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A DIRECT METHOD FOR BOUNDARY INTEGRAL EQUATIONS ON A CONTOUR WITH A PEAK

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In memoriam V.D. Kupradze

Abstract. Boundary integral equations in the logarithmic potential theory are studied by the direct method under the assumption that the contour has a peak.

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1. Introduction

In this paper we apply the so-called direct variant of the method of boundary integral equations to solve the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \ \Omega, \quad u = \varphi \ \text{ on } \ \Gamma \setminus \{O\} \tag{1}$$

and the Neumann problem

$$\Delta u = 0 \ \text{ in } \ \Omega, \quad \partial u/\partial n = \psi \ \text{ on } \ \Gamma \setminus \{O\} \tag{2}$$

with boundary data ψ satisfying $\int_\Gamma \psi ds = 0$, where Ω is a plane simply connected domain having compact closure and the boundary Γ with a peak at $z = 0$. Here and elsewhere we assume that the normal n is directed outwards.

The classical method for solving boundary value problems is their reduction to boundary integral equations by using potentials. In the case of the Dirichlet and Neumann problems for the Laplace equation the solutions of these problems are represented in the form of simple and double layer potentials whose densities satisfy boundary integral equations. However, there exists another way of reduction when solutions of integral equations are represented explicitly by the solutions of the boundary value problems (1) and (2). In this case the integral equations can be obtained from the integral representation for a harmonic function:

$$u(z) = \frac{1}{2\pi}\Big(V\frac{\partial u}{\partial n}\Big)(z) - \frac{1}{2\pi}\big(Wu\big)(z), \quad z \in \Omega,$$

where V is the simple layer potential

$$\big(V\sigma\big)(z) = \int_\Gamma \sigma(q) \log \frac{|z|}{|z-q|} ds_q$$

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and W is the double layer potential

$$\big(W\sigma\big)(z) = \int_\Gamma \sigma(q)\frac{\partial}{\partial n}\log\frac{1}{|z-q|}\,ds_q\,.$$

By making use of the continuity of the simple layer potential and the limit relation for the double layer potential we obtain

$$\pi u(z) = \left(V\frac{\partial u}{\partial n}\right)(z) - \big(Wu\big)(z), \quad z \in \Gamma\setminus\{O\}, \tag{3}$$

where $\big(Wu\big)(z)$ is the direct value of Wu at the point z on $\Gamma\setminus\{O\}$. By inserting the known values φ of u on $\Gamma\setminus\{O\}$ into (1) we obtain that $\partial u/\partial n$ satisfies the integral equation of first kind

$$V\gamma = \pi\varphi + W\varphi \ \text{ on } \ \Gamma\setminus\{O\}.$$

The normal derivative on $\Gamma\setminus\{O\}$ of the solution u of (2) is defined by the boundary data ψ. From (3) it follows that u on $\Gamma\setminus\{O\}$ is a solution of the integral equation

$$\pi\sigma + W\sigma = V\psi\,.$$

Using the results of [1]–[3] we study the equations obtained by the direct reduction of problems (1) and (2) for a domain with a peak to integral equations. For every integral equation we choose a pair of function spaces and prove the solvability of the equation in one of these spaces with a right-hand side from another. We also describe solutions of the corresponding homogeneous equation.

We mention recent articles [4], [5] dealing with boundary integral equations in weighted L_p-spaces on contours with peaks. The solvability in Smirnov's classes of boundary value problems in domains with piecewise smooth boundaries was studied in [6], [7] where the reduction to the Riemann-Hilbert problem for analytic functions in the unit disc was used.

Let Ω be a plane simply connected domain with boundary Γ which has a peak at the origin O. We assume that $\Gamma\setminus\{O\}$ belongs to the class C^2. We say that O is an outward (inward) peak if Ω (the exterior domain Ω') is given near O by the inequalities $\kappa_-(x) < y < \kappa_+(x)$, $0 < x < \delta$, where

$$x^{-\mu-1}\kappa_\pm(x) \in C^2[0,\delta], \quad \lim_{x\to+0} x^{-\mu-1}\kappa_\pm(x) = \alpha_\pm$$

with $\mu > 0$ and $\alpha_+ > \alpha_-$.

By $\Gamma_\pm$ we denote the arcs $\{(x,\kappa_\pm(x)) : x \in [0,\delta]\}$. Points on Γ_+ and Γ_- with equal abscissas are denoted by q_+ and q_-.

We say that φ belongs to $\mathcal{L}_{p,\beta}(\Gamma)$ if $|q|^\beta\varphi \in L_p(\Gamma)$. The norm in this space is given by

$$\parallel \varphi \parallel_{\mathcal{L}_{p,\beta}(\Gamma)} = \parallel |q|^\beta\varphi \parallel_{L_p(\Gamma)}\,.$$

Let $\mathcal{L}^1_{p,\beta}(\Gamma)$ be the space of absolutely continuous functions on $\Gamma\setminus\{O\}$ with the finite norm

$$\parallel \varphi \parallel_{\mathcal{L}^1_{p,\beta}(\Gamma)} = \parallel (\partial/\partial s)\varphi \parallel_{\mathcal{L}_{p,\beta}(\Gamma)} + \parallel \varphi \parallel_{\mathcal{L}_{p,\beta-1}(\Gamma)}\,.$$

We introduce the space $\mathfrak{N}_{p,\beta}(\Gamma)$ of absolutely continuous functions φ on $\Gamma\backslash\{O\}$ with the finite norm

$$\| \varphi \|_{\mathfrak{N}_{p,\beta}(\Gamma)} = \Big(\int\limits_{\Gamma_+ \cup \Gamma_-} |\varphi(q_+) - \varphi(q_-)|^p |q|^{p(\beta-\mu)} ds_q \Big)^{1/p} + \| \varphi \|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)} .$$

By $\mathfrak{N}^{-1}_{p,\beta}(\Gamma)$ we denote the space of functions on $\Gamma\backslash\{O\}$ represented in the form $\varphi = (d/ds)\psi$, where $\psi \in \mathfrak{N}_{p,\beta}(\Gamma)$ and $\psi(z_0) = 0$ for a fixed point $z_0 \in \Gamma \setminus \{O\}$. A norm on $\mathfrak{N}^{-1}_{p,\beta}(\Gamma)$ is defined by

$$\| \varphi \|_{\mathfrak{N}^{-1}_{p,\beta}(\Gamma)} = \| \psi \|_{\mathfrak{N}_{p,\beta}(\Gamma)} .$$

Furthermore, we introduce the space $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ of absolutely continuous functions φ on $\Gamma \setminus \{O\}$ supplied with the norms

$$\| \varphi \|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)} = \Big(\int\limits_{\Gamma_+ \cup \Gamma_-} |\varphi(q_+) + \varphi(q_-)|^p |q|^{p(\beta-\mu)} ds_q \Big)^{1/p} + \| \varphi \|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)} .$$

Let $\mathfrak{P}(\Gamma)$ denote the space of restrictions to $\Gamma \setminus \{O\}$ of functions of the form $p(z) = \sum_{k=0}^{m} t^{(k)} \mathrm{Re}\, z^k$, where $m = [\mu - \beta - p^{-1} + 2^{-1}]$ and $t^{(k)}$ are real numbers. We endow $\mathfrak{P}(\Gamma)$ with the norm

$$\| p \|_{\mathfrak{P}(\Gamma)} = \sum_{k=0}^{m} | t^{(k)} | .$$

The space $\mathfrak{M}_{p,\beta}(\Gamma)$ is defined as the direct sum of $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $\mathfrak{P}(\Gamma)$.

Now, we can describe our results. We assume that $p > 1$ and $0 < \beta + p^{-1} < \min\{\mu, 1\}$.

Let Ω have an outward peak. We introduce the double layer potential in Ω' by setting

$$\big(W^{ext}\sigma\big)(z) = \int\limits_{\Gamma} \sigma(q) \Big(\frac{\partial}{\partial n_q} \log \frac{1}{|z-q|} + 1 \Big) ds_q, \quad z \in \Omega' .$$

The value of this potential at the point $z \in \Gamma \setminus \{O\}$ will be also denoted by $\big(W^{ext}\sigma\big)(z)$. In Theorem 1 we prove that the integral equation

$$\pi\sigma + W^{ext}\sigma = V\psi \tag{4}$$

with the function $\psi \in \mathfrak{N}^{-1}_{p,\beta}(\Gamma)$ on the right-hand side has a unique solution σ in $\mathfrak{N}_{p,\beta}(\Gamma)$ satisfying $\int_\Gamma \sigma ds = 0$.

As is shown in Theorem 3, the integral equation of first kind

$$V\gamma = \pi\varphi + W\varphi \tag{5}$$

with the function $\varphi \in \mathfrak{N}_{p,\beta}(\Gamma)$ on the right-hand side has a unique solution γ in $\mathfrak{N}^{-1}_{p,\beta}(\Gamma)$.

For Ω with an inward peak, we show in Theorem 2 that the integral equation

$$\pi\sigma + W^{ext}\sigma = V\psi' , \tag{6}$$

with the function $\psi \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$ is solvable in $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ and that the solution σ satisfies $\int_\Gamma \sigma ds = 0$. In Theorem 4 we prove that the integral equation

$$V\gamma = \pi\varphi + W\varphi \tag{7}$$

with the function $\varphi \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$ has a solution $\gamma \in \mathcal{L}_{p,\beta+1}(\Gamma)$.

In Theorems 2 and 4 we prove that the homogeneous equations (6) and (7) have only trivial solutions for $0 < \beta + p^{-1} < 1/2$ and a one-dimensional space of solutions for $1/2 < \beta + p^{-1} < 1$.

We shall use the following statements proved in [1]–[3].

