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A generalisation of fixed point theorems in a 2-metric space [1]

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Abstract

Here we generalise, improve and unify the fixed point theorems due to Delbosco[1], Skof[8], Khan et al.[5] and several other fixed point theorems for a single map and common fixed point theorems ([6], [7]) for a pair of mappings in a setting of 2-metric space.

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1 Introduction

Delbosco[1] and Skof[8] have established a fixed point theorem for self maps of complete metric spaces by introducing a class Φ of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) $\phi : [0,\infty) \to [0,\infty)$ is continuous in R^+ and strictly increasing in R^+.

(ii) $\phi(t) = 0$ if and only if $t = 0$.

(iii) $\phi(t) \geq Mt^\mu$ for every $t > 0$, $\mu > 0$ are constants.

In 1977, F.Skof[8] gave the following theorem.

Theorem 1 *Let T be a self map of a complete metric space (X, d) and $\phi \in \Phi$ such that for every $x, y \in X$*

$$(1) \qquad \phi(d(Tx,Ty)) \leq a\phi(d(x,y)) + b\phi(d(x,Tx)) + c\phi(d(y,Ty))$$

where a, b and c are three nonnegative constants satisfying $a+b+c < 1$. Then T has a unique fixed point.

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In 1984, Khan et al.[5] generalised the Theorem 1 by using much extensive condition than (1) and removed the condition (iii). They proved the following theorem as follows.

Theorem 2 *Let T be a self map of a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore , let a, b, c be three decreasing funtions from R^+ into $[0, 1)$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose T satisfies the folowing condition*

$$\begin{aligned} \phi\left(d\left(Tx,Ty\right)\right) \leq\ & a\left(d\left(x,y\right)\right)\phi\left(d\left(x,y\right)\right)+b\left(d\left(x,y\right)\right)\left[\phi\left(d\left(x,Tx\right)\right)\right. \\ & \left.+\phi\left(d\left(y,Ty\right)\right)\right]+c\left(d\left(x,y\right)\right)\min\left\{\phi\left(d\left(x,Ty\right)\right),\right. \\ & \left.\phi\left(d\left(x,Ty\right)\right)\right\} \end{aligned} \tag{2}$$

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

We first give a 2-metric analogue of Theorem 2. In this connection we need some preliminary ideas about 2-metric space.

2 Preliminaries

In Sixties, Gähler([2]-[3]) first defined 2-metric space as follows: Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

(I) given distinct elements x,y of X, there exists an element z of X such that $d(x, y, z) \neq 0$

(II) $d(x, y, z) = 0$ when at least two of x, y, z are equal,

(III) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all x, y, z in X, and

(IV) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X.

When d is a 2-metric on X, then the ordered pair (X, d) is called a 2-metric space.

Definition 1 *A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $a \in X$, $\lim d(x_n, x_m, a) = 0$ as $n, m \to \infty$.*

Definition 2 *A sequence $\{x_n\}$ in X is convergent to an element $x \in X$ if for each $a \in X$, $\lim\limits_{n\to\infty} d(x_n, x, a) = 0$*

Definition 3 *A complete 2-metric space is one in which every Cauchy sequence in X converges to an element of X.*

3 Main Results

Theorem 3 *Let T be a self map of a complete 2-metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore , let a, b, c be three decreasing funtions from R^+ into $[0,1)$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose T satisfies the folowing condition*

$$\begin{aligned} \phi\left(d\left(Tx, Ty, u\right)\right) &\leq a\left(d\left(x, y, u\right)\right)\phi\left(d\left(x, y, u\right)\right) \\ &+b\left(d\left(x, y, u\right)\right)\left[\phi\left(d\left(x, Tx, u\right)\right) + \phi\left(d\left(y, Ty, u\right)\right)\right] \\ &+c\left(d\left(x, y, u\right)\right)\min\left\{\phi\left(d\left(x, Ty, u\right)\right), \phi\left(d\left(y, Tx, u\right)\right)\right\} \end{aligned} \tag{3}$$

