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BASIC BOUNDARY VALUE PROBLEMS OF THERMOELASTICITY FOR ANISOTROPIC BODIES WITH CUTS. I

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Abstract. The three-dimensional problems of the mathematical theory of thermoelasticity are considered for homogeneous anisotropic bodies with cuts. It is assumed that the two-dimensional surface of a cut is a smooth manifold of an arbitrary configuration with a smooth boundary. The existence and uniqueness theorems for boundary value problems of statics and pseudo-oscillations are proved in the Besov ($\mathbb{B}^s_{p,q}$) and Bessel-potential ($\mathbb{H}^s_{p,q}$) spaces by means of the classical potential methods and the theory of pseudodifferential equations on manifolds with boundary. Using the embedding theorems, it is proved that the solutions of the considered problems are Hölder continuous. It is shown that the displacement vector and the temperature distribution function are C^α-regular with any exponent $\alpha < 1/2$.

This paper consists of two parts. In this part all the principal results are formulated. The forthcoming second part will deal with the auxiliary results and proofs.

Introduction

Three-dimensional crack problems evoke much interest in engineering applications. In this paper we investigate the three-dimensional boundary value problems (BVPs) of thermoelasticity in certain function spaces when the anisotropic elastic body under consideration contains any number of nonintersecting cuts in the form of two-dimensional smooth surfaces with smooth boundaries.

For domains bounded by smooth closed manifolds of the class $C^{2+\gamma}$ the basic BVPs were completely investigated using the potential method by V. D. Kupradze and his collaborators [1] in the isotropic case and by D. Natroshvili [2] in the anisotropic case.

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Analogous three-dimensional crack problems of classical elasticity for isotropic bodies were treated in the Bessel-potential spaces $\mathbb{H}_2^s$ by M. Costabel and E. Stephan in [3]. The BVPs for homogeneous anisotropic bodies were studied in the Besov ($\mathbb{B}_{p,q}^s$) and Bessel-potential ($\mathbb{H}_p^s$) spaces by R. Duduchava and collaborators in [4] who established more precise results on the regularity of solutions (C^α-regularity with $\alpha < 1/2$). Each of the quoted papers contains an ample bibliography to the above-mentioned problems.

To illustrate our approach we consider two basic BVPs for an infinite domain. All the results obtained here remain valid (with minor modifications) for a bounded domain with interior cuts, i.e., when the cut surface does not touch the domain boundary.

This paper consists of two parts. The first part contains three sections.

In the first section we formulate the problems and introduce the spaces of functions and distributions needed for proving the unique solvability of the problems and, further on, the regularity properties of solutions.

In the second section we show the mapping properties of single- and double-layer potentials of thermoelasticity (both on the surface and from the surface to the space; see Theorems 2 and 4) and derive the integral representations of regular solutions.

The third section contains the formulations of the main theorems of the paper, concerning the existence and uniqueness of solutions of the problems discussed in the first section, the regularity of such solutions, and the explicit solvability properties of the corresponding boundary integral equations (see Theorems 5–9).

The proofs of Theorems 7 and 8 will be given in the forthcoming second part of the paper after recalling some auxiliary results.

§ 1. Formulation of the Problems

Let Ω^+ be a bounded domain in $\mathbb{R}^3$ with the smooth boundary $\partial\Omega^+ = \Sigma$ and S be the connected part of Σ with the smooth boundary curve $\partial S = \ell \neq \varnothing$. Then S is a two-dimensional surface of an arbitrary configuration with the boundary ℓ. It is assumed that $\Omega^- := \mathbb{R}^3\backslash\overline{\Omega}^+$, where $\overline{\Omega}^+ = \Omega^+\cup\Sigma$, $\mathbb{R}_S^3 = \mathbb{R}\backslash\overline{S}$ and $\overline{S} = S\cup\ell$.

Let $\mathbb{R}_S^3$ be filled with some homogeneous anisotropic elastic material having density ρ, elastic coefficients

$$c_{kjpq} = c_{pqkj} = c_{jkpq}, \tag{1.1}$$

heat conductivity coefficients

$$\lambda_{ij} = \lambda_{ji} \tag{1.2}$$

and thermal capacity c_0.

In what follows $\mathbb{R}^3_S$ is treated as an infinite elastic body with a cut along the surface $\overline{S}$. For simplicity Σ, S and ℓ will further be assumed to be C^∞-regular.

By $u = (u_1, u_2, u_3)$ and u_4 we denote the displacement vector field and the temperature field, respectively. The components of the thermal stress vector calculated on a surface element with the unit normal vector $n = (n_1, n_2, n_3)$ have the form

$$[\mathbf{P}(D_x, n)U]_k := [\mathbf{T}(D_x, n)u]_k - \beta_{kj} n_j u_4, \quad k = 1, 2, 3,$$

where $U = (u_1, u_2, u_3, u_4)$, $[\mathbf{T}(D_x, n)u]_k := c_{kjpq} n_j D_q u_p$ are the components of the classical stress vector, $D_x := (D_1, D_2, D_3)$, $D_p = \partial/\partial x_p$, and the constants $\beta_{ij} = \beta_{ji}$ are expressed in terms of the thermal and the elastic constants (cf. [5]). Here and in what follows, summation from 1 to 3 over repeated indices is meant.