Theorem A (see [1]). *Let Ω have either an outward or an inward peak and let $0 < \beta + p^{-1} < \min\{\mu, 1\}$. Then the operator*

$$\mathcal{L}_{p,\beta+1}(\Gamma) \times \mathbb{R} \ni (\sigma, t) \xrightarrow{\mathcal{V}} V\sigma + c \in \mathfrak{N}_{p,\beta}(\Gamma)$$

is continuous and, if $\beta + p^{-1} \neq 2^{-1}$, it is surjective. The kernel of $\mathcal{V}$ is trivial for $0 < \beta + p^{-1} < 1/2$ and one-dimensional for $1/2 < \beta + p^{-1} < 1$. If Ω has an outward peak and $1/2 < \beta + p^{-1} < 1$, then

$$\ker \mathcal{V} = \left\{ \frac{t}{\pi} \frac{\partial}{\partial n} \operatorname{Im} \frac{1}{\gamma^{(out)}} \, , \, t \operatorname{Im} \frac{1}{\gamma^{(out)}(\infty)} \right\},$$

where $t \in \mathbb{R}$ and $\gamma^{(out)}$ is the conformal mapping of Ω' onto $\mathbb{R}^2_+ = \{z : \operatorname{Im} z > 0\}$ subject to $\gamma^{(out)}(0) = 0$ and $\gamma^{(out)}(\infty) = i$.

If Ω has an inward peak and $1/2 < \beta + p^{-1} < 1$ then

$$\ker \mathcal{V} = \left\{ \frac{t}{\pi} \frac{\partial}{\partial n} \operatorname{Im} \frac{1}{\gamma^{(in)}}, 0 \right\},$$

where $t \in \mathbb{R}$ and $\gamma^{(in)}$ is the conformal mapping of Ω onto $\mathbb{R}^2_+$ subject to the conditions $\gamma^{(in)}(0) = 0$ and $\gamma^{(in)}(z_0) = i$ with a fixed point $z_0 \in \Omega$.

Theorem B (see [2]). *Let Ω have an inward peak and let $0 < \beta + p^{-1} < \min\{\mu, 1\}$. Then the operator*

$$\mathcal{L}^1_{p,\beta+1}(\Gamma) \ni \sigma \xrightarrow{\mathcal{W}_{in}} (\pi I + W^{ext})\sigma \in \mathfrak{N}_{p,\beta}(\Gamma)$$

is continuous and, provided $\beta + p^{-1} \neq 2^{-1}$, it is surjective. The operator $\mathcal{W}$ has a trivial kernel for $0 < \beta + 1/p < 1/2$ and a one-dimensional kernel for $1/2 < \beta + 1/p < 1$. In the latter case

$$\operatorname{Ker} \mathcal{W} = \left\{ t \operatorname{Re} \frac{1}{\gamma_0} \right\},$$

where $t \in \mathbb{R}$ and γ_0 is the conformal mapping of Ω onto $\mathbb{R}^2_+$ subject to

$$\gamma_0(0) = 0, \quad \int_\Gamma \operatorname{Re} \frac{1}{\gamma_0} ds = 0 \quad \text{and} \quad \operatorname{Im} \gamma_0(z_0) = 1\,,$$

with a fixed point z_0 in Ω.

We introduce the functions $\mathcal{I}_k^{ext}$, $k = 1, 2, \ldots$, by setting

$$\mathcal{I}_k^{ext}(z) = \operatorname{Im}\left(\frac{zz_0}{z_0 - z}\right)^{k-1/2}, \quad z \in \Omega',$$

where z_0 is a fixed point in Ω.

Theorem C (see [3]). *Let Ω have an outward peak and let $0 < \beta + p^{-1} < \min\{\mu, 1\}$ and $m = \left[\mu - \beta - p^{-1} + 2^{-1}\right]$. Then the operator*

$$\mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m \ni (\sigma, t) \xmapsto{\mathcal{W}_{out}} \pi\sigma + W^{ext}\sigma + \sum_{k=1}^{m} t^{(k)}\mathcal{I}_k^{ext} \in \mathfrak{M}_{p,\beta}(\Gamma)$$

is continuous and injective. In case $\mu - \beta - p^{-1} + 2^{-1} \notin N$ and $\beta + p^{-1} \neq 2^{-1}$ the operator $\mathcal{W}_\mu$ is bijective.

In the proof of the surjectivity of $\mathcal{W}_{out}$ we use the next proposition.

Proporision 1. *Let Ω have an inward peak and let $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$, where*

$$0 < \beta + p^{-1} < \min\{\mu, 1\}, \quad \beta + p^{-1} \neq 2^{-1} \ \textit{ and } \ \mu - \beta - p^{-1} + 2^{-1} \notin \mathbb{N}.$$

Then there exists a harmonic extension of φ onto Ω' with the normal derivative in the space $\mathcal{L}_{p,\beta+1}(\Gamma)$ such that the conjugate function g with $g(z_0) = 0$ for a fixed point $z_0 \in \Gamma \setminus \{O\}$ can be written as

$$\sum_{k=1}^{m} c_k(\varphi) \operatorname{Re} z^{k-1/2} + g^{\#}(z),$$

where $c_k(\varphi)$ are linear continuous functionals in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $g^{\#}$ satisfies

$$\|g^{\#}\|_{\mathfrak{N}^{(-)}_{p,\beta}(\Gamma)} \leq c\, \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}.$$

with a constant c independent of φ.

The proof of Proposition 1 in [3] was not complete. In particular, the condition $\beta + p^{-1} \neq 2^{-1}$ was omitted. In Appendix we give a complete proof of this proposition.

2. The Dirichlet and Neumann problems for domains with peaks

2.1. Let the operator T be defined by

$$Tf(x) = \int_{\mathbb{R}} K(x, y) f(y) dy,$$

where

$$|K(x, y)| \leq c \frac{1}{|x - y|} \frac{1}{(1 + |x - y|^J)}, \quad J \geq 0.$$

Here and elsewhere by c we denote different positive constants.

We introduce the space $L_{p,\alpha}(\mathbb{R})$ of functions on $\mathbb{R}$ with the norm

$$\|\varphi\|_{L_{p,\alpha}(\mathbb{R})} = \|(1 + x^2)^{\alpha/2}\varphi\|_{L_p(\mathbb{R})}.$$

The following lemma can be proved in the same way as Stein's theorem on the boundedness of a singular integral operator in a weighted L_p-space [8].

Lemma 1. *If $T : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, is bounded and $-J < \alpha + p^{-1} < J + 1$, then T is a continuous operator in $L_{p,\alpha}(\mathbb{R})$.*

2.2. Let $G \subset \mathbb{R}^2$ be a domain with C^2-boundary such that the set $\{(\tau,\nu) \in G : \tau \le 0\}$ has compact closure and $\{(\tau,\nu) \in G : \tau > 0\} = \{(\tau,\nu) : \tau > 0, |\nu| < 1\}$.

As usual, by $C_0^\infty(G)$ we mean the space of infinitely differentiable functions with compact supports in G. By $W_p^k(G)$, $k = 0,1,2$, $p \in (1,\infty)$, we denote the Sobolev space of functions in $L_p(G)$ with derivatives up to order k in $L_p(G)$. The notation $\mathring{W}_p^k(G)$ stands for the completion of $C_0^\infty(G)$ in $W_p^k(G)$. Let $W_p^{k-1/p}(\partial G)$ be the space of traces on ∂G of functions in $W_p^k(G)$. We introduce also the space $W_p^{-1}(G)$ of distributions on G with the finite norm

$$\| \varphi \|_{W_p^{-1}(G)} = \inf \sum_{j=0}^{2} \| \varphi_j \|_{L_p(G)},$$

where the infimum is taken over all representations $\varphi = \varphi_0 + (\partial/\partial\tau)\varphi_1 + (\partial/\partial\nu)\varphi_2$ with $\varphi_j \in L_p(G)$, $j = 0,1,2$.

Let $\alpha \in \mathbb{R}$. We say that $\varphi \in W_{p,\alpha}^k(G)$ if $(1+\tau^2)^{\alpha/2}\varphi \in W_p^k(G)$, $k = -1,0,1\ldots$, and define the norm

$$\| \varphi \|_{W_{p,\alpha}^k(G)} = \| (1+\tau^2)^{\alpha/2}\varphi \|_{W_p^k(G)} .$$

The spaces $W_{p,\alpha}^k(\partial G)$ and $W_{p,\alpha}^{k-1/p}(\partial G)$ are introduced in the same way. By $L_{p,\alpha}(\partial G)$ we denote the space of functions with the finite norm

$$\| \varphi \|_{L_{p,\alpha}(\partial G)} = \| (1+\tau^2)^{\alpha/2}\varphi \|_{L_p(\partial G)} .$$

We shall make use of the same definitions for the strip $\Pi = \{(\tau,\nu) : \tau \in \mathbb{R}, |\nu| < 1\}$.

The following lemma is contained in more general results of [9].

Lemma 2. *The operator*

$$W_{p,\alpha}^k(G) \ni u \to \{\Delta u, u|_{\partial G}\} \in W_{p,\alpha}^{k-2}(G) \times W_{p,\alpha}^{k-1/p}(\partial G)$$

performs an isomorphism for every real α and $k = 1,2$. The same is true for $k = 1,2$ if G is replaced by the strip Π.

2.3. In this section we consider the Dirichlet problem in Ω with an outward peak. The following proposition is an improvement of Proposition 1 from [1].