where $x, y, u \in X$, each two of x, y and u are distinct. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary.
Define $x_{n+1} = Tx_n$; $n = 0, 1, 2, ...$, also let $\alpha_n = d\left(x_n, x_{n+1}, u\right)$ for $n = 0, 1, 2, ...$; and $\beta_n = \phi\left(\alpha_n\right)$. Then we have

$$\begin{aligned} \beta_{n+1} &= \phi\left(\alpha_{n+1}\right) \\ &= \phi\left(d\left(x_{n+1}, x_{n+2}, u\right)\right) \\ &= \phi\left(d\left(Tx_n, Tx_{n+1}, u\right)\right) \\ &\leq a\left(d\left(x_n, x_{n+1}, u\right)\right)\phi\left(d\left(x_n, x_{n+1}, u\right)\right) \\ &\quad +b\left(d\left(x_n, x_{n+1}, u\right)\right)\left[\phi\left(d\left(x_n, Tx_n, u\right)\right) + \phi\left(d\left(x_{n+1}, Tx_{n+1}, u\right)\right)\right] \\ &\quad +c\left(d\left(x_n, x_{n+1}, u\right)\right)\min\left\{\phi\left(d\left(x_n, Tx_{n+1}, u\right)\right),\right. \\ &\quad \left.\phi\left(d\left(x_{n+1}, Tx_n, u\right)\right)\right\} \\ &= a\left(d\left(x_n, x_{n+1}, u\right)\right)\phi\left(d\left(x_n, x_{n+1}, u\right)\right) \\ &\quad +b\left(d\left(x_n, x_{n+1}, u\right)\right)\left[\phi\left(d\left(x_n, x_{n+1}, u\right)\right) + \phi\left(d\left(x_{n+1}, x_{n+2}, u\right)\right)\right] \\ &\quad +c\left(d\left(x_n, x_{n+1}, u\right)\right)\min\left\{\phi\left(d\left(x_n, x_{n+2}, u\right)\right), \phi\left(d\left(x_{n+1}, x_{n+1}, u\right)\right)\right\} \\ &= a\left(\alpha_n\right)\phi\left(\alpha_n\right) + b\left(\alpha_n\right)\left[\phi\left(\alpha_n\right) + \phi\left(\alpha_{n+1}\right)\right] \end{aligned}$$

$$\text{implies} \quad \beta_{n+1} \leq \frac{a\left(\alpha_n\right) + b\left(\alpha_n\right)}{1 - b\left(\alpha_n\right)}\beta_n \tag{4}$$

Since $a(t) + 2b(t) + c(t) < 1$, $a\left(\alpha_n\right) + 2b\left(\alpha_n\right) < 1$ which implies

$$\frac{a\left(\alpha_n\right) + b\left(\alpha_n\right)}{1 - b\left(\alpha_n\right)} < 1$$

If we set

$$r = \frac{a\left(\alpha_n\right) + b\left(\alpha_n\right)}{1 - b\left(\alpha_n\right)}$$

then from (4) we get $\beta_{n+1} \leq r\beta_n$ where $r < 1$. So $\beta_n \leq r^n \beta_0$, such that $\beta_n \to 0$ as $n \to \infty$. Since $\beta_n < \beta_{n-1}$ and ϕ is strictly increasing, $\alpha_n < \alpha_{n-1}$, $n = 1, 2, \ldots$ Thus $\alpha_n \to \alpha$ (say). Then $\beta_n = \phi(\alpha_n) \to \phi(\alpha)$, since ϕ is continuous. So $\phi(\alpha) = 0$ and hence by (ii), $\alpha = 0$ implies $\alpha_n \to 0$.