The strain tensor components e_{kj} are defined by the formulas

$$e_{kj} = \frac{1}{2}(D_j u_k + D_k u_j), \quad k, j = 1, 2, 3,$$

while the stress tensor components are related to e_{kj} as follows (Hooke's law):

$$\tau_{kj} = c_{kjpq} e_{pq} = c_{kjpq} D_p u_q.$$

Potential energy in classical elasticity reads

$$2W = e_{kj}\tau_{kj} = c_{kjpq} e_{kj} e_{pq}.$$

From the physical standpoint, potential energy is assumed to be a positive definite quadratic form with respect to the variables $e_{kj} = e_{jk}$:

$$2W \geq \delta e_{kj} e_{kj}, \quad \delta = const > 0. \tag{1.3}$$

Combining the static and the pseudo-oscillation cases, we consider the following system of equations of thermoelasticity:

$$\mathbf{A}(D_x, \tau)U(x) = F(x, \tau), \quad x \in \mathbb{R}_S, \tag{1.4}$$

where $F = (F_1, \dots, F_4)$ is a given vector with a compact support, $\mathbf{I}_m = \|\delta_{kj}\|_{m\times m}$ is the unit matrix,

$$\mathbf{A}(D, \tau) = \left\| \begin{matrix} & \mathbf{C}(D) - \rho\tau^2 \mathbf{I}_3 & & \begin{matrix} -\beta_{1j} D_j \\ -\beta_{2j} D_j \\ -\beta_{3j} D_j \end{matrix} \\ \alpha_{1j} D_j, & \alpha_{2j} D_j, & \alpha_{3j} D_j, & \mathbf{\Lambda}(D) - c_0\tau \end{matrix} \right\|_{4\times 4}, \tag{1.5}$$

$$\mathbf{C}(D) := \|\mathbf{C}_{kp}(D)\|_{3\times 3}, \quad \mathbf{C}_{kp}(D) := c_{kjpq} D_j D_q,$$

$$\mathbf{\Lambda}(D) := \lambda_{pq} D_p D_q, \quad \alpha_{kj} = -\tau T_0 \beta_{kj}, \tag{1.6}$$

$T_0 = const > 0$ is a temperature of the medium in the natural state (cf. [5]), $\tau = \sigma + i\omega$; $\tau = 0$ corresponds to the static case, while $\tau = \sigma + i\omega$, $\sigma \geq \sigma_0 > 0$ corresponds to the pseudo-oscillation case (equation (1.4) is obtained from the dynamic equations of thermoelasticity upon applying the Laplace transform).

Treating S as a double-sided surface, we consider two basic BVPs for equation (1.4) with a Dirichlet type boundary condition

$$[U(x)]^{\pm} = \varphi^{\pm}(x,\tau), \quad x \in S, \tag{1.7}$$

and with a Neumann type boundary condition

$$[\mathbf{B}(D_x, n(x))U(x)]^{\pm} = \psi^{\pm}(x,\tau), \quad x \in S, \tag{1.8}$$

where the symbol $[\cdot]^{\pm}$ denotes the limiting value on S of a function (vector) from $\Omega^{\pm}$, $n(x)$ is the unit normal at $x \in S$ outward with respect to Ω^+, $\varphi^{\pm} = (\varphi_1^{\pm}, \dots, \varphi_4^{\pm})$ and $\psi^{\pm} = (\psi_1^{\pm}, \dots, \psi_4^{\pm})$ are the known vector-functions,

$$\mathbf{B}(D_x, n(x)) = \left\| \begin{matrix} \mathbf{T}(D_x, n(x)) & \begin{matrix} -\beta_{1j}n_j(x) \\ -\beta_{2j}n_j(x) \\ -\beta_{3j}n_j(x) \end{matrix} \\ 0, \quad 0, \quad 0, & \lambda_{pq}n_p(x)D_q \end{matrix} \right\|_{4\times 4}, \tag{1.9}$$

$$\mathbf{T}(D_x, n(x)) := \|\mathbf{T}_{kp}(D_x, n(x))\|_{3\times 3}, \quad \mathbf{T}_{kp}(D_x, n(x)) := c_{kjpq}n_j(x)D_q.$$

In problems (1.7) and (1.8) it is required that

$$u_k(x) = \begin{cases} o(1) & \text{for } \tau = 0, \\ O(|x|^N) & \text{for } \operatorname{Re}\tau > 0, \end{cases} \quad k = 1,2,3,4, \tag{1.10}$$

for a sufficiently large $|x|$ and some positive number N.

These conditions imply (see [6], [7])

$$D^{\alpha}u_k(x) = \begin{cases} O(|x|^{-1-|\alpha|}) & \text{for } \tau = 0, \\ O(|x|^{-\nu}) & \text{for } \operatorname{Re}\tau > 0, \end{cases} \tag{1.11}$$

as $|x| \to \infty$, where α is an arbitrary multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and ν is an arbitrary positive number.