Proporision 2. *Let φ belong to space $\mathfrak{N}_{p,\beta}(\Gamma)$, where $0 < \beta + p^{-1} < \min\{\mu, 1\}$. Then there exists a harmonic extension u of φ onto Ω with the normal derivative $\partial u/\partial n$ in $\mathfrak{N}_{p,\beta}^{-1}(\Gamma)$ satisfying*

$$\| \partial u/\partial n \|_{\mathfrak{N}_{p,\beta}^{-1}(\Gamma)} \le c \, \| \varphi \|_{\mathfrak{N}_{p,\beta}(\Gamma)} . \qquad (8)$$

Proof. It is sufficient to obtain (8) under the assumption $\varphi = 0$ in a neighborhood of the peak.

We start with the case where $\varphi \in \mathfrak{N}_{p,\beta}(\Gamma)$ vanishes on $\Gamma \cap \{|q| < \delta/2\}$. Let θ be a conformal mapping of the unit disk D onto Ω. We introduce the harmonic extension F of the continuous function $\varphi \circ \theta$ onto D. The normal derivative $\partial F/\partial n$ on ∂D has the form

$$\frac{\partial F}{\partial n}(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d}{dt} \varphi(\theta(e^{it})) \operatorname{cotan} \frac{s-t}{2}\, dt,\ \zeta = e^{is}.$$

Hence $u = F \circ \theta^{-1}$ satisfies

$$\| \partial u/\partial n \|_{\mathcal{L}_{p,\gamma+1}(\Gamma)} \leq c_1 \ \| \partial F/\partial n \|_{L_p(\partial D)} \leq c_2 \ \| (\varphi \circ \theta)' \|_{L_p(\partial D)} \leq c_3 \ \| \varphi \|_{\mathfrak{N}_{p,\gamma}(\Gamma)}$$

for every real γ. Since $\int_\Gamma (\partial u/\partial n) ds = 0$, $\partial u/\partial n$ is represented in the form $(\partial/\partial s)v$, where the function v satisfies

$$\| v \|_{\mathcal{L}_{p,\beta-\mu}(\Gamma)} + \| v' \|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \leq c \, \| \varphi \|_{\mathfrak{N}_{p,\beta}(\Gamma)} .$$

It remains to prove (8) for $\varphi \in \mathfrak{N}_{p,\beta}(\Gamma)$ vanishing outside $\Gamma_+ \cup \Gamma_-$. Let G be the domain defined in 2.2.

By $z = \omega(\tau + i\nu)$ we denote a conformal mapping of G onto Ω such that $\omega(\infty) = 0$ and $\omega(\partial G \cap \{(x, y) : x > 1\}) \supset \Gamma_+ \cup \Gamma_-$. It is well-known and follows essentially from Warschawski's asymptotic formula for conformal mappings [10] that

$$\omega(\tau + i\nu) = c\,(\tau + i\nu)^{-1/\mu}(1 + o(1)), \quad |\omega'(\tau + i\nu)| \leq c\,|\tau + i\nu|^{-1-1/\mu} \tag{9}$$

as $\tau \to +\infty$, $\tau + i\nu \in G$, and that

$$\omega^{-1}(z) = c\,z^{-\mu}(1 + o(1)), \quad |\big(\omega^{-1}\big)'(z)| \leq c\,|z|^{-1-\mu} \tag{10}$$

as $x \to 0$, $z = x + iy \in \Omega$. We introduce the functions $\Phi_\pm$ on $\mathbb{R}$ by

$$\Phi_\pm(\tau) = \varphi(\omega(\tau \pm i)) \ \text{ for } \ \tau \geq 0 \ \text{ and } \ \Phi_\pm(\tau) = 0 \ \text{ for } \ \tau < 0.$$

The following estimates hold:

$$\begin{aligned} c_1\, \|\varphi\|_{\mathfrak{N}_{p,\beta}(\Gamma)} &\leq \| \Phi_+ - \Phi_- \|_{L_{p,1-\alpha}(\mathbb{R})} \\ &+ \sum_{\pm} \Big(\|\Phi_\pm\|_{L_{p,-\alpha}(\mathbb{R})} + \big\|\Phi'_\pm\big\|_{L_{p,1-\alpha}(\mathbb{R})} \Big) \leq c_2\, \|\varphi\|_{\mathfrak{N}_{p,\beta}(\Gamma)} \end{aligned} \tag{11}$$

with α defined by $\beta + p^{-1} = \mu(\alpha - p^{-1})$.

Let r be a measurable function on $(0, \infty)$ subject to $|r(\tau)| \leq 1$. We have

$$\begin{aligned} &\int_0^\infty |\Phi_\pm(\tau) - \Phi_\pm(\tau + r(\tau))|^p\, \tau^{(1-\alpha)p} d\tau \\ &\leq c_1 \int_0^\infty \Bigg(\int_{\tau-1}^{\tau+1} |\Phi'_\pm(\nu)| d\nu \Bigg)^p \tau^{(1-\alpha)p} d\tau \leq c_2 \int_0^\infty |\Phi'_\pm(\tau)|^p \tau^{(1-\alpha)p} d\tau. \end{aligned} \tag{12}$$

For z, $z_1 \in \Omega$ with $|z_1 - z| \le x^{\mu+1}$ the distance between $\omega^{-1}(z_1)$ and $\omega^{-1}(z)$ is bounded by (10). Hence and by (12) we obtain the left inequality in (11).

Let h be a measurable function on $[0, \delta]$ such that $|h(x)| \le x^{\mu+1}$. By the theorem on the boundedness of the Hardy–Littlewood maximal operator in weighted L^p-spaces (see [11]) we have

$$\int_0^\delta |\varphi(x) - \varphi(x + h(x))|^p x^{(\beta-\mu)p} dx$$

$$\le c \int_0^\delta \left(\frac{1}{x^{\mu+1}} \int_{x-x^{\mu+1}}^{x+x^{\mu+1}} |t\, \varphi'(t)| dt \right)^p x^{\beta p} dx \le c \int_0^\delta |\varphi'(t)|^p t^{(\beta+1)p} dt. \tag{13}$$

By using (9) we obtain that for $\zeta, \zeta_1 \in G$ with $|\zeta - \zeta_1| < 1$ the distance between $\omega(\zeta)$ and $\omega(\zeta_1)$ does not exceed $c\, x^{\mu+1}$. Hence and by (13) the right inequality in (11) follows.

Let Π denote the same strip as in 2.2. We introduce the bounded harmonic function $\Phi^{(+)}$ on Π taking the same value $(\Phi_+(\tau) + \Phi_-(\tau))/2$ at the points $(\tau, 1)$ and $(\tau, -1)$ of $\partial\Pi$. The Fourier transform of $\Phi^{(+)}(\tau, \nu)$ with respect to τ is given by

$$c \left(\widehat{\Phi_+}(\xi) + \widehat{\Phi_-}(\xi) \right) \cosh{(\nu\xi)} \left(\cosh\, \xi\right)^{-1},$$

where $\widehat{\Phi_\pm}$ denote the Fourier transform of $\Phi_\pm$. Therefore the Fourier transform of $(\partial/\partial n)\Phi^{(+)}(\tau, \pm1)$ is equal to

$$c\, i \left(\widehat{\Phi'_+}(\xi) + \widehat{\Phi'_-}(\xi) \right) \tanh\, \xi\, .$$

Hence

$$\frac{\partial \Phi^{(+)}}{\partial n}(\tau, \pm 1) = c \int_{\mathbb{R}} \frac{d}{dt} \left(\Phi_+(t) + \Phi_-(t)\right) \left(\sinh\left(\frac{\pi}{2}(\tau - t) \right) \right)^{-1} dt\, .$$

By Lemma 1,

$$\| \, \partial\Phi^{(+)}/\partial n \, \|_{L_{p,1-\alpha}(\partial\Pi)} \le c \, \| \, (\Phi_+ + \Phi_-)' \, \|_{L_{p,1-\alpha}(\partial\Pi)} \, .$$

We rewrite the Fourier transform of $(\partial/\partial n)\Phi^{(+)}(\tau, \nu)$ with respect to τ in the form

$$c\, i\xi \left(\widehat{\Phi'_+}(\xi) + \widehat{\Phi'_-}(\xi) \right) \frac{\sinh\, \xi}{\xi\, \cosh\, \xi}\, .$$

Clearly, the function $(\partial/\partial n)\Phi^{(+)}(\tau, \pm1)$ can be represented as $(\partial/\partial s)Y^{(-)}(\tau, \pm1)$, where $Y^{(-)}(\tau, +1) = -Y^{(-)}(\tau, -1)$. By a theorem on the Fourier multipliers in weighted L_p-spaces (see [12]) it follows that

$$\| \, Y^{(-)} \, \|_{L_{p,1-\alpha}(\partial\Pi)} + \| \, (\partial/\partial\tau) Y^{(-)} \, \|_{L_{p,1-\alpha}(\partial\Pi)} \le c \, \| \, (\partial/\partial\tau)(\Phi_+ + \Phi_-) \, \|_{L_{p,1-\alpha}(\partial\Pi)} \, .$$

Let $\Phi^{(-)}$ be the bounded harmonic function on Π taking the opposite values $(\Phi_+(\tau) - \Phi_-(\tau))/2$ and $(\Phi_-(\tau) - \Phi_+(\tau))/2$ at the points $(\tau, 1)$ and $(\tau, -1)$ of

$\partial\Pi$. The Fourier transform of $\Phi^{(-)}(\tau,\nu)$ with respect to τ has the form

$$c\left(\widehat{\Phi_+}(\xi)-\widehat{\Phi_-}(\xi)\right)\sinh(\nu\xi)\,(\sinh\,\xi)^{-1}\,.$$