We now show that $\{x_n\}$ is a Cauchy sequence. We prove it by contradiction. Then for every positive integer ϵ and for every positive integer k there exist two positive integers $m(k)$ and $n(k)$ such that

$$k < n(k) < m(k) \text{ and } d\left(x_{m(k)}, x_{n(k)}, u\right) > \epsilon \tag{5}$$

For each integer k, let $m(k)$ be the least integer for which $m(k) > n(k) > k$,

$$d\left(x_{n(k)}, x_{m(k)-1}, u\right) \leq \epsilon \quad \text{and} \quad d\left(x_{n(k)}, x_{m(k)}, u\right) > \epsilon$$

Then we have

$$\begin{aligned} \epsilon &< d\left(x_{n(k)}, x_{m(k)}, u\right) \\ &\leq d\left(x_{n(k)}, x_{m(k)}, x_{m(k)-1}\right) \\ &+ d\left(x_{n(k)}, x_{m(k)-1}, u\right) + d\left(x_{m(k)-1}, x_{m(k)}, u\right) \end{aligned} \tag{6}$$

Now by (3), we have

$$\begin{aligned} \phi\left(d\left(x_{n(k)}, x_{m(k)}, x_{m(k)-1}\right)\right) &= \phi\left(d\left(Tx_{n(k)-1}, Tx_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\leq a\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad \phi\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad +b\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad \left[\phi\left(d\left(x_{n(k)-1}, Tx_{n(k)-1}, x_{m(k)-1}\right)\right)\right. \\ &\quad \left.+\phi\left(d\left(x_{m(k)-1}, Tx_{m(k)-1}, x_{m(k)-1}\right)\right)\right] \\ &\quad +c\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad \min\left\{\phi\left(d\left(x_{n(k)-1}, Tx_{m(k)-1}, x_{m(k)-1}\right)\right),\right. \\ &\quad \left.\phi\left(d\left(x_{m(k)-1}, Tx_{n(k)-1}, x_{m(k)-1}\right)\right)\right\} \\ &= a\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad \phi\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad +b\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad \left[\phi\left(d\left(x_{n(k)-1}, x_{n(k)}, x_{m(k)-1}\right)\right)\right. \\ &\quad \left.+\phi\left(d\left(x_{m(k)-1}, x_{m(k)}, x_{m(k)-1}\right)\right)\right] \\ &\quad +c\left(d\left(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}\right)\right) \\ &\quad \min\left\{\phi\left(d\left(x_{n(k)-1}, x_{m(k)}, x_{m(k)-1}\right)\right),\right. \\ &\quad \left.\phi\left(d\left(x_{m(k)-1}, x_{n(k)}, x_{m(k)-1}\right)\right)\right\} \\ &= 0 \end{aligned}$$

which implies by (ii)

$$d\left(x_{n(k)}, x_{m(k)}, x_{m(k)-1}\right) = 0 \tag{7}$$

So by (6) and (7) we get, $\epsilon < d\left(x_{n(k)}, x_{m(k)}, u\right) \leq 0 + \epsilon + \alpha_{m(k)-1}$. Since $\{\alpha_n\}$ converges to 0, $d\left(x_{n(k)}, x_{m(k)}, u\right) \to \epsilon$ as $k \to \infty$. Again

$$\begin{aligned} d\left(x_{n(k)+1}, x_{m(k)}, u\right) &\leq d\left(x_{n(k)+1}, x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)+1}, x_{n(k)}, u\right) \\ &\quad + d\left(x_{n(k)}, x_{m(k)}, u\right) \\ &= \alpha_{n(k)} + d\left(x_{n(k)}, x_{m(k)}, u\right), \end{aligned}$$

since $d\left(x_{n(k)+1}, x_{m(k)}, x_{n(k)}\right)$ can be made 0 as we have done in equation (7). So $d\left(x_{n(k)+1}, x_{m(k)}, u\right) \leq \alpha_{n(k)} + d\left(x_{n(k)}, x_{m(k)}, u\right) \to \epsilon$ as $k \to \infty$. In the similar way

$$\begin{aligned} d\left(x_{n(k)+2}, x_{m(k)}, u\right) &\leq d\left(x_{n(k)+2}, x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+2}, x_{n(k)+1}, u\right) \\ &\quad + d\left(x_{n(k)+1}, x_{m(k)}, u\right) \\ &= \alpha_{n(k)+1} + d\left(x_{n(k)+1}, x_{m(k)}, u\right), \end{aligned}$$