The symmetry properties of coefficients (1.1) and the positive definiteness of the energy quadratic form (1.3) imply that the operator $\mathbf{C}(D)$ defined by (1.6) is a formally self-adjoint strongly elliptic matrix differential operator (see [8])

$$\begin{gathered} \operatorname{Re}\big(\mathbf{C}(\xi)\eta, \eta\big) = \big(\mathbf{C}(\xi)\eta, \eta\big) = c_{kjpq}\xi_j\xi_q\eta_p\overline{\eta}_k \geq \delta_0|\xi|^2|\eta|^2, \\ \delta_0 = const > 0, \quad \xi \in \mathbb{R}^3, \quad \eta \in \mathbb{C}^3, \end{gathered} \tag{1.12}$$

where $\mathbb{C}^3$ is the three-dimensional complex Euclidean space, the bar designates complex conjugation, and $(a,b)=a\overline{b}=a_k\overline{b}_k$ for $a,b\in\mathbb{C}^3$.

In contrast to $\mathbf{C}(D)$ the operator $\mathbf{A}(D,\tau)$ is elliptic but not self-adjoint. Denote by $\mathbf{A}^*(D,\tau)$ the operator formally adjoint to $\mathbf{A}(D,\tau)$. It is obvious that

$$\mathbf{A}^*(D,\tau)=\mathbf{A}^T(-D,\overline{\tau}), \tag{1.13}$$

where the superscript T denotes the transposition operator.

Note that the quadratic form $\mathbf{\Lambda}(\xi)$ defined by (1.6) is also positive definite (see [5]),

$$\mathbf{\Lambda}(\xi)=\lambda_{pq}\xi_p\xi_q\geq\delta_1|\xi|^2,\quad \xi\in\mathbb{R}^3,\quad \delta_1=const>0. \tag{1.14}$$

A function $f:\overline{\Omega}^{\pm}\to\mathbb{R}^1$ is said to be regular in $\Omega^{\pm}$ if $f\in C^2(\Omega^{\pm})\cap C^1(\overline{\Omega}^{\pm})$. A vector $v=(v_1,\dots,v_m)$ (matrix $\widetilde{A}=\|\widetilde{a}_{kj}\|_{m\times m}$) is said to be regular in $\Omega^{\pm}$ if all its components (entries) are regular functions in $\Omega^{\pm}$. In general, $v\in P$ ($\widetilde{A}\in P$) means that all components of v (all entries of $\widetilde{A}$) belong to the space P.

Let $C^{k+\gamma}(\overline{\Omega}^{\pm})$, where $k\geq 0$ is an integer and $0<\gamma<1$, denote the space of functions u defined on $\overline{\Omega}^{\pm}$ whose derivatives $D^\alpha u$ of order $|\alpha|=k$ are Hölder continuous with the exponent γ. The space $C^{k+\gamma}(\Sigma)$ is defined similarly (cf., for example, [1]).

Assume that $U=(u_1,\dots,u_4)$ and $V=(v_1,\dots,v_4)$ are the regular vectors in $\Omega^{\pm}$ satisfying the conditions $\mathbf{A}U$, $\mathbf{A}^*V\in\mathbb{L}_1(\Omega^{\pm})$ and (1.11). Then we have (see [2])

$$\int\limits_{\Omega^{\pm}}(\mathbf{A}U,V)\,dx=\pm\int\limits_{\partial\Omega^{\pm}}\big([\mathbf{B}U]^{\pm},[V]^{\pm}\big)\,dS-\int\limits_{\Omega^{\pm}}E(U,V)\,dx, \tag{1.15}$$

$$\begin{aligned}\int\limits_{\Omega^{\pm}}\big\{(\mathbf{A}U,V)-(U,\mathbf{A}^*V)\big\}dx=\pm\int\limits_{\partial\Omega^{\pm}}\big\{\big([\mathbf{B}U]^{\pm},[V]^{\pm}\big)dS-\\ -\big([U]^{\pm},[\overline{\mathbf{Q}}V]^{\pm}\big)\big\}dS,\end{aligned} \tag{1.16}$$

$$\begin{aligned}\int\limits_{\Omega^{\pm}}\Big\{(\mathbf{A}U)_k\overline{U}_k+\frac{1}{\overline{\tau}T_0}(\overline{\mathbf{A}U})_4U_4\Big\}dx=-\int\limits_{\Omega^{\pm}}\Big\{c_{ijkl}D_lu_kD_j\overline{u}_i+\\ +\rho\tau^2u_k\overline{u}_k+\frac{1}{\overline{\tau}T_0}\lambda_{ij}D_ju_4D_i\overline{u}_4+\frac{c_0}{T_0}\,u_4\overline{u}_4\Big\}dx\pm\\ \pm\int\limits_{\partial\Omega^{\pm}}\Big\{[\mathbf{B}U]_k^{\pm}[U_k]^{\pm}+\frac{1}{\overline{\tau}T_0}\,[U_4]^{\pm}\Big[\frac{\partial}{\partial\nu}\overline{U}_4\Big]^{\pm}\Big\}dS,\end{aligned} \tag{1.17}$$