Therefore the Fourier transform of $(\partial/\partial n)\Phi^{(-)}(\tau,\pm 1)$ is equal to

$$\pm c\,\xi\left(\widehat{\Phi_+}(\xi)-\widehat{\Phi_-}(\xi)\right)\cosh\xi\,(\sinh\,\xi)^{-1}$$
$$=\pm c\Big\{\left(\widehat{\Phi_+}(\xi)-\widehat{\Phi_-}(\xi)\right)\frac{\xi}{\sinh\,\xi}+i\left(\widehat{\frac{d}{dt}\Phi_+}(\xi)-\widehat{\frac{d}{dt}\Phi_-}(\xi)\right)\tanh\,\xi\Big\}.$$

and hence

$$\frac{\partial\Phi^{(-)}}{\partial n}(\tau,\pm 1)=\pm c_1\int\limits_{\mathbb{R}}(\Phi_+(t)-\Phi_-(t))\left(\cosh\frac{\pi}{2}(\tau-t)\right)^{-2}dt$$
$$\pm c_2\int\limits_{\mathbb{R}}\left(\frac{d\Phi_+}{dt}(t)-\frac{d\Phi_-}{dt}(t)\right)(\sinh\,\pi(\tau-t))^{-1}\,dt\,.$$

By Lemma 1,

$$\|\,\partial\Phi^{(-)}/\partial n\,\|_{L_{p,1-\alpha}(\partial\Pi)}\le c\,\|\,\Phi_+-\Phi_-\,\|_{W^1_{p,1-\alpha}(\partial\Pi)}\,.$$

It is clear that $(\partial/\partial n)\Phi^{(-)}$ can be written as $(\partial/\partial s)Y^{(+)}(\tau,\pm 1)$, where $Y^{(+)}(\tau,+1)=Y^{(+)}(\tau,-1)$ and according to the theorem on the Fourier multipliers in weighted L_p-spaces (see [12]) the following estimate is valid:

$$\|\,Y^{(+)}\,\|_{L_{p,-\alpha}(\partial\Pi)}+\|\,(\partial/\partial\tau)Y^{(+)}\,\|_{L_{p,1-\alpha}(\partial\Pi)}\le c\,\|\,(\Phi_+-\Phi_-)\,\|_{W^1_{p,1-\alpha}(\partial\Pi)}\,.$$

Let $\chi\in C^\infty(\mathbb{R})$ be equal to 1 for $t>1$ and vanish for $t<0$, and let $\Psi=\Delta(\chi(\Phi^{(-)}+\Phi^{(+)}))$. Using Lemma 2 we have

$$\|\,\Psi\,\|_{L_p(\Pi)}\le c\Big(\|\,\Phi^{(-)}\,\|_{W^{1-1/p}_{p,-\alpha}(\partial\Pi)}+\|\,\Phi^{(+)}\,\|_{W^{1-1/p}_{p,-\alpha}(\partial\Pi)}\Big).$$

By the inclusion $W^1_p(\partial\Pi)\subset W^{1-1/p}_p(\partial\Pi)$ the right-hand side has the majorant

$$c\left(\|\,\Phi^{(-)}\,\|_{W^1_{p,1-\alpha}(\partial\Pi)}+\|\,\Phi^{(+)}\,\|_{L_{p,-\alpha}(\partial\Pi)}+\|\,d\Phi^{(+)}/dt\,\|_{L_{p,1-\alpha}(\partial\Pi)}\right).$$

Applying Lemma 2 with $k=2$, we obtain that the Dirichlet problem

$$\Delta Z=-\Psi\ \text{ in }\ G,\quad Z=0\ \text{ on }\ \partial G$$

has a solution satisfying

$$\|\,\partial Z/\partial n\,\|_{L_{p,\gamma}(\partial G)}\le c_1\,\|\,Z\,\|_{W^2_{p,\gamma}(G)}\le c_2\,\|\,\Psi\,\|_{L_p(G)}$$

for every real γ. Since $\int_{\partial G}(\partial Z/\partial n)ds=0$, the function $\partial Z/\partial n$ can be represented in the form $\partial X/\partial s$, where X satisfies

$$\|\,X\,\|_{\mathcal{L}_{p,1-\alpha}(\Gamma)}+\|\,X'\,\|_{\mathcal{L}_{p,1-\alpha}(\Gamma)}\le c\,\|\,\Phi_+-\Phi_-\,\|_{L_{p,1-\alpha}(\mathbb{R})}$$
$$+\sum_{\pm}\left(\|\,\Phi_\pm\,\|_{L_{p,-\alpha}(\mathbb{R})}+\|\,\Phi'_\pm\,\|_{L_{p,1-\alpha}(\mathbb{R})}\right)\,.$$

We set $F = Z + \chi(\Phi^{(-)} + \Phi^{(+)})$. By (12) the function $u = F \circ \omega^{-1}$ is the required harmonic extension of φ onto Ω. □

Since $1 \in \mathfrak{N}_{p,\beta}(\Gamma)$, the next proposition follows from the Cauchy–Riemann conditions.

Proporision 3. *Let ψ belong to $\mathfrak{N}^{-1}_{p,\beta}(\Gamma)$, where $0 < \beta + p^{-1} < 1$. Then the Neumann problem in Ω with boundary data ψ has a solution v in $\mathfrak{N}_{p,\beta}(\Gamma)$ satisfying*

$$\| v \|_{\mathfrak{N}_{p,\beta}(\Gamma)} \leq c \| \psi \|_{\mathfrak{N}^{-1}_{p,\beta}(\Gamma)} .$$

3. An Integral Equation of the Neumann Problem

Theorem 1. *Let Ω have an outward peak, and let $0 < \beta + p^{-1} < \min\{\mu, 1\}$. Then, for any $\psi \in \mathfrak{N}^{-1}_{p,\beta}(\Gamma)$, the integral equation*

$$\pi\sigma + W^{ext}\sigma = V\psi \tag{14}$$

has a unique solution σ in $\mathfrak{N}_{p,\beta}(\Gamma)$, satisfying $\int_\Gamma \sigma\, ds = 0$.

Proof. Let ψ belong to $C_0^\infty(\Gamma \setminus \{O\})$. By h we denote a solution of the Neumann problem in Ω with boundary data ψ as in Proposition 3. From the integral representation of the harmonic function h in Ω and the limit relation for the double layer potential we obtain

$$h(z) + \frac{1}{\pi}\int_\Gamma h(q)\frac{\partial}{\partial n_q}\log\frac{1}{|z-q|}ds_q = \frac{1}{\pi}\int_\Gamma \log\frac{|z|}{|z-q|}\psi(q)ds_q \tag{15}$$

We choose h so that $\int_\Gamma h(q)ds_q = 0$. According to Proposition 3, h belongs to $\mathfrak{N}_{p,\beta}(\Gamma)$. From (15) it follows that $\sigma = h \in \mathfrak{N}_{p,\beta}(\Gamma)$ is a solution of (14).

Now let ψ be an arbitrary function in $\mathfrak{N}^{-1}_{p,\beta}(\Gamma)$. For $\beta > -p^{-1}$ there exists a sequence $\{\psi_n\}_{n\geq 1}$ of smooth functions on $\Gamma \setminus \{O\}$ vanishing near the peak and approaching ψ in $\mathfrak{N}^{-1}_{p,\beta}(\Gamma)$. By σ_n we denote the constructed solution of (14) with ψ_n on the right-hand side which is unique by Theorem C. Since the operator

$$\mathcal{L}_{p,\beta+1}(\Gamma) \ni \psi \longmapsto V\psi \in \mathfrak{N}_{p,\beta}(\Gamma)$$

is continuous (see Theorem A), it follows that $\{V\psi_n\}$ converges in $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ to the limit $V\psi$. According to Proposition 3, the sequence $\{\sigma_n\}$ converges in $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ to the limit σ. Since the operator

$$\mathcal{L}^1_{p,\beta+1}(\Gamma) \ni \sigma \xmapsto{\mathcal{W}} (\pi I + W^{ext})\sigma \in \mathfrak{M}_{p,\beta}(\Gamma) \subset \mathcal{L}^1_{p,\beta+1}(\Gamma)$$

is continuous (see Theorem C), we obtain that σ is a solution of (14) by passing to the limit.

The kernel of $\mathcal{W}$ in $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ is trivial. Therefore equation (14) is uniquely solvable in $\mathfrak{N}_{p,\beta}(\Gamma)$. □

Remark 1. Under the assumptions of Theorem 1 and provided $\mu - \beta - p^{-1} + 2^{-1} \notin N$, $\beta + p^{-1} \neq 2^{-1}$, $\psi \in \mathfrak{N}^{-1}_{p,\beta}(\Gamma)$ we conclude that $V\psi$ belongs to $\mathfrak{N}_{p,\beta}(\Gamma) \cap \mathfrak{M}_{p,\beta}(\Gamma)$ and satisfies the "orthogonality" conditions

$$\int_\Gamma \frac{\partial}{\partial s}(V\psi)\,\mathrm{Re}\,\frac{1}{\zeta^{2k-1}}\,ds = 0, \quad k = 1, \dots, m\,,$$

where ζ is the conformal mapping of Ω' onto $\mathbb{R}^2_+$ subject to

$$\zeta(0) = 0, \quad \mathrm{Re}\,\zeta(\infty) = 0 \quad \text{and} \quad \mathrm{Re}\,(1/\zeta(z)) = \pm x^{-1/2} + O(1).$$