since $d\left(x_{n(k)+2}, x_{m(k)}, x_{n(k)+1}\right)$ can be made 0 as we have done in equation (7). So $d\left(x_{n(k)+2}, x_{m(k)}, u\right) \leq \alpha_{n(k)+1} + d\left(x_{n(k)+1}, x_{m(k)}, u\right) \to \epsilon$ as $k \to \infty$ and in similar fashion we can show $d\left(x_{n(k)+2}, x_{m(k)+1}, u\right) \to \epsilon$ as $k \to \infty$. Using (3), we deduce that

$$\begin{aligned}
\phi\left(d\left(x_{n(k)+2}, x_{m(k)+1}, u\right)\right) &= \phi\left(d\left(Tx_{n(k)+1}, Tx_{m(k)}, u\right)\right) \\
&\leq a\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad \phi\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad +b\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad \left[\phi\left(d\left(x_{n(k)+1}, Tx_{n(k)+1}, u\right)\right)\right. \\
&\quad \left.+\phi\left(d\left(x_{m(k)}, Tx_{m(k)}, u\right)\right)\right] \\
&\quad +c\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad \min\left\{\phi\left(d\left(x_{n(k)+1}, Tx_{m(k)}, u\right)\right),\right. \\
&\quad \left.\phi\left(d\left(x_{m(k)}, Tx_{n(k)+1}, u\right)\right)\right\} \\
&= a\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad \phi\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad +b\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad \left[\phi\left(d\left(x_{n(k)+1}, x_{n(k)+2}, u\right)\right)\right. \\
&\quad \left.+\phi\left(d\left(x_{m(k)}, x_{m(k)+1}, u\right)\right)\right] \\
&\quad +c\left(d\left(x_{n(k)+1}, x_{m(k)}, u\right)\right) \\
&\quad \min\left\{\phi\left(d\left(x_{n(k)+1}, x_{m(k)+1}, u\right)\right),\right. \\
&\quad \left.\phi\left(d\left(x_{m(k)}, x_{n(k)+2}, u\right)\right)\right\}
\end{aligned}$$

Letting $k \to \infty$, we get

$$\phi(\epsilon) \leq a(\epsilon)\,\phi(\epsilon) + c(\epsilon)\,\phi(\epsilon) = \{a(\epsilon) + c(\epsilon)\}\,\phi(\epsilon) < \phi(\epsilon)$$

which is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since X is complete 2-metric space, $\lim_n x_n = z \in X$. Now we shall show that $Tz = z$.
Again using (3) we have

$$\begin{aligned}
\phi\left(d\left(x_{n(k)+1}, Tz, u\right)\right) &= \phi\left(d\left(Tx_{n(k)}, Tz, u\right)\right) \\
&\leq a\left(d\left(x_{n(k)}, z, u\right)\right)\phi\left(d\left(x_{n(k)}, z, u\right)\right) \\
&\quad +b\left(d\left(x_{n(k)}, z, u\right)\right)\left[\phi\left(d\left(x_{n(k)}, Tx_{n(k)}, u\right)\right)\right. \\
&\quad \left.+\phi\left(d\left(z, Tz, u\right)\right)\right] + c\left(d\left(x_{n(k)}, z, u\right)\right) \\
&\quad \min\left\{\phi\left(d\left(x_{n(k)}, Tz, u\right)\right), \phi\left(d\left(z, Tx_{n(k)}, u\right)\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
\text{implies } \phi\left(d\left(x_{n(k)+1}, Tz, u\right)\right) \leq\ & a\left(d\left(x_{n(k)}, z, u\right)\right)\phi\left(d\left(x_{n(k)+1}, z, u\right)\right) \\
& +b\left(d\left(x_{n(k)}, z, u\right)\right) \\
& \left[\phi\left(d\left(x_{n(k)}, x_{n(k)+1}, u\right)\right)\right. \\
& \left.+\phi\left(d\left(z, Tz, u\right)\right)\right] \\
& +c\left(d\left(x_{n(k)}, z, u\right)\right) \\
& \min\left\{\phi\left(d\left(x_{n(k)}, Tz, u\right)\right),\right. \\
& \left.\phi\left(d\left(z, x_{n(k)+1}, u\right)\right)\right\}
\end{aligned}
$$