where

$$\begin{aligned} E(U,V) &= c_{kjpq}D_q u_p D_j \overline{v}_k + \rho\tau^2 u_k \overline{v}_k - \beta_{kp} u_4 D_q \overline{v}_k + \\ &\quad + \lambda_{pq} D_q u_4 D_p \overline{v}_4 + c_0 \tau u_4 \overline{v}_4 + \tau T_0 \beta_{pq} D_p u_p \overline{v}_4, \qquad (1.18) \\ \frac{\partial}{\partial\nu(x)}\, u_4 &= \lambda_{pq} n_p(x) D_q u_4(x), \\ \mathbf{Q}(D_x, n(x)) &:= \left\| \begin{matrix} \mathbf{T}(D_x, n(x)) & \begin{matrix} T_0\tau\beta_{1j}n_j(x) \\ T_0\tau\beta_{2j}n_j(x) \\ T_0\tau\beta_{3j}n_j(x) \end{matrix} \\ 0, \quad 0, \quad 0, & \lambda_{pq} n_p(x) D_q \end{matrix} \right\|_{4\times 4}. \end{aligned}$$

In what follows the BVPs (1.7) and (1.8) will be investigated in different functional spaces.

To formulate these problems in exact terms we need the Sobolev $\mathbb{W}^k_p(\mathbb{R}^3)$, $\mathbb{W}^k_p(\Omega)$, $\mathbb{W}^k_p(\Sigma)$, Sobolev–Slobodecky $\mathbb{W}^s_p(\mathbb{R}^3)$, $\mathbb{W}^s_p(\Omega)$, $\mathbb{W}^s_p(\Sigma)$, the Bessel-potential $\mathbb{H}^s_p(\mathbb{R}^3)$, $\mathbb{H}^s_p(\Omega)$, $\mathbb{H}^s_p(\Sigma)$, and the Besov $\mathbb{B}^s_{p,q}(\mathbb{R}^3)$, $\mathbb{B}^s_{p,q}(\Omega)$, $\mathbb{B}^s_{p,q}(\Sigma)$ spaces ($k = 0, 1, 2, \dots$, $-\infty < s < \infty$, $1 < p < \infty$, $1 \le q \le \infty$). For the definitions of these spaces see [9].

Let $\mathbb{X}(\mathbb{R}^3)$ be one of the above-mentioned function spaces. For an arbitrary unbounded domain $\Omega^- \subset \mathbb{R}^3$ (with a smooth boundary $\partial\Omega^-$) we denote by $\mathbb{X}_{loc}(\Omega^-)$ the subset of distributions $\varphi \in D'(\Omega^-)$ with

$$\varphi\big|_{\Omega^-_R} \in \mathbb{X}(\Omega^-_R), \quad \Omega^-_R = \big\{x \in \Omega \,:\, |x| < R\big\}, \quad \forall R > 0,$$

and by $\mathbb{X}_{comp}(\Omega^-)$ the set of functions $\varphi \in \mathbb{X}(\Omega^-)$ with compact supports.

$\mathbb{X}_{loc}(\mathbb{R}^3_S)$ denotes the subset of distributions $\varphi \in D'(\mathbb{R}^3_S)$ satisfying the conditions

$$\varphi\big|_{\Omega^+} \in \mathbb{X}(\Omega^+), \quad \varphi\big|_{\Omega_R} \in \mathbb{X}(\Omega_R), \quad \Omega_R = \big\{x \in \mathbb{R}^3 \backslash \overline{\Omega}^+, \ |x| \le R\big\},$$

for an arbitrary $R > 0$ and any bounded domain Ω^+ with $S \subset \partial\Omega^+$; here $\varphi|_\Omega$ is the restriction to Ω.

In particular, $\mathbb{W}^1_{p.loc}(\mathbb{R}^3_S)$ denotes the Sobolev space of functions φ on $\mathbb{R}^3_S$ which are p-integrable on $\Omega\backslash S$ for each compact domain $\Omega \subset \mathbb{R}^3$ together with their generalized derivatives of order 1

$$\big\|\varphi; \mathbb{W}^1_p(\Omega\backslash S)\big\| = \Big\{ \int\limits_{\Omega\backslash S} \big(|\varphi(x)|^p + |\nabla\varphi(x)|^p\big) dx \Big\}^{1/p} < \infty,$$
$$\nabla\varphi := (D_1\varphi, D_2\varphi, D_3\varphi).$$

For the open surface $S \subset \Sigma$ the spaces $\mathbb{H}_p^t(S)$, $\widetilde{\mathbb{H}}_p^t(S)$ are defined as

$$\mathbb{H}_p^t(S) = \big\{ \mathbf{r}_S f \;:\; f \in \mathbb{H}_p^t(\Sigma) \big\},$$
$$\widetilde{\mathbb{H}}_p^t(S) = \big\{ f \in \mathbb{H}_p^t(\Sigma) \;:\; \operatorname{supp} f \subset \overline{S} \big\} \subset \mathbb{H}_p^t(\Sigma),$$

where $\mathbf{r}_S f = f|_S$ is the restriction.

The spaces $\mathbb{B}_{p,q}^t(S)$ and $\widetilde{\mathbb{B}}_{p,q}^t(S)$ are defined similarly. Note that $\mathbb{H}_2^s = \mathbb{W}_2^s = \mathbb{B}_{2,2}^s$, $\mathbb{W}_p^t = \mathbb{B}_{p,p}^t$, and $\mathbb{H}_p^k = \mathbb{W}_p^k$ hold for any $-\infty < s < \infty$, for any positive and non-integer t, and for any non-negative integer $k = 0, 1, 2, \dots,$ respectively.