Proof. By Theorem 1 we have $V\psi \in \mathfrak{N}_{p,\beta}(\Gamma)$, since $\psi \in \mathcal{L}_{p,\beta+1}(\Gamma)$, and $V\psi \in \mathfrak{M}_{p,\beta}(\Gamma)$. Therefore $V\psi = \varphi + \sum_{k=0}^m d^{(k)}\mathrm{Re}\,z^k$, where $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$. According to Proposition 1, there exists a harmonic extension of φ onto Ω' such that the conjugate function g satisfying $g(\infty) = 0$ has the representation

$$\sum_{k=1}^m c^{(k)}(\varphi)\mathrm{Re}\,z^{k-1/2} + g^{\#}(z),$$

where $c^{(k)}(\varphi)$ are linear continuous functionals in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and $g^{\#} \in \mathfrak{N}_{p,\beta}(\Gamma)$. We apply the Green formula to the functions g and $\mathrm{Re}\,\zeta^{1-2k}$ in $\Omega' \cap \{|z| < \varepsilon\}$. Passing to the limit as $\varepsilon \to 0$, we obtain

$$c^{(k)}(\varphi) = c_k \int_\Gamma \frac{\partial g}{\partial n}\,\mathrm{Re}\,\frac{1}{\zeta^{2k-1}}\,ds = c_k \int_\Gamma \varphi'_s \mathrm{Re}\,\frac{1}{\zeta^{2k-1}}\,ds.$$

Let $(\sigma, t) \in \mathcal{L}^1_{p,\beta+1}(\Gamma) \times \mathbb{R}^m$ be the solution of the equation

$$(\pi I + W^{ext})\sigma + \sum_{k=1}^m t^{(k)}\mathcal{I}_k = V\psi$$

with the right-hand side from $\mathfrak{M}_{p,\beta}(\Gamma)$. As is shown in Theorem 2 [3], the components $t^{(k)}$, are equal to $c^{(k)}(\varphi)$, $k = 1, \dots, m$. Since the operator $\mathcal{W}$ is surjective and since

$$\int_\Gamma \frac{\partial}{\partial n}\,\mathrm{Im}\,z^k\,\mathrm{Re}\,\frac{1}{\zeta^{2k-1}}\,ds = 0\,,$$

it follows that $V\psi \in \mathfrak{N}_{p,\beta}(\Gamma) \cap \mathfrak{M}_{p,\beta}(\Gamma)$ and $c^{(k)}(V\psi) = 0$, $k = 1, \dots, m$. □

Theorem 2. *Let Ω have an inward peak, and let $0 < \beta + p^{-1} < \min\{\mu, 1\}$ and $\beta + p^{-1} \neq 1/2$. Then, for any $\psi \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$, the boundary integral equation*

$$\pi\sigma + W^{ext}\sigma = V\psi' \tag{16}$$

has a solution $\sigma \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$ satisfying $\int_\Gamma \sigma\,ds = 0$. The homogeneous equation (16) has only a trivial solution for $0 < \beta + p^{-1} < 1/2$ and a one-dimensional space of solutions for $1/2 < \beta + p^{-1} < 1$, given by

$$\Big\{t\,\mathrm{Re}\frac{1}{\gamma_0}\Big\},$$

where $t \in \mathbb{R}$ and γ_0 is the conformal mapping of Ω onto $\mathbb{R}^2_+$ subject to

$$\gamma_0(0) = 0, \quad \int\limits_\Gamma \operatorname{Re} \frac{1}{\gamma_0}\, ds = 0 \quad \textit{and} \quad \operatorname{Im}\gamma_0(z_0) = 1$$

with a fixed point $z_0 \in \Omega$.

Proof. Let $\psi \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$. Then $V\psi' \in \mathfrak{N}_{p,\beta}(\Gamma)$ (see Theorem A). According to Theorem B, equation (16) is solvable in $\mathcal{L}^1_{p,\beta+1}(\Gamma)$. Since the harmonic extension of $V\psi'$ vanishes at infinity, we have $\int_\Gamma \sigma\, ds = 0$. By Theorem B, the set of solutions to the homogeneous equation (16) in $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ is one-dimensional for $1/2 < \beta + p^{-1} < 1$ and trivial for $0 < \beta + p^{-1} < 1/2$. The set of solutions to the homogeneous equation (16) is described in Theorem B. □

4. An Integral Equation of the Dirichlet Problem

Theorem 3. *Let Ω have an outward peak and let $0 < \beta + p^{-1} < \min\{\mu, 1\}$. Then the boundary integral equation*

$$V\gamma = \pi\varphi + W\varphi \tag{17}$$

has a solution $\gamma \in \mathfrak{N}^{-1}_{p,\beta}(\Gamma)$ for every $\varphi \in \mathfrak{N}_{p,\beta}(\Gamma)$. This solution is unique for $\beta + p^{-1} \neq 2^{-1}$.

Proof. Let $\varphi \in C_0^\infty(\Gamma \setminus \{O\})$. By u we denote the bounded harmonic extension of φ onto Ω constructed in Proposition 2. By the integral representation of the harmonic function u on Ω and by the limit relation for the simple layer potential we obtain for $z \in \Gamma \setminus \{O\}$

$$\int\limits_\Gamma \log \frac{|z|}{|z-q|} \frac{\partial u}{\partial n}(q)\, ds_q = \pi\varphi(z) + \int\limits_\Gamma \varphi(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|}\, ds_q\,. \tag{18}$$

Since $\partial u/\partial n$ belongs to $\mathfrak{N}^{-1}_{p,\beta}(\Gamma)$, from (18) it follows that $\gamma = \partial u/\partial n$ satisfies (17).

Now let φ be an arbitrary function in $\mathfrak{N}_{p,\beta}(\Gamma)$. There exists a sequence $\{\varphi_n\}_{n\geq 1}$ of smooth functions on $\Gamma \setminus \{O\}$ vanishing near the peak and converging to φ in $\mathfrak{N}_{p,\beta}(\Gamma)$. By γ_n we denote the constructed solution of (17) with φ_n instead of φ on the right-hand side. Since the operator $(\pi I + W) : \mathcal{L}^1_{p,\beta+1}(\Gamma) \to \mathfrak{M}_{p,\beta}(\Gamma)$ is continuous (see Theorem C), we obtain by taking the limit that $\{\pi\varphi_n + W\varphi_n\}_{n\geq 1}$ converges in $\mathcal{L}^1_{p,\beta+1}(\Gamma)$. In view of Proposition 2, the sequence $\{\partial u_n/\partial n\}$, where u_n is the bounded extension of φ_n onto Ω, converges in $\mathcal{L}_{p,\beta+1}(\Gamma)$. According to Theorem A,

$$\mathcal{L}_{p,\beta+1}(\Gamma) \ni \gamma \overset{\mathcal{V}}{\longmapsto} V\gamma \in \mathfrak{N}_{p,\beta}(\Gamma)$$

is continuous. Then by passing to the limit in the equation $V\gamma_n = \pi\varphi_n + W\varphi_n$ we obtain that γ is a solution of (17). For Ω with an outward peak Theorem 1 implies that $\operatorname{Ker} \mathcal{V}$ is trivial provided $\beta + p^{-1} \neq 2^{-1}$. Therefore the just obtained solution of (17) is unique. □

Theorem 4. *Let Ω have an inward peak, and let $0 < \beta + p^{-1} < \min\{\mu, 1\}$, $\beta + p^{-1} \neq 1/2$. Then the integral equation*

$$V\gamma = \pi\varphi + W\varphi \tag{19}$$

has a solution $\gamma \in \mathcal{L}_{p,\beta+1}(\Gamma)$ for every $\varphi \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$. This solution is unique for $0 < \beta+p^{-1} < 1/2$ and the homogeneous equation (19) has a one-dimensional space of solutions for $1/2 < \beta + p^{-1} < 1$ given by

$$\Big\{ t\, \frac{\partial}{\partial n} \operatorname{Im} \frac{1}{\gamma^{(in)}} \Big\},$$

where $t \in \mathbb{R}$ and $\gamma^{(in)}$ is the conformal mapping of Ω onto $\mathbb{R}^2_+$ subject to the conditions $\gamma^{(in)}(0) = 0$ and $\gamma^{(in)}(z_0) = i$ with a fixed point $z_0 \in \Omega$.

Proof. Let $\varphi \in \mathcal{L}^1_{p,\beta+1}(\Gamma)$. According to Theorem B, $(\pi I + W)\varphi$ belongs to $\mathfrak{N}_{p,\beta}(\Gamma)$ and its harmonic extension onto Ω'

$$\big(W\sigma\big)(z) = \int\limits_\Gamma \sigma(q) \frac{\partial}{\partial n_q} \log \frac{1}{|z-q|}\, ds_q, \quad z \in \Omega',$$

vanishes at infinity. The range of the operator

$$\mathcal{L}_{p,\beta+1}(\Gamma) \ni \gamma \longmapsto V\gamma \in \mathfrak{N}_{p,\beta}(\Gamma)$$

consists of the elements of $\mathfrak{N}_{p,\beta}(\Gamma)$ whose harmonic extensions to Ω' vanish at infinity (see Theorem A). Therefore equation (19) has a solution in $\mathcal{L}_{p,\beta+1}(\Gamma)$. The homogeneous equation (19) has only a trivial solution in $\mathcal{L}_{p,\beta+1}(\Gamma)$ for $0 < \beta + p^{-1} < 1/2$ and a one-dimensional space of solutions for $1/2 < \beta + p^{-1} < 1$. The set of solutions to the homogeneous equation (19) is described in Theorem A. $\square$

5. Appendix: Proof of Proposition 1

(i) We shall make use of the representation of a conformal mapping θ of $\mathbb{R}^2_+ = \{\zeta = \xi + i\eta : \ \eta > 0\}$ onto Ω

$$\theta(\xi) = \begin{cases} \sum\limits_{k=2}^{[2\mu]+1} B^{(k)}\xi^k + B^{([2\mu]+2)}\xi^{[2\mu]+2}\log|\xi| + B^{(\pm)}|\xi|^{2\mu+2} & \\ \qquad +O\big(\xi^{2\mu+2+\gamma}\big) & \text{if } \ 2\mu \in \mathbb{N}, \\ \sum\limits_{k=2}^{[2\mu]+2} B^{(k)}\xi^k + B^{(\pm)}|\xi|^{2\mu+2} + B^{([2\mu]+3)}\xi^{[2\mu]+3} & \\ \qquad +O\big(\xi^{2\mu+2+\gamma}\big) & \text{if } \ 2\mu \notin \mathbb{N}, \end{cases} \tag{20}$$

as $\xi \to \pm 0$, where $B^{(k)}$, $k = 2, \dots, [2\mu]+2$, are real coefficients and $0 < \gamma < \min(\mu, 1)$. Decomposition (20) can be differentiated at least once (see [3]).