Passing limit as $n \to \infty$ on bothsides of the inequality we get, $\phi(d(z, Tz, u)) = 0$ which gives by (ii), $d(z, Tz, u) = 0$ i.e. $Tz = z$. Next let w be another fixed point of T. Then

$$
\begin{aligned}
\phi(d(z, w, u)) &= \phi(d(Tz, Tw, u)) \\
&\leq a(d(z, w, u))\phi(d(z, w, u)) \\
&\quad +b(d(z, w, u))[\phi(d(z, Tz, u)) + \phi(d(w, Tw, u))] \\
&\quad +c(d(z, w, u))\min\{\phi(d(z, Tw, u)), \\
&\quad \phi(d(w, Tz, u))\} \\
&= [a(d(z, w, u)) + c(d(z, w, u))]\phi(d(z, w, u)) \\
&< \phi(d(z, w, u)), \qquad \text{since } a(t) + c(t) < 1
\end{aligned}
$$

which is a contradiction leads to the fact that $z = w$ and thus completes the proof.

Next we verify the Theorem (3) by a proper example.

Example 1. Let $X = R^+ \times R^+$ and d be a 2-metric which expresses $d(x, y, u)$ as the area of the Euclidean triangle with vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $u = (u_1, u_2)$. Then (X, d) is a complete 2-metric space[6].
Now take $x = (1, 0)$, $y = (2, 0)$ and $u = (1, 1)$ also let $T : X \to X$ be a mapping such that

$$
\begin{aligned}
Tx &= (2, 0) \text{ where } x = (1, 0) \in X \text{ and} \\
Ty &= (3, 0) \text{ where } y = (2, 0) \in X
\end{aligned}
$$

Now setting $a(t) = \frac{2}{5}$, $b(t) = \frac{1}{5}$, $c(t) = \frac{1}{6}$ and $\phi(t) = t^2$; $t \in R^+$. We observe that all the conditions of Theorem (3) satisfied except the condition (3). Also it is very clear that T has no fixed point in X in this case.

Next we establish a common fixed point theorem in this line.

Theorem 4 *Let S and T be self mappings of a complete 2-metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore , let a, b, c be three decreasing* *funtions from R^+ into $[0, 1)$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose S and T satisfy the folowing condition*

$$\begin{aligned} \phi\left(d\left(Sx, Ty, u\right)\right) \;\leq\; & a\left(d\left(x, y, u\right)\right)\phi\left(d\left(x, y, u\right)\right) \\ & + b\left(d\left(x, y, u\right)\right)\left[\phi\left(d\left(x, Sx, u\right)\right) + \phi\left(d\left(y, Ty, u\right)\right)\right] \\ & + c\left(d\left(x, y, u\right)\right)\min\left\{\phi\left(d\left(x, Ty, u\right)\right),\right. \\ & \left.\phi\left(d\left(y, Sx, u\right)\right)\right\} \end{aligned} \tag{8}$$

where $x, y, u \in X$, each two of x, y and u are distinct. Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Define $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$; $n = 0, 1, 2, ...$, also let $\alpha_n = d\left(x_n, x_{n+1}, u\right)$ for $n = 0, 1, 2, ...$; and $\beta_n = \phi\left(\alpha_n\right)$. We also assume that $\alpha_n > 0$ for every n. Now for an even integer n, we have