In contrast to closed surfaces, even for infinitely smooth S, ℓ, $\varphi^\pm$, and $\psi^\pm$ the solutions of problems $\{(1.4),\ F = 0,\ (1.7)\}$ and $\{(1.4),\ F = 0,\ (1.8)\}$ have in general no C^α-smoothness with $\alpha > 1/2$ in the vicinity of $\ell = \partial S$ but are infinitely differentiable elsewhere.

Hence we seek solutions of the BVPs (1.7) and (1.8) from the Sobolev space $\mathbb{W}_{p,loc}^1(\mathbb{R}_S^3)$ provided that

$$\varphi^\pm \in \mathbb{B}_{p,p}^{1-1/p}(S), \quad \varphi^0 = \varphi^+ - \varphi^- \in \widetilde{\mathbb{B}}_{p,p}^{1-1/p}(S) \tag{1.19}$$

for the Dirichlet type problem (1.7) and

$$\psi^\pm \in \mathbb{B}_{p,p}^{-1/p}(S), \quad \psi^0 = \psi^+ - \psi^- \in \widetilde{\mathbb{B}}_{p,p}^{-1/p}(S) \tag{1.20}$$

for the Neumann type problem (1.8).

Further the problems $\{(1.4),\ F = 0,\ (1.7),\ (1.10),\ (1.19)\}$ and $\{(1.4),\ F = 0,\ (1.8),\ (1.10),\ (1.20)\}$ will be referred to as Problem $\mathcal{D}$ and Problem $\mathcal{N}$, respectively. Note that property (1.11) holds for solutions of Problems $\mathcal{D}$ and $\mathcal{N}$ as well.

For $s > 1/p$ by the trace theorem (see [9], Theorem 3.3.3) we have

$$\begin{aligned} &g^\pm \in \mathbb{B}_{p,p}^{s-1/p}(S) \quad \text{if} \quad g \in \mathbb{H}_{p,loc}^s(\mathbb{R}_S^3) \cup \mathbb{W}_{p,loc}^s(\mathbb{R}_S^3), \\ &g^\pm \in \mathbb{B}_{p,q}^{s-1/p}(S) \quad \text{if} \quad g \in \mathbb{B}_{p,q,loc}^s(\mathbb{R}_S^3). \end{aligned} \tag{1.21}$$

Therefore (1.7) and (1.19) are compatible and correctly defined if $U \in \mathbb{W}_{p,loc}^1(\mathbb{R}_S^3)$.

As to (1.8), (1.20), we should give some additional explanation, since $D_j U_k \in \mathbb{L}_{p,loc}(\mathbb{R}_S^3) = \mathbb{W}_{p,loc}^0(\mathbb{R}_S^3)$ and they have no traces on S in general.

We can make condition (1.8) meaningful for $U \in \mathbb{W}_{p,loc}^1(\mathbb{R}_S^3)$ with $\mathbf{A}U \in \mathbb{L}_{p,loc}(\mathbb{R}_S^3)$ using equality (1.15). Indeed, it can be rewritten as

$$\begin{aligned} &\langle [\mathbf{B}(D_x, n(x))U]^+, V^+ \rangle_\Sigma = \\ &\int\limits_{\Omega^+} \big(\mathbf{A}(D)U, V\big) dx + \int\limits_{\Omega^+} E(U,V)\, dx, \end{aligned} \tag{1.22}$$

for $U \in \mathbb{W}^1_p(\Omega^+)$, $\mathbf{A}U \in \mathbb{L}_p(\Omega^+)$, $\forall V \in \mathbb{W}^1_{p'}(\Omega^+)$;

$$\langle [\mathbf{B}(D_x, n(x))U]^-, V^-\rangle_\Sigma = -\int\limits_{\Omega^-} \big(\mathbf{A}(D)U, V\big)\, dx - \int\limits_{\Omega^-} E(U,V)\, dx, \quad (1.23)$$

for $U \in \mathbb{W}^1_{p,loc}(\Omega^-)$, $\mathbf{A}U \in \mathbb{L}_p(\Omega^-)$, $\forall V \in \mathbb{W}^1_{p',comp}(\Omega^-)$. Here $p' = p/(p-1)$ and $\langle\cdot,\cdot\rangle_\Sigma$ defines the duality between $\mathbb{B}^{-1/p}_{p,p}(\Sigma)$ and $\mathbb{B}^{1/p}_{p',p'}(\Sigma)$ given by

$$\langle f, g\rangle_\Sigma = \int\limits_\Sigma f g\, dS$$

for the smooth functions f and g.

Relations (1.22) and (1.23) define $[B(D, n(x))U]^\pm \in \mathbb{B}^{-1/p}_{p,p}(S)$ correctly, since by virtue of (1.21) their right-hand side expressions exist for any $V \in \mathbb{W}^1_{p'}(\Omega^+)$ and $V \in \mathbb{W}^1_{p',comp}(\Omega^-)$, respectively, and $V^\pm \in \mathbb{B}^{1/p}_{p',p'}(\Sigma)$.