By D we denote the image of Ω under the mapping $u+iv=(x+iy)^{1/2}$. Let $\widetilde{\theta}$ denote a conformal mapping of $\mathbb{R}^2_+$ onto D normalized by $\widetilde{\theta}(0)=0$. Then

$$\widetilde{\theta}(\xi)=\begin{cases}\displaystyle\sum_{k=1}^{[2\mu]} b^{(k)}\xi^k+b^{([2\mu]+1)}\xi^{[2\mu]+1}\log|\xi|+b^{(\pm)}|\xi|^{2\mu+1}+O\big(\xi^{2\mu+1+\gamma}\big) & \text{if } 2\mu\in\mathbb{N},\\ \displaystyle\sum_{k=1}^{[2\mu]+1} b^{(k)}\xi^k+b^{(\pm)}|\xi|^{2\mu+1}+b^{([2\mu]+2)}\xi^{[2\mu]+2}+O\big(\xi^{2\mu+1+\gamma}\big) & \text{if } 2\mu\notin\mathbb{N},\end{cases}$$

The inverse mapping $\theta^{-1}(z)$ restricted to $\Gamma_\pm$ has the form

$$\xi=\sum_{k=1}^{[2\mu]}(\pm1)^k\beta^{(k)}x^{k/2}+(\pm1)^{[2\mu]+1}\beta^{([2\mu]+1)}x^{([2\mu]+1)/2}\log\frac{1}{x}+\beta^{(\pm)}x^{\mu+1/2}+o\left(x^{\mu+1/2}\right)$$

if $2\mu\in\mathbb{N}$, and

$$\xi=\sum_{k=1}^{[2\mu]+1}(\pm1)^k\beta^{(k)}x^{k/2}+\beta^{(\pm)}x^{\mu+1/2}+o\left(x^{\mu+1/2}\right) \tag{21}$$

if $2\mu\notin\mathbb{N}$. Here $\beta^{(k)}$, $k=1,\dots[2\mu]+1$, are real coefficients. We notice that there exists a function of the form

$$d_0(\zeta)=\begin{cases}\displaystyle\zeta+\sum_{k=2}^{[2\mu]}a^{(k)}\zeta^k & \text{if } 2\mu\in\mathbb{N},\\ \displaystyle\zeta+\sum_{k=2}^{[2\mu]+1}a^{(k)}\zeta^k & \text{if } 2\mu\notin\mathbb{N},\end{cases}$$

defined on $\mathbb{R}^2_+$ and satisfying

$$(\widetilde{\theta}\circ d_0)(\zeta)=\zeta+\gamma(\zeta)\,.$$

Here $\gamma(\zeta)$ is the holomorphic function on a neighborhood of $\zeta=0$ in $\mathbb{R}^2_+$ taking real values in a neighborhood of $\xi=0$ in $\mathbb{R}$ and having the representation

$$\gamma(\xi)=\begin{cases}b^{[2\mu]+1}\xi^{[2\mu]+1}\log|\xi|+O\big(|\xi|^{2\mu+1}\big) & \text{if } 2\mu\in\mathbb{N},\\ O\big(|\xi|^{2\mu+1}\big) & \text{if } 2\mu\notin\mathbb{N}\,.\end{cases}$$

Then the function

$$d(\zeta)=\begin{cases}d_0(\zeta)+\gamma(\zeta) & \text{if } 2\mu\in\mathbb{N},\\ d_0(\zeta) & \text{if } 2\mu\notin\mathbb{N}\end{cases}$$

satisfies

$$\big(\widetilde{\theta}\circ d\big)(\xi)=\xi+O\big(|\xi|^{2\mu+1}\big)\,.$$

It is clear that $\theta_0=\big(\widetilde{\theta}\circ d\big)^2$ is a conformal mapping of a neighborhood of $\zeta=0$ in $\mathbb{R}^2_+$ onto a neighborhood of the peak in Ω and admitting the representation

$$x=\operatorname{Re}\theta_0(\xi)=\xi^2+O\big(|\xi|^{2\mu+2}\big)\quad\text{as}\quad \xi\to\pm0\,. \tag{22}$$

The inverse mapping θ_0^{-1} has the form

$$\xi = \operatorname{Re}\theta_0^{-1}(z) = \pm x^{1/2} + O\big(x^{\mu+1/2}\big) \quad \text{on} \quad \Gamma_\pm\,.$$

By diminishing δ in the definition of $\Gamma_\pm$, we can assume that θ_0 is defined on $\Gamma_+ \cup \Gamma_-$.

(ii) Now, let $\varphi \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ vanish outside $\Gamma_+ \cup \Gamma_-$. We extend the function $\Phi(\tau) = (\varphi \circ \theta_0)(\tau)$ by zero outside a small neighborhood of O.

We first prove the estimate

$$c_1 \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)} \le \|\Phi^{(+)}\|_{L_{p,2\beta-2\mu+1/p}(\mathbb{R})} + \|(d\Phi/d\tau)\|_{L_{p,2\beta+1+1/p}(\mathbb{R})} + \|\Phi\|_{L_{p,2\beta+1/p}(\mathbb{R})} \le c_2 \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}, \quad (23)$$

where $\Phi^{(+)}(\xi) = (\Phi(\xi) + \Phi(-\xi))/2$.

Let r be a measurable function on $(0,\infty)$ subject to $|r(\xi)| \le |\xi|^{2\mu+1}$. We choose $\ell \in [0,1]$ such that

$$\ell/2 < \beta + p^{-1} < (\ell+1)/2\,.$$

Then, from the boundedness of the Hardy–Littlewood maximal operator in a weighted L_p-space (see [11]), we obtain

$$\int_{\mathbb{R}} |\Phi(\xi) - \Phi(\xi + r(\xi))|^p |\xi|^{2\beta p - 2\mu p + 1} d\xi \le c \int_{\mathbb{R}} \left(\frac{1}{|\xi|^{2\mu+1}} \int_{\xi - c|\xi|^{2\mu+1}}^{\xi + c|\xi|^{2\mu+1}} \big|\tau^{1+\ell} (d\Phi/d\tau)(\tau)\big| d\tau \right)^p |\xi|^{2\beta p - \ell p + 1} d\xi \le c \int_{\mathbb{R}} |(d\Phi/d\tau)(\xi)|^p |\xi|^{2\beta p + p + 1} d\xi\,. \quad (24)$$

For $z \in \Gamma_+$ we have

$$|\theta_0^{-1}(z) + \theta_0^{-1}(z_-)| \le c\,|\xi|^{2\mu+1}\,.$$

Hence and by (24) the left inequality in (23) follows.

Let h be a measurable function on $[0,\delta]$ such that $|h(x)| \le x^{\mu+1}$. Similarly to (24) we have

$$\int_0^\delta |\varphi(x) - \varphi(x + h(x))|^p x^{(\beta-\mu)p} dx \le c \int_0^\delta \left(\frac{1}{x^{\mu+1}} \int_{x - cx^{\mu+1}}^{x + cx^{\mu+1}} t\,|(d\varphi/dt)(t)| dt \right)^p x^{\beta\, p} dx \le c \int_0^\delta |(d\varphi/dt)(t)|^p x^{(\beta+1)p} dx\,. \quad (25)$$

By using (22) we obtain that for ξ in a small neighborhood of the origin the distance between $\theta_0(\xi)$ and $\theta_0(-\xi)$ does not exceed $c\,x^{\mu+1}$. Hence and by (25) the right inequality in (23) follows.

We introduce a function $\mathcal{H}$ by

$$\mathcal{H}(\zeta) = \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{d\Phi}{d\tau}(\tau)\,\mathrm{Re}\,\log \frac{\zeta-\tau}{\zeta} d\tau, \quad \zeta = \xi + i\eta \in \mathbb{R}^2_+\,.$$

From the norm inequality for the Hilbert transform of even functions in the space $L_{p,2\beta-1+p^{-1}}(\mathbb{R})$ (see [13]) it follows that the function

$$(\partial/\partial\xi)\mathcal{H}^{(+)}(\xi) = \frac{1}{\pi\xi^2} \int\limits_{\mathbb{R}} \frac{d}{d\tau}\Phi^{(-)}(\tau)\frac{\tau^2 d\tau}{\xi-\tau}$$

satisfies

$$\|(\partial/\partial\xi)\mathcal{H}^{(+)}\|_{L_{p,2\beta+1+1/p}(\mathbb{R})} \le c\,\|(d/d\xi)\Phi^{(-)}\|_{L_{p,2\beta+1+1/p}(\mathbb{R})}\,, \tag{26}$$

where $\Phi^{(-)}(\xi) = (\Phi(\xi) - \Phi(-\xi))/2$ and $\mathcal{H}^{(+)}(\xi)$ denote the function $(\mathcal{H}(\xi) + \mathcal{H}(-\xi))/2$.