$$\begin{aligned} \beta_n &= \phi\left(\alpha_n\right) \\ &= \phi\left(d\left(x_n, x_{n+1}, u\right)\right) \\ &= \phi\left(d\left(Sx_{n-1}, Tx_n, u\right)\right) \\ &\leq a\left(d\left(x_{n-1}, x_n, u\right)\right)\phi\left(d\left(x_{n-1}, x_n, u\right)\right) \\ &\quad + b\left(d\left(x_{n-1}, x_n, u\right)\right)\left[\phi\left(d\left(x_{n-1}, Sx_{n-1}, u\right)\right) + \phi\left(d\left(x_n, Tx_n, u\right)\right)\right] \\ &\quad + c\left(d\left(x_{n-1}, x_n, u\right)\right)\min\left\{\phi\left(d\left(x_{n-1}, Tx_n, u\right)\right), \phi\left(d\left(x_n, Sx_{n-1}, u\right)\right)\right\} \\ &= a\left(d\left(x_{n-1}, x_n, u\right)\right)\phi\left(d\left(x_{n-1}, x_n, u\right)\right) \\ &\quad + b\left(d\left(x_{n-1}, x_n, u\right)\right)\left[\phi\left(d\left(x_{n-1}, x_n, u\right)\right) + \phi\left(d\left(x_n, x_{n+1}, u\right)\right)\right] \\ &\quad + c\left(d\left(x_{n-1}, x_n, u\right)\right)\min\left\{\phi\left(d\left(x_{n-1}, x_{n+1}, u\right)\right), \phi\left(d\left(x_n, x_n, u\right)\right)\right\} \\ &= a\left(\alpha_{n-1}\right)\phi\left(\alpha_{n-1}\right) + b\left(\alpha_{n-1}\right)\left[\phi\left(\alpha_{n-1}\right) + \phi\left(\alpha_n\right)\right] \end{aligned}$$

$$\text{implies} \quad \beta_n \leq \frac{a\left(\alpha_{n-1}\right) + b\left(\alpha_{n-1}\right)}{1 - b\left(\alpha_{n-1}\right)}\beta_{n-1} \tag{9}$$

Since $a(t) + 2b(t) + c(t) < 1$, $a\left(\alpha_{n-1}\right) + 2b\left(\alpha_{n-1}\right) < 1$ which implies

$$\frac{a\left(\alpha_{n-1}\right) + b\left(\alpha_{n-1}\right)}{1 - b\left(\alpha_{n-1}\right)} < 1$$

If we set

$$r = \frac{a\left(\alpha_{n-1}\right) + b\left(\alpha_{n-1}\right)}{1 - b\left(\alpha_{n-1}\right)}$$

then from (3.9) we get $\beta_n \leq r\beta_{n-1}$ where $r < 1$. So $\beta_n \leq r^n\beta_0$, such that $\beta_n \to 0$ as $n \to \infty$. Since $\beta_n < \beta_{n-1}$ and ϕ is strictly increasing, $\alpha_n < \alpha_{n-1}$,

$n = 1, 2, ...$ Thus $\alpha_n \to \alpha$ (say). Then $\beta_n = \phi(\alpha_n) \to \phi(\alpha)$, since ϕ is continuous. So $\phi(\alpha) = 0$ and hence by (ii), $\alpha = 0$ implies $\alpha_n \to 0$.
We now show that $\{x_n\}$ is a Cauchy sequence. We prove it by contradiction. Then for every positive integer ϵ and for every positive integer k there exist two positive integers $2p(k)$ and $2q(k)$ such that

$$k < 2q(k) < 2p(k) \quad \text{and} \quad d\left(x_{2p(k)}, x_{2q(k)}, u\right) > \epsilon \tag{10}$$

For each integer k, let $2p(k)$ be the least integer for which $2p(k) > 2q(k) > k$,

$$d\left(x_{2q(k)}, x_{2p(k)-2}, u\right) \leq \epsilon \quad \text{and} \quad d\left(x_{2q(k)}, x_{2p(k)}, u\right) > \epsilon$$

Then we have

$$\begin{aligned}\epsilon < d\left(x_{2q(k)}, x_{2p(k)}, u\right) &\leq d\left(x_{2q(k)}, x_{2p(k)}, x_{2p(k)-2}\right) + d\left(x_{2q(k)}, x_{2p(k)-2}, u\right) \\ &\quad + d\left(x_{2p(k)-2}, x_{2p(k)}, u\right)\end{aligned}$$

Since we can easily show that $d\left(x_{2q(k)}, x_{2p(k)}, x_{2p(k)-2}\right) = 0$ as we have shown in equation (7) of Theorem (3).