§ 2. Properties of Fundamental Solutions and Potentials

By $\mathcal{A}(\xi,\tau)$ denote the symbol matrix of the operator $\mathbf{A}(D_x,\tau)$ (see (1.5)). Obviously,

$$\mathcal{A}(\xi,\tau) = \mathbf{A}(-i\xi,\tau)$$

and the matrix distribution

$$\Phi(x,\tau) := (2\pi)^{-3} \int\limits_{\mathbb{R}^3} e^{-i\xi\tau} \mathcal{A}^{-1}(\xi,\tau)\, d\xi \quad (2.1)$$

represents the fundamental matrix of the operator $\mathbf{A}(D_x,\tau)$, i.e.,

$$\mathbf{A}(D_x,\tau)\Phi(x,\tau) = \delta(x)\mathbf{I}_4$$

where $\delta(\cdot)$ is the Dirac distribution. The fundamental matrix of the formally adjoint operator reads

$$\Phi^*(x,\tau) = \Phi^T(-x,\tau). \quad (2.2)$$

The entries of these matrices are of the class $C^\infty(\mathbb{R}^3\backslash\{0\})$ and for $\operatorname{Re}\tau > 0$ they, together with all their derivatives, decay faster than any negative power of $|x|$ at infinity. For $\tau = 0$ we have

$$D^\alpha \Phi_{kj}(x,0) = O(|x|^{-1-|\alpha|}) \quad \text{as} \quad |x| \to \infty$$

(see [2]).

Near the origin the main singular parts of the matrices $\Phi(x,\tau)$ and $\Phi^*(x,\tau)$ coincide and have the form (see [2], [10])

$$\Phi(x) := \begin{Vmatrix} & & & 0 \\ & \Gamma(x) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \gamma(x) \end{Vmatrix}_{4\times 4}, \tag{2.3}$$

where $\Gamma(\cdot)$ is the fundamental matrix of the classical operator $\mathbf{C}(D)$ while $\gamma(\cdot)$ is the fundamental function of the operator $\mathbf{\Lambda}(D)$

$$\begin{aligned} &\Gamma(x) = (2\pi)^{-3}\int\limits_{\mathbb{R}^3} e^{ix\xi}\mathcal{C}^{-1}(\xi)\,d\xi = (8\pi^2|x|)^{-1}\int\limits_0^{2\pi}\mathcal{C}^{-1}(a\widetilde{\eta})\,d\varphi, \\ &\mathcal{C}(\xi) = \|c_{kjpq}(-i\xi_j)(-i\xi_q)\|_{3\times 3}, \\ &\gamma(x) = (2\pi)^{-3}\int\limits_{\mathbb{R}^3} e^{ix\xi}\mathbf{\Lambda}^{-1}(-i\xi)\,d\xi = -\{4\pi|L|^{1/2}(L^{-1}x,x)^{1/2}\}^{-1}, \\ &\mathbf{\Lambda}(\xi) = \lambda_{pq}\xi_q\xi_p \end{aligned} \tag{2.4}$$

(see [11]) with $\widetilde{\eta} = (\cos\varphi, \sin\varphi, 0)$, $L = \|\lambda_{pq}\|_{3\times 3}$, $|L| = \det L$; here $a = \|a_{jk}\|_{3\times 3}$ is an orthogonal matrix with the property $a^T x = (0,0,|x|)$.

It is evident that the equalities

$$\Phi(tx) = t^{-1}\Phi(x), \quad \Phi(x) = \Phi^T(x) = \Phi(-x) \tag{2.5}$$

hold for any positive $t > 0$. Near the origin $\Phi_{kj}(x,\tau)$ has the asymptotics

$$D^\alpha[\Phi_{kj}(x,\tau) - \Phi_{kj}(x)] = \begin{cases} O(\ln|x|), & \text{if } \alpha = 0, \\ O(|x|^{-|\alpha|}), & \text{if } |\alpha| > 0. \end{cases} \tag{2.6}$$

The properties of generalized potentials corresponding to these matrices in the case of closed surfaces were studied in [2], [10]. Due to these results from now on we shall assume without loss of generality that $F = 0$ in (1.4), as the particular solution of (1.4) can be written explicitly using the generalized Newtonian potentials (see, for example, [12]).

On account of (1.16) we obtain the following integral representation of a regular vector:

$$\begin{aligned} &\int\limits_{\Omega^\omega} \Phi(x-y,\tau)\mathbf{A}(D_y,\tau)U(y)\,dy \pm \int\limits_{\partial\Omega^\pm} \big\{\big[\mathbf{Q}(D_y,n(y))\Phi^T(x-y,\tau)\big]^T \times \\ &\qquad \times [U(y)]^\pm - \Phi(x-y,\tau)\big[\mathbf{B}(D_y,n(y))U(y)\big]^\pm\big\}d_yS = \\ &\qquad\qquad = \begin{cases} U(x), & x \in \Omega^\pm, \\ 0, & x \in \Omega^\mp. \end{cases} \end{aligned} \tag{2.7}$$

