Corollary 3.** *Let $f,\ g_k \in \mathcal{A}$ $(k = 1,\ 2)$ and $\lambda > 0$. If*

$$\Re\left(1 + \frac{z\psi_k''(z)}{\psi_k'(z)}\right) > -\delta \qquad \left(z \in \mathbb{U};\ \psi_k(z) := \frac{H_{\lambda+1,l,m}(\alpha_j)g_k(z)}{z} \quad (k = 1,\ 2)\right),$$

*where* $\delta$ *is given by* (3.2), *also let the function* $\frac{H_{\lambda+1,l,m}(\alpha_j)f}{z}$ *is univalent in* $\mathbb{U}$ *and* $\frac{H_{\lambda,l,m}(\alpha_j)f}{z} \in Q$, *then the following subordination relationship*

$$\frac{H_{\lambda+1,l,m}(\alpha_j)g_1(z)}{z} \prec \frac{H_{\lambda+1,l,m}(\alpha_j)f(z)}{z} \prec \frac{H_{\lambda+1,l,m}(\alpha_j)g_2(z)}{z} \qquad (z \in \mathbb{U})$$

*implies that*

$$\frac{H_{\lambda,l,m}(\alpha_j)g_1(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)f(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)g_2(z)}{z} \qquad (z \in \mathbb{U}).$$

*Furthermore, the functions* $\frac{H_{\lambda,l,m}(\alpha_j)g_1}{z}$ *and* $\frac{H_{\lambda,l,m}(\alpha_j)g_2}{z}$ *are, respectively, the best subordinant and the best dominant.*

**Corollary 4.** *Let* $f,\ g_k \in \mathcal{A}$ $(k = 1,\, 2)$, $\lambda > 0$ *and* $\Re(\alpha_j + 1) > 0$. *If*

$$\Re\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\eta \qquad \left(z \in \mathbb{U};\ \varphi_k(z) := \frac{H_{\lambda,l,m}(\alpha_j)g_k(z)}{z} \quad (k = 1,\, 2)\right),$$

*where* $\eta$ *is given by* (3.15), *also let the function* $\frac{H_{\lambda,l,m}(\alpha_j)f}{z}$ *is univalent in* $\mathbb{U}$ *and* $\frac{H_{\lambda,l,m}(\alpha_j+1)f}{z} \in Q$, *then the following subordination relationship*

$$\frac{H_{\lambda,l,m}(\alpha_j)g_1(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)f(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j)g_2(z)}{z} \qquad (z \in \mathbb{U})$$

*implies that*

$$\frac{H_{\lambda,l,m}(\alpha_j+1)g_1(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j+1)f(z)}{z} \prec \frac{H_{\lambda,l,m}(\alpha_j+1)g_2(z)}{z} \qquad (z \in \mathbb{U}).$$

*Furthermore, the functions* $\frac{H_{\lambda,l,m}(\alpha_j+1)g_1}{z}$ *and* $\frac{H_{\lambda,l,m}(\alpha_j+1)g_2}{z}$ *are, respectively, the best subordinant and the best dominant.*

## Acknowledgements

The present investigation was supported by the *Scientific Research Fund of Huaihua University* and the *Scientific Research Fund of Hunan Provincial Education Department* under Grant 07C507 of People's Republic of China.

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