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# SOME OPEN PROBLEMS ABOUT THE SOLUTIONS OF THE DELAY DIFFERENCE EQUATION $x_{n+1} = A/x_n^2 + 1/x_{n-k}^p$

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Abstract. We discuss the dynamics of the positive solutions of the delay difference equation in the title for some special values of the parameters $A$, $p$ and $k$ and we pose a conjecture and two open problems.

**1. Introduction.** Consider the difference equation

$$x_{n+1} = \frac{A}{x_n^2} + \frac{1}{\sqrt{x_{n-1}}}, \quad n = 0, 1, \dots , \tag{1}$$

where $A \in (0, \infty)$ and the initial conditions $x_{-1}$ and $x_0$ are arbitrary positive numbers. The following conjecture is predicted by computer simulations.

**2. Conjecture.** *Let $\bar{x}$ denote the unique positive equilibrium of Eq.* (1).
(a) *Show that when*

$$0 < A < \frac{15}{4} \tag{2}$$

*the positive equilibrium of Eq.* (1) *is globally asymptotically stable.*
(b) *Show that when*

$$A > \frac{15}{4} \tag{3}$$

*there exists a periodic cycle with period two which is asymptotically stable.*

With the use of a computer one can easily experiment with difference equations and one can easily discover that such equations possess fascinating properties with a great deal of structure and regularity. Of course all computer observations and predictions must also be proven analytically. Therefore this is a fertile area of research, still in its infancy, with deep and important results which require our attention.

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For some developments on the global behavior of solutions of delay difference equations the reader is referred to the forthcoming monograph by Kocic and Ladas [2]. See also [1] and [3].

Although we are unable to establish the above conjecture, we have proven the following result.

**Theorem 1.** (*a) Assume that* (2) *holds. Then the positive equilibrium* $\bar{x}$ *of Eq.* (1) *is locally asymptotically stable.*

(*b) Assume that* (3) *holds. Then Eq.* (1) *has a periodic solution with period two.*

*Proof.* (a) Set $\varrho = \sqrt{\bar{x}}$. Then the linearized equation of Eq. (1) about $\bar{x}$ is

$$y_{n+1} + \frac{2A}{\varrho^6} y_n + \frac{1}{2\varrho^3} y_{n-1} = 0, \quad n = 0, 1, \dots . \tag{4}$$

From the well-known Schur-Cohn criterion, Eq. (4) is asymptotically stable provided that

$$\frac{2A}{\varrho^6} < 1 + \frac{1}{2\varrho^3} < 2. \tag{5}$$

Note that $\varrho$ satisfies the equation

$$\varrho^2 = \frac{A}{\varrho^4} + \frac{1}{\varrho}. \tag{6}$$

Hence $\varrho > 1$ and (5) is satisfied if and only if

$$2A < \varrho^6 + \frac{1}{2}\varrho^3 = A + \frac{3}{2}\varrho^3,$$

that is,

$$\varrho > \left(\frac{2A}{3}\right)^{1/3}. \tag{7}$$

Set $f(t) = t^6 - t^3 - A$ and observe that $f(t) < 0$ if $0 < t < \varrho$ and $f(t) > 0$ if $t > \varrho$. Hence (7) is equivalent to $f\big((\frac{2A}{3})^{1/3}\big) < 0$; that is

$$A < \frac{15}{4}.$$

(b) Eq. (1) has a periodic solution of the form $\{p, q, p, q, \dots\}$ or $\{q, p, q, p, \dots\}$ if and only if

$$p = \frac{A}{q^2} + \frac{1}{\sqrt{p}} \quad \text{and} \quad q = \frac{A}{p^2} + \frac{1}{\sqrt{q}}. \tag{8}$$

Set $x = \sqrt{p}$ and $y = \sqrt{q}$. Then the system of algebraic equations (8) is equivalent to

$$\left.\begin{aligned} x^2 &= \frac{A}{y^4} + \frac{1}{x} \\ y^2 &= \frac{A}{x^4} + \frac{1}{y} \end{aligned}\right\} \text{ with } x, y > 0. \tag{9}$$