We introduce the generalized single- and double-layer potentials

$$\mathbf{P}^1_\Sigma g(x) := \int\limits_\Sigma \Phi(x-y,\tau)g(y)\,d_yS, \tag{2.8}$$

$$\mathbf{P}^2_\Sigma g(x) := \int\limits_\Sigma \big[\mathbf{Q}(D_y,n(y))\Phi^T(x-y,\tau)\big]^T g(y)\,d_yS, \tag{2.9}$$

$$x \in \mathbb{R}^3_\Sigma = \mathbb{R}^3 \backslash \Sigma,$$

which are the solutions of the homogeneous equation (1.4), i.e., for $F=0$. The same notation will be used for the direct values of $\mathbf{P}^j_\Sigma g(x)$, $x \in \Sigma$ $(j=1,2)$. Note in this connection that for $x\in\Sigma$ integral (2.9) exists only in the sense of the Cauchy principal value, while (2.8) exists as a usual improper integral (see [2]). In a similar manner we define

$$(\mathbf{P}^3_\Sigma g)(x) := \int\limits_\Sigma \mathbf{B}(D_x,n(x))\Phi(x-y,\tau)g(y)\,d_yS \tag{2.10}$$

for $x\in\mathbb{R}^3_\Sigma$ and $x\in\Sigma$; here $n(x)$ is the C^∞_0-extension of the exterior unit normal vector from Σ onto $\mathbb{R}^3$.

Lemma 1. *The equalities*

$$[\mathbf{P}^2_\Sigma g]^\pm(x) = \pm\tfrac{1}{2}g(x) + \mathbf{P}^2_\Sigma g(x), \quad k\ge 0, \tag{2.11}$$

$$[\mathbf{B}(D_x,n(x))\mathbf{P}^1_\Sigma g]^\pm(x) = \mp\tfrac{1}{2}g + \mathbf{P}^3_\Sigma g(x), \quad k\ge 0, \tag{2.12}$$

$$[\mathbf{P}^1_\Sigma g]^+(x) = [\mathbf{P}^1_\Sigma g]^-(x) = \mathbf{P}^1_\Sigma g(x), \quad k\ge 0, \tag{2.13}$$

$$[\mathbf{B}(D_x,n(x))\mathbf{P}^2_\Sigma g]^+(x) = [\mathbf{B}(D_x,n(x))\mathbf{P}^2_\Sigma g]^-(x) \equiv \\ = \mathbf{P}^4_\Sigma g(x), \quad k\ge 1, \tag{2.14}$$

are fulfilled for any $g\in C^{k+\gamma}(\Sigma)$, $0<\gamma<1$ and $x\in\Sigma$.

The operators

$$\mathbf{P}^j_\Sigma \;:\; C^{k+\gamma}(\Sigma)\to C^{k+2-j+\gamma}(\overline{\Omega}^\pm), \quad j=1,2, \tag{2.15}$$

$$\mathbf{P}^j_\Sigma \;:\; C^{k+\gamma}(\Sigma)\to C^{k+2-j+\gamma}(\Sigma), \quad j=1,2, \tag{2.16}$$

$$\mathbf{P}^3_\Sigma \;:\; C^{k+\gamma}(\Sigma)\to C^{k+\gamma}(\Sigma), \tag{2.17}$$

$$\mathbf{P}^4_\Sigma \;:\; C^{k+1+\gamma}(\Sigma)\to C^{k+\gamma}(\Sigma) \tag{2.18}$$

are bounded for $0<\gamma<1$ and any integer $k\ge 0$.

We retain notation of type (2.8)–(2.10) and (2.14) for the potentials and the corresponding operators when the closed surface Σ is replaced by the open surface S. The potentials $\mathbf{P}^j$ possess essentially different properties and require an especially careful approach.

Theorem 2. *Let* $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. $\mathbf{P}^1_\Sigma$, $\mathbf{P}^2_\Sigma$, $\mathbf{P}^3_\Sigma$ *and* $\mathbf{P}^4_\Sigma$ *are the pseudodifferential operators of orders* -1, 0, 0 *and* 1, *respectively.*

The following operators are bounded (*cf.* (2.15)–(2.18))*:*

$$\begin{aligned} \mathbf{P}^1_\Sigma &: \mathbb{H}^s_p(\Sigma) \to \mathbb{H}^{s+1}_p(\Sigma), \\ \mathbf{P}^1_\Sigma &: \mathbb{B}^s_{p,q}(\Sigma) \to \mathbb{B}^{s+1}_{p,q}(\Sigma), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathbf{P}^2_\Sigma,\ \mathbf{P}^3_\Sigma &: \mathbb{H}^s_p(\Sigma) \to \mathbb{H}^s_p(\Sigma), \\ \mathbf{P}^2_\Sigma,\ \mathbf{P}^3_\Sigma &: \mathbb{B}^s_{p,q}(\Sigma) \to \mathbb{B}^s_{p,q}(\Sigma), \end{aligned} \tag{2.20}$$

$$\begin{aligned} \mathbf{P}^4_\Sigma &: \mathbb{H}^{s+1}_p(\Sigma) \to \mathbb{H}^s_p(\Sigma), \\ \mathbf{P}^4_\Sigma &: \mathbb{B}^{s+1}_{p,q}(\Sigma) \to \mathbb{B}^s_{p,q}(\Sigma). \end{aligned} \tag{2.21}$$