Using again the norm inequality for the Hilbert transform of even functions in the space $L_{p,2\beta+p^{-1}}(\mathbb{R})$ (see [13]), we obtain that the function

$$(\partial/\partial\xi)\mathcal{H}^{(-)}(\xi) = \frac{1}{\pi\xi} \int\limits_{\mathbb{R}} \frac{d}{d\tau}\Phi^{(+)}(\tau)\frac{\tau d\tau}{\xi-\tau}$$

satisfies

$$\|(\partial/\partial\xi)\mathcal{H}^{(-)}\|_{L_{p,2\beta+1+1/p}(\mathbb{R})} \le c\,\|(d/d\xi)\Phi^{(+)}\|_{L_{p,2\beta+1+1/p}(\mathbb{R})} \tag{27}$$

for $0 < \beta + \frac{1}{p} < \frac{1}{2}$. Here $\Phi^{(+)}(\xi) = (\Phi(\xi) + \Phi(-\xi))/2$ and $\mathcal{H}^{(-)}(\xi) = (\mathcal{H}(\xi) - \mathcal{H}(-\xi))/2$.

Let ν be an even C_0^∞-function on $\mathbb{R}$ with a small support, vanishing inside a neighborhood of $\xi = 0$ and subjected to $\int_{\mathbb{R}} \nu(u)du = 1$. We remark that $\int_{\mathbb{R}} \Phi^{(+)}(\tau)d\tau$ is a linear continuous functional on $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$. We introduce the function $\varphi_0 := \Phi^{(+)} \circ \theta^{-1} - (\nu \circ \theta^{-1}) \int_{\mathbb{R}} \Phi^{(+)}\tau d\tau$ and, for $\frac{1}{2} < \beta + \frac{1}{p} < 1$, we consider the function $\varphi - \varphi_0 \in \mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ for which we keep the same notation φ. Then the function $(\partial/\partial\xi)\mathcal{H}^{(-)}$ can be represented in the form

$$(\partial/\partial\xi)\mathcal{H}^{(-)}(\xi) = \frac{1}{\pi\xi^2} \int\limits_{\mathbb{R}} \frac{d}{d\tau}\Phi^{(+)}(\tau)\frac{\tau^2 d\tau}{\xi-\tau}\,.$$

From the norm inequality for the Hilbert transform of odd functions in $L_{p,2\beta-1+p^{-1}}(\mathbb{R})$ (see [13]) it follows that

$$\|(\partial/\partial\xi)\mathcal{H}^{(-)}\|_{L_{p,2\beta+1+1/p}(\mathbb{R})} \le c\,\|(d/d\xi)\Phi^{(+)}\|_{L_{p,2\beta+1+1/p}(\mathbb{R})}\,. \tag{28}$$

Let n_0 be the integer subject to the inequalities

$$n_0 - 1 \le 2(\mu - \beta - p^{-1}) < n_0\,.$$

Then $m = [\mu - \beta - p^{-1} + 2^{-1}]$ is the largest integer satisfying $2m \le n_0$.

We represent the odd function $\mathcal{H}^{(-)}$ on $\mathbb{R}$ in the form

$$\mathcal{H}^{(-)}(\xi) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d}{d\tau} \Phi^{(+)}(\tau) \log\Big| \frac{\xi - \tau}{\xi} \Big| d\tau$$

$$= \frac{\xi^{n_0}}{\pi} \int_{\mathbb{R}} \frac{\Phi^{(+)}(\tau)}{\tau^{n_0}(\xi - \tau)} d\tau - \sum_{k=0}^{n_0 - 1} \frac{\xi^k}{\pi} \int_{\mathbb{R}} \frac{\Phi^{(+)}(\tau)}{\tau^{k+1}} d\tau \, .$$

Since $\Phi^{(+)} \in L_{p,2\beta-2\mu+1/p}(\mathbb{R})$ and since

$$0 < 2\beta - 2\mu + n_0 + 2p^{-1} < 1 \;\text{ for even } n_0$$

and

$$0 < 2\beta - 2\mu + n_0 + 2p^{-1} < 2 \;\text{ for odd } n_0,$$

it follows from the boundedness of the Hilbert transform in weighted L_p-spaces (see [13]) that $L_{p,2\beta-2\mu+1/p}(\mathbb{R})$–norm of

$$\frac{\xi^{n_0}}{\pi} \int_{\mathbb{R}} \frac{\Phi^{(+)}(\tau)}{\tau^{n_0}(\xi - \tau)} d\tau$$

does not exceed $\|\Phi^{(+)}\|_{L_{p,2\beta-2\mu+1/p}(\mathbb{R})}$ provided $\mu - \beta - p^{-1} + 2^{-1} \notin N$.

Hence by (26), (27) and (28) we obtain that the function $h(z) = \mathcal{H} \circ \theta_0^{-1}(z)$ can be represented in the form

$$\sum_{k=1}^{m} a_k(\varphi) \mathrm{Re} z^{k-1/2} + h^{\#}(z) \tag{29}$$

for $z \in \Omega$ lying in a small neighborhood of the peak. Here

$$a_k(\varphi) = \int_{\mathbb{R}} \Phi^{(+)}(\tau) \tau^{-2k} d\tau \, , \;\; 1 \le k \le m \, ,$$

are linear continuous functionals in $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$, and $h^{\#}$ belongs to $\mathfrak{N}_{p,\beta}(\Gamma)$ and satisfies

$$\|h^{\#}\|_{\mathfrak{N}_{p,\beta}(\Gamma_+ \cup \Gamma_-)} \le c \, \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}.$$

Now let $æ \in C^\infty(\mathbb{R}^2)$ be equal to 1 for $|z| < \delta$ and vanish for $|z| > \delta$. We extend $æh$ by zero outside a small neighborhood of O and set

$$\psi_1(z) = -\Delta(æh)(z), \;\; z \in \Omega',$$

$$\varphi_1(z) = (\partial/\partial s)\varphi(z) - (\partial/\partial n)(æh)(z), \;\; z \in \Gamma.$$

Since $(\partial/\partial\eta)\mathcal{H} = (\partial/\partial\xi)\Phi \in L_{p,\beta+1}(\mathbb{R})$, it follows that $\partial h/\partial n$ belongs to $\mathcal{L}_{p,\beta+1}(\Gamma_+ \cup \Gamma_-)$ We consider the boundary value problem

$$\Delta\mathcal{F}(\zeta) = \mathcal{Q}(\zeta), \;\; \zeta \in \mathbb{R}^2_+, \;\; (\partial/\partial n)\mathcal{F}(\xi + i0) = \mathcal{T}(\xi), \;\; \xi \in \mathbb{R}, \tag{30}$$

where $\mathcal{Q}(\zeta) = (\psi_1 \circ \theta)(\zeta)\,|\theta'(\zeta)|^2$ and $\mathcal{T}(\xi) = (\varphi_1 \circ \theta)(\xi + i0)\,|\theta'(\xi + i0)|$. For $0 < \beta + 1/p < 1/2$, by using the estimates

$$|\mathrm{grad}\,\mathcal{H}(\zeta)| \le \frac{c}{|\zeta|}\Big|\int\limits_{\mathbb{R}} \frac{\tau\,(d\Phi/d\tau)(\tau)}{\zeta - \tau}d\tau\Big|, \quad |\mathcal{H}(\zeta)| \le \Big|\int\limits_{\mathbb{R}} \frac{\Phi(\tau)}{\zeta - \tau}d\tau\Big|, \tag{31}$$

and the theorems on the boundedness of the Hardy–Littlewood maximal operator and the Hilbert transform in weighted L_p-spaces [11], [14], we obtain

$$\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)} \le c\left(\|d\Phi/d\tau\|_{L_{p,2\beta+1+1/p}(\mathbb{R})} + \|\Phi\|_{L_{p,2\beta+1/p}(\mathbb{R})}\right). \tag{32}$$

In order to prove (32) for $1/2 < \beta + 1/p < 1$, we use (31) for $\Phi = \Phi^{(-)}$ and the estimates

$$|\mathrm{grad}\,\mathcal{H}^{(-)}(\zeta)| \le \frac{c}{|\zeta|^2}\Big|\int\limits_{\mathbb{R}} \frac{\tau^2\,(d\Phi^{(+)}/d\tau)(\tau)}{\zeta - \tau}d\tau\Big|, \ \ |\mathcal{H}^{(-)}(\zeta)| \le \frac{1}{|\zeta|}\Big|\int\limits_{\mathbb{R}} \frac{\tau\,\Phi^{(+)}(\tau)}{\zeta - \tau}d\tau\Big|.$$

A solution of (30) is given by

$$\mathcal{F}(\zeta) = \int\limits_{\mathbb{R}} \mathcal{T}(u)\mathfrak{G}(u,\zeta)\,du - \int\limits_{\mathbb{R}^2_+} \mathcal{Q}(w)\mathfrak{G}(w,\zeta)\,du\,dv, \quad w = u + iv,$$

with the Green function

$$\mathfrak{G}(w,\zeta) = \frac{1}{2\pi}\log\left|\left(1 - \frac{w}{\zeta}\right)\left(1 - \frac{w}{\overline{\zeta}}\right)\right|.$$

We rewrite $\mathcal{F}$ on $\mathbb{R}$ in the form

$$\begin{gathered}\mathcal{F}(\xi) = t_{-1}(\varphi)\log|\xi| + t_0(\varphi) \\ +\frac{1}{\pi}\int\limits_{\mathbb{R}} \mathcal{T}(u)\log\Big|1 - \frac{\xi}{u}\Big|du - \frac{1}{\pi}\int\limits_{\mathbb{R}^2_+} \mathcal{Q}(w)\log\Big|1 - \frac{\xi}{w}\Big|dudv\,,\end{gathered} \tag{33}$$

where

$$t_{-1}(\varphi) = -\frac{1}{\pi}\int\limits_{\mathbb{R}} \mathcal{T}(u)du + \frac{1}{\pi}\int\limits_{\mathbb{R}^2_+} \mathcal{Q}(w)dudv,$$

and

$$t_0(\varphi) = \frac{1}{\pi}\int\limits_{\mathbb{R}} \mathcal{T}(u)\log|u|du - \frac{1}{\pi}\int\limits_{\mathbb{R}^2_+} \mathcal{Q}(w)\log|w|dudv\,.$$