$$\begin{aligned}\epsilon < d\left(x_{2q(k)}, x_{2p(k)}, u\right) &\leq d\left(x_{2q(k)}, x_{2p(k)-2}, u\right) + d\left(x_{2p(k)-2}, x_{2p(k)}, u\right) \\ &\leq d\left(x_{2q(k)}, x_{2p(k)-2}, u\right) \\ &\quad + d\left(x_{2p(k)-2}, x_{2p(k)}, x_{2p(k)-1}\right) \\ &\quad + d\left(x_{2p(k)-2}, x_{2p(k)-1}, u\right) + d\left(x_{2p(k)-1}, x_{2p(k)}, u\right)\end{aligned}$$

Again we can show like equation (7) of Theorem (3),
$d\left(x_{2p(k)-2}, x_{2p(k)}, x_{2p(k)-1}\right) = 0$. Thus

$$\epsilon < d\left(x_{2q(k)}, x_{2p(k)}, u\right) \leq \epsilon + 0 + \alpha_{2p(k)-2} + \alpha_{2p(k)-1} \tag{11}$$

Since $\{\alpha_n\}$ converges to 0, $d\left(x_{2q(k)}, x_{2p(k)}, u\right) \to \epsilon$.

$$\begin{aligned}\text{Now} \quad d\left(x_{2q(k)}, x_{2p(k)+1}, u\right) &\leq d\left(x_{2q(k)}, x_{2p(k)+1}, x_{2p(k)}\right) \\ &\quad + d\left(x_{2q(k)}, x_{2p(k)}, u\right) \\ &\quad + d\left(x_{2p(k)}, x_{2p(k)+1}, u\right) \\ &\leq d\left(x_{2q(k)}, x_{2p(k)}, u\right) + \alpha_{2p(k)}\end{aligned}$$

since we can show that $d\left(x_{2q(k)}, x_{2p(k)+1}, x_{2p(k)}\right) = 0$ as we have done in equation (7) of Theorem (3).

$$\text{So} \quad d\left(x_{2q(k)}, x_{2p(k)+1}, u\right) \to \epsilon \quad \text{as } k \to \infty \tag{12}$$

Again

$$\begin{aligned} d\left(x_{2q(k)}, x_{2p(k)+2}, u\right) &\leq d\left(x_{2q(k)}, x_{2p(k)+2}, x_{2p(k)+1}\right) + d\left(x_{2q(k)}, x_{2p(k)+1}, u\right) \\ &\quad + d\left(x_{2p(k)+1}, x_{2p(k)+2}, u\right) \\ &\leq d\left(x_{2q(k)}, x_{2p(k)+1}, u\right) + d\left(x_{2p(k)+1}, x_{2p(k)+2}, u\right), \\ &\quad \text{since } d\left(x_{2q(k)}, x_{2p(k)+2}, x_{2p(k)+1}\right) = 0 \text{ for similar} \\ &\quad \text{reason as of equation (7) of Theorem (3)} \\ &\leq d\left(x_{2q(k)}, x_{2p(k)+1}, x_{2p(k)}\right) + d\left(x_{2q(k)}, x_{2p(k)}, u\right) \\ &\quad + d\left(x_{2p(k)}, x_{2p(k)+1}, u\right) + d\left(x_{2p(k)+1}, x_{2p(k)+2}, u\right) \\ &\leq 0 + d\left(x_{2q(k)}, x_{2p(k)}, u\right) + \alpha_{2p(k)} + \alpha_{2p(k)+1} \end{aligned}$$

which gives

$$d\left(x_{2q(k)}, x_{2p(k)+2}, u\right) \to \epsilon \quad \text{as} \quad k \to \infty \tag{13}$$

$$\text{Similarly,} \quad d\left(x_{2q(k)+1}, x_{2p(k)+2}, u\right) \to \epsilon \quad \text{as } k \to \infty \tag{14}$$