Proof. The first claim of the theorem is proved in [13], while the other follows from the well-known properties of the boundedness of pseudodifferential operators (see, for example, [14]). □

Let Σ_0 be an m-dimensional C^∞-smooth compact manifold without boundary and embedded in $\mathbb{R}^n$ $(n \ge m)$. Consider the distribution $v \times \delta_{\Sigma_0} \in \mathbb{S}'(\mathbb{R}^n)$ defined by the formula

$$\langle v \times \delta_{\Sigma_0}, \varphi \rangle := \langle v, \varphi\big|_{\Sigma_0} \rangle, \quad \varphi \in \mathbb{S}(\mathbb{R}^n), \tag{2.22}$$

for any $v \in \mathbb{H}^s_p(\Sigma_0)$ $(v \in \mathbb{B}^s_{p,q}(\Sigma_0)$, $1 < p < \infty$, $1 \le q \le \infty)$.

The above definition is correct, since the restriction $\varphi|_{\Sigma_0} \in C^\infty(\Sigma_0)$.

Lemma 3. *Let* $v \in \mathbb{B}^s_{p,p}(\Sigma_0)$, $(v \in \mathbb{B}^s_{p,q}(\Sigma_0))$, $1 < p < \infty$, $1 \le q \le \infty$, $s < 0$. *Then* $v \times \delta_{\Sigma_0} \in \mathbb{H}^{s-\frac{n-m}{p'}}_p(\mathbb{R}^n)$ $(v \times \delta_{\Sigma_0} \in \mathbb{B}^{s-\frac{n-m}{p'}}_{p,q}(\mathbb{R}^n))$, $p' = p/(p-1)$.

Proof. Applying the trace theorem (see [9], Theorem 3.3.3), we conclude that any function ψ from $\mathbb{H}^{-s+\frac{n-m}{p'}}_{p'}(\mathbb{R}^n)$ (from $\mathbb{B}^{-s+\frac{n-m}{p'}}_{p',q'}(\mathbb{R}^n)$) has the trace $\psi|_{\Sigma_0} \in \mathbb{B}^{-s}_{p',p'}$ $(\psi|_{\Sigma_0} \in \mathbb{B}^{-s}_{p',q'}(\Sigma_0))$. Hence, by virtue of definition (2.22), $v \times \delta_{\Sigma_0}$ represents a bounded functional on $\mathbb{H}^{-s+\frac{n-m}{p'}}_{p'}(\mathbb{R}^n)$ (on $\mathbb{B}^{-s+\frac{n-m}{p'}}_{p',q'}(\mathbb{R}^n)$, $1 < q < \infty$) and by the duality property (see [9], Theorem 2.11.1) we get the proof for $1 < q < \infty$.

For $q = 1, \infty$ the proof is accomplished by interpolation (see [9], Theorem 3.3.6). □

Theorem 4. *Let* $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $j = 1, 2$. *Then the operators*

$$\mathbf{P}^j_\Sigma : \mathbb{B}^s_{p,p}(\Sigma) \to \mathbb{H}^{s+2-j+1/p}_p(\Omega^+) \cap C^\infty(\Omega^+),$$
$$\mathbf{P}^j_\Sigma : \mathbb{B}^s_{p,q}(\Sigma) \to \mathbb{B}^{s+2-j+1/p}_{p,q}(\Omega^+) \cap C^\infty(\Omega^+), \tag{2.23}$$

$$\mathbf{P}^j_\Sigma : \mathbb{B}^s_{p,p}(\Sigma) \to \mathbb{H}^{s+2-j+1/p}_{p,loc}(\Omega^-) \cap C^\infty(\Omega^-),$$
$$\mathbf{P}^j_\Sigma : \mathbb{B}^s_{p,q}(\Sigma) \to \mathbb{B}^{s+2-j+1/p}_{p,q,loc}(\Omega^-) \cap C^\infty(\Omega^-) \tag{2.24}$$

are bounded and for these extended operators formulas (2.11)–(2.14) *remain valid in the corresponding spaces.*

Representation (2.7) *holds for* $U \in \mathbb{W}^1_{p,loc}(\Omega^\pm)$ *if, in addition,* $\mathbf{A}(D_x, \tau)U = 0$ *in* $\Omega^\pm$.

Proof. Let us first consider $\mathbf{P}^1_\Sigma$ and $s < 0$. Assume that $\varphi \in C^\infty_0(\mathbb{R}^3)$, $\varphi(\xi) = 1$ for $|\xi| \le 1$. If $g \in \mathbb{B}^s_{p,p}(\Sigma)$ ($g \in \mathbb{B}^s_{p,q}(\Sigma)$), we have

$$\begin{aligned}\mathbf{P}^1_\Sigma g &= \Phi * (g \times \delta_\Sigma) = -\mathcal{F}^{-1}\mathcal{A}^{-1}\mathcal{F}(g \times \delta_\Sigma) = \\ &= -\mathcal{F}^{-1}\mathcal{A}^{-1}(\xi)[1 - \varphi(\xi)]\mathcal{F}(g \times \delta_\Sigma) - \\ &- \mathcal{F}^{-1}\mathcal{A}^{-1}(\xi)\varphi(\