Hence

$$\frac{\partial\mathcal{F}}{\partial\xi}(\xi) - \frac{t_{-1}(\varphi)}{\xi} = \frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{\mathcal{T}(u)}{\xi - u}du - \frac{1}{\pi}\int\limits_{\mathbb{R}^2_+} \frac{(\xi - u)\mathcal{Q}(w)}{|\xi - w|^2}dudv\,.$$

By the boundedness of the Hilbert transform in $L_p(\mathbb{R})$ and the Minkowski inequality we find that

$$(\partial\mathcal{F}/\partial\xi)(\xi) - t_{-1}(\varphi)\,\xi^{-1} \in L_p(\mathbb{R}) \tag{34}$$

and that the L_p-norm of this function does not exceed

$$\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}.$$

It is clear that in a neighborhood of infinity

$$(\partial\mathcal{F}/\partial\xi)(\xi) = R_\infty(\xi)\,\xi^{-2}, \tag{35}$$

where

$$|R_\infty(\xi)| \le c\left(\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}\right) \text{ for large } |\xi|.$$

Set $f = \mathcal{F}\circ\theta$. From (34) and (35) it follows that $\partial f/\partial s$ belongs to $\mathcal{L}_{p,\beta+1}(\Gamma)$ and satisfies

$$\|\partial f/\partial s\|_{\mathcal{L}_{p,\beta+1}(\Gamma)} \le c\,\|\varphi\|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)}\,. \tag{36}$$

By the Taylor decomposition of the integral terms in (33) we obtain

$$\mathcal{F}(\xi) = t_{-1}(\varphi)\log|\xi| + t_0(\varphi) + \sum_{k=1}^{n_0-1} t_k(\varphi)\xi^k + |\xi|^{n_0}R_{n_0}(\xi)\,, \tag{37}$$

where

$$|t_k(\varphi)| \le c\left(\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}\right),\ \ k=-1,\dots,n_0-1,$$

and

$$|R_{n_0}(\xi)| \le c\left(\|\mathcal{T}\|_{L_p(\mathbb{R})} + \|\mathcal{Q}\|_{L_p(\mathbb{R}^2_+)}\right)$$

for small $|\xi|$. Taking into account the asymptotic representations (20), (21) of θ^{-1} and the inequality $2(\mu-\beta-p^{-1}) < n_0$, it follows from (36) and (37) that f is represented in the form

$$f(z) = \sum_{k=1}^{m} b_k(\varphi)\mathrm{Re}\, z^{k-1/2} + f^{\#}(z),\ \ z\in\Omega\,, \tag{38}$$

where $f^{\#} \in \mathfrak{N}_{p,\beta}(\Gamma)$, and $b_k(\varphi)$, $k=1,\dots,m$, are linear combinations of the coefficients $t_\ell(\varphi)$ in (37).

According to (29) and (38), the function $g = æh + f$ is harmonic in Ω and can be written as

$$g(z) = \sum_{k=1}^{m} c_k(\varphi)\mathrm{Re}\, z^{k-1/2} + g^{\#}(z),\ \ z\in\Omega',$$

with $c_k(\varphi) = a_k(\varphi) + b_k(\varphi)$. Moreover,

$$\sum_{k=1}^{m} |c^{(k)}| + \|g^{\#}\|_{\mathfrak{N}_{p,\beta}(\Gamma)} \le c\,\|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}$$

and $(g\circ\theta)(\infty) = 0$ by the definition of g. Owing to $(\partial/\partial s)g \in \mathcal{L}_{p,\beta+1}(\Gamma)$, one of the functions conjugate to $-g$ is a harmonic extension of φ onto Ω with the normal derivative in $\mathcal{L}_{p,\beta+1}(\Gamma)$.

(iii) Now let φ belong to $\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)$ and let φ vanish on $\Gamma\cap\{|q|<\delta/2\}$. We introduce the function

$$\Phi(\xi) = (\varphi\circ\theta)(1/\xi),\quad \xi\in\mathbb{R},$$

which vanishes outside a certain interval. Set

$$\mathcal{G}(\zeta) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \Phi(\tau) \mathrm{Re}\, \frac{\zeta}{\tau(\zeta - \tau)} d\tau, \quad \zeta \in \mathbb{R}^2_+. \tag{39}$$

It is clear that one of the conjugate functions $\widetilde{\mathcal{G}}$ is a harmonic extension of $-\Phi$ onto $\mathbb{R}^2_+$. It follows from the boundedness of the Hilbert transform in L_p-spaces that $g = \mathcal{G}\big(1/\theta^{-1}\big)$ belongs to $\mathcal{L}^1_{p,\beta+1}(\Gamma)$ and satisfies

$$\| \, g \, \|_{\mathcal{L}^1_{p,\beta+1}(\Gamma)} \leq c \ \| \, \varphi \, \|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)} \ . \tag{40}$$

Further, we represent $\mathcal{G}$ on $\mathbb{R}$ in the form

$$\mathcal{G}(\xi) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \Phi(\tau)\tau^{-1} d\tau + \sum_{k=1}^{n_0-1} \frac{1}{\pi \xi^k} \int_{\mathbb{R}} \Phi(\tau)\tau^{k-1} d\tau$$

$$+ \frac{1}{\pi \xi^{n_0-1}} \int_{\mathbb{R}} \frac{\Phi(\tau)\, \tau^{n_0-1}}{\xi - \tau} d\tau = \sum_{k=0}^{n_0-1} t_k(\varphi)\xi^k + \mathcal{G}^{\#}(\xi),$$

where

$$t_k(\varphi) = \frac{1}{\pi} \int_{\mathbb{R}} \Phi(\tau)\tau^{k-1} d\tau \ \text{ and } \ \mathcal{G}^{\#}(\xi) = \frac{1}{\pi \xi^{n_0-1}} \int_{\mathbb{R}} \frac{\Phi(\tau)\, \tau^{n_0-1}}{\xi - \tau} d\tau, \ \ \xi \in \mathbb{R}. \tag{41}$$

Since $-1/p < 2\mu - 2\beta - 3/p - n_0 + 1 < 1 - 1/p$, we have

$$\sum_{k=0}^{n_0-1} |t_k(\varphi)| + \left\| \mathcal{G}^{\#} \right\|_{L_{p,2\mu-2\beta-3/p}(\mathbb{R})} \leq c \left\| \Phi \right\|_{L_{p,2\mu-2\beta-3/p}(\mathbb{R})} .$$

Hence it follows from (40) that g is represented in the form

$$g(z) = \sum_{k=1}^{m} c_k(\varphi) \mathrm{Re}\, z^{k-1/2} + g^{\#}(z), \quad z \in \Omega,$$

where $g^{\#} \in \mathfrak{N}_{p,\beta}(\Gamma)$, and $c_k(\varphi)$, $k = 1, \ldots, m$, are linear combinations of the coefficients $t_\ell(\varphi)$, $\ell = 1, \ldots, n_0 - 1$ in (41). These coefficients and the function $g^{\#}$ satisfy

$$\sum_{k=1}^{m} |c_k(\varphi)| + \| g^{\#} \|_{\mathfrak{N}_{p,\beta}(\Gamma)} \leq c \, \|\varphi\|_{\mathfrak{N}^{(+)}_{p,\beta}(\Gamma)}$$

and the conjugate function $\widetilde{g} = \widetilde{\mathcal{G}}(1/\theta^{-1})$ is the harmonic extension of φ onto Ω with the normal derivative in $\mathcal{L}_{p,\beta+1}(\Gamma)$. □

References

1. V. Maz'ya and A. Soloviev, L_p-theory of a boundary integral equation on a cuspidal contour. *Appl. Anal.* **65**(1997), 289–305.
2. V. Maz'ya and A. Soloviev, L_p-theory of boundary integral equations on a contour with outward peak. *Integral Equations Operator Theory* **32**(1998), 75–100.

3. V. Maz'ya and A. Soloviev, L_p-theory of boundary integral equations on a contour with inward peak. *Z. Anal. Anwendungen* **17**(1998), No. 3, 641–673.
4. R. Duduchava, T. Latsabidze, and A. Saginashvili, Singular integral operators with the complex conjugation on curves with peaks. *Integral Equation Operator Theory* **22**(1995), 1–36.
5. R. Duduchava and B. Silbermann, Boundary value problems in domain with peaks. *Mem. Differential Equations Math. Phys.* **21**(2000), 1–122.
6. V. Kokilashvili and V. Paatashvili, On the Riemann-Hilbert problem in the domain with a nonsmooth boundary. *Georgian Math. J.* **4**(1997), No. 3, 279–302.
7. G. Khuskivadze, V. Kokilashvili, and V. Paatashvili, Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14**,(1998), 1–195.
8. E. M. Stein, Note on singular integrals. *Proc. Amer. Math. Soc.* **8**(1957), 250–254.
9. V. G. Maz'ya and B. A. Plamenevskiĭ, Estimates in L_p and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary. *Amer. Math. Soc. Transl. (2)* **123**(1984), 1–56.
10. S. E. Warschawski, On conformal mapping of infinite strips. *Trans. Amer. Math. Soc.* **51**(1941), 280–335.
11. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165**(1972), 207–226.
12. D. S. Kurtz and R. L. Wheeden, Results on weighted norm inequalities for multipliers. *Trans. Amer. Math. Soc.* **255**(1979), 343–362.
13. K. F. Andersen, Weighted norm inequalities for Hilbert transforms and conjugate functions of even and odd functions. *Proc. Amer. Math. Soc.* **56**(1976), 99–107.
14. R. A. Hunt, B. Muckenhoupt, and R. L. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176**(1973), 227–251.

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