Set $\xi = x + y$, $\eta = xy$ and $\zeta = \eta^3$. Then $x$ and $y$ are the roots of the quadratic equation $\lambda^2 - \xi\lambda + \eta = 0$ and these roots are real, positive, and distinct if and only if

$$\xi, \eta \in (0, \infty) \quad \text{and} \quad \eta < \frac{1}{4}\xi^2. \tag{10}$$

Cancel the denominators in (9), then multiply the first equation by $x$ and the second by $y$, equate the terms $x^4y^4$, and divide by $x - y$. This leads to

$$A\xi = \eta(\xi^2 - \eta). \tag{11}$$

Cancel the denominators in (9), subtract and then divide by $x - y$. This yields

$$\eta^3 = -A + \xi(\xi^2 - 2\eta). \tag{12}$$

Subtract from the first equation in (9), the second, and use (12) to obtain

$$\xi = \frac{(A-1)\eta^3 + A^2}{\eta^4}. \tag{13}$$

By substituting (13) into (11) we find

$$G(\zeta) = \zeta^3 + (A-1)\zeta^2 + A^2(2-A)\zeta - A^4 = 0. \tag{14}$$

Note that

$$G(z) < 0 \quad \text{if} \quad z < \zeta \quad \text{and} \quad G(z) > 0 \quad \text{if} \quad z > \zeta. \tag{15}$$

In view of (10) and (13) we obtain

$$4\zeta^3 < (A-1)^2\zeta^2 + 2A^2(A-1)\zeta + A^4$$

and so by using (14) we find

$$H(\zeta) = (A+3)(A-1)\zeta^2 + 2A^2(3-A)\zeta - 3A^4 > 0.$$

The positive root of this quadratic equation is $\zeta = 3A^2/(A+3)$ and so $H(\zeta) > 0$ if and only if $G(3A^2/(A+3)) < 0$, that is

$$A > \frac{15}{4}.$$

The proof of the theorem is complete. □

**3. Open problems.** A related difference equation is

$$x_{n+1} = \frac{a}{x_n^2} + \frac{1}{x_{n-1}}, \quad n = 0, 1, \dots , \tag{16}$$

where $a \in (0, \infty)$ and $x_{-1}$, $x_0 \in (0, \infty)$.

One can show that the following result holds.

**Theorem 2.** *The following statements are true:*

(*a*) *The unique positive equilibrium $\bar{x}$ of Eq.* (16) *is locally asymptotically stable if*

$$a < 2\sqrt{3} \tag{17}$$

*and unstable if*

$$a > 2\sqrt{3}. \tag{18}$$

(*b*) *When* (18) *holds, Eq.* (16) *has a periodic cycle with period two,* $\{p, q, p, q, \dots\}$.

*Furthermore*

$$p = \frac{a + \sqrt{a^2 + 2 - 2\sqrt{1 + 4a^2}}}{2} \quad \text{and} \quad q = \frac{a - \sqrt{a^2 + 2 - 2\sqrt{1 + 4a^2}}}{2}.$$

**Open problem 1.** (a) For what values of $a$ is the positive equilibrium $\bar{x}$ of Eq. (16) globally asymptotically stable?

(b) For what values of $a$ is the periodic cycle $\{p, q, p, q, \dots\}$ of Eq. (16) asymptotically stable? What is its basin of attraction?

Eqs. (1) and (16) are special cases of the delay difference equation

$$x_{n+1} = \frac{A}{x_n^2} + \frac{1}{x_{n-k}^p} \quad n = 0, 1, \dots \tag{19}$$

where $A$, $p \in (0, \infty)$ and $k \in \{0, 1, \dots\}$ and the initial conditions $x_{-k}, \dots , x_0$ are arbitrary positive numbers.

**Open problem 2.** (a) Obtain conditions on $A$, $p$ and $k$ under which the positive equilibrium of Eq. (19) is globally asymptotically stable.

(b) Obtain conditions on $A$, $p$ and $k$ under which Eq. (19) has periodic cycles of period two. Under what conditions on $A$, $p$ and $k$ are these periodic cycles stable? What is the basin of attraction?

(c) Do there exist periodic cycles of period greater than two?

## References

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