Now from (8) we get

$$\begin{aligned} \phi\left(d\left(x_{2p(k)+2}, x_{2q(k)+1}, u\right)\right) &= \phi\left(d\left(Sx_{2p(k+)1}, Tx_{2q(k)}, u\right)\right) \\ &\leq a\left(d\left(x_{2p(k)+1}, x_{2q(k)}, u\right)\right) \\ &\quad \phi\left(d\left(x_{2p(k)+1}, x_{2q(k)}, u\right)\right) \\ &\quad + b\left(d\left(x_{2p(k)+1}, x_{2q(k)}, u\right)\right) \\ &\quad \left[\phi\left(d\left(x_{2p(k)+1}, Sx_{2p(k)+1}, u\right)\right)\right. \\ &\quad \left. + \phi\left(d\left(x_{2q(k)}, Tx_{2q(k)}, u\right)\right)\right] \\ &\quad + c\left(d\left(x_{2p(k)+1}, x_{2q(k)}, u\right)\right) \\ &\quad \min\left\{\phi\left(d\left(x_{2p(k)+1}, Tx_{2q(k)}, u\right)\right),\right. \\ &\quad \left.\phi\left(d\left(x_{2q(k)}, Sx_{2p(k)+1}, u\right)\right)\right\} \end{aligned}$$

Passing limit as $k \to \infty$ we get by (12), (13) and (14),

$$\phi(\epsilon) \leq a(\epsilon)\,\phi(\epsilon) + c(\epsilon)\,\phi(\epsilon) = \{a(\epsilon) + c(\epsilon)\}\,\phi(\epsilon) < \phi(\epsilon)$$

which is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since X is complete 2-metric space, $\lim_n x_n = z \in X$. Again using (8) we have

$$\begin{aligned}
\phi\left(d\left(x_{2p(k)+2}, Tz, u\right)\right) &= \phi\left(d\left(Sx_{2p(k)+1}, Tz, u\right)\right) \\
&\leq a\left(d\left(x_{2p(k)+1}, z, u\right)\right)\phi\left(d\left(x_{2p(k)+1}, z, u\right)\right) \\
&\quad +b\left(d\left(x_{2p(k)+1}, z, u\right)\right) \\
&\quad \left[\phi\left(d\left(x_{2p(k)+1}, Sx_{2p(k)+1}, u\right)\right)\right. \\
&\quad \left.+\phi\left(d\left(z, Tz, u\right)\right)\right] + c\left(d\left(x_{2p(k)+1}, z, u\right)\right) \\
&\quad \min\left\{\phi\left(d\left(x_{2p(k)+1}, Tz, u\right)\right),\right. \\
&\quad \left.\phi\left(d\left(z, Sx_{2p(k)+1}, u\right)\right)\right\}
\end{aligned}$$

Taking limit as $k \to \infty$ we get $\phi(d(z, Tz, u)) = 0$ implies $d(z, Tz, u) = 0$ by property (ii). Hence $Tz = z$. Similarly it can be shown that $Sz = z$. So S and T have a common fixed point $z \in X$. We now show that z is the unique common fixed point of S and T. If not, then let w be another fixed point of S and T. Then

$$\begin{aligned}
\phi(d(z, w, u)) &= \phi(d(Sz, Tw, u)) \\
&\leq a(d(z, w, u))\,\phi(d(z, w, u)) \\
&\quad +b(d(z, w, u))\left[\phi(d(z, Sz, u)) + \phi(d(w, Tw, u))\right] \\
&\quad +c(d(z, w, u))\min\{\phi(d(z, Tw, u)), \\
&\quad \phi(d(w, Sz, u))\} \\
&= \left[a(d(z, w, u)) + c(d(z, w, u))\right]\phi(d(z, w, u)) \\
&< \phi(d(z, w, u)), \qquad \text{since } a(t) + c(t) < 1
\end{aligned}$$

which is a contradiction. Hence $z = w$ and thus completes the proof.

Remark 1. In the same way we can verify the Theorem (4) by setting $S(1,0) = (2,0)$ and $T(2,0) = (3,0)$ taking all the values same on the complete 2-metric space (X, d) as described in Example 1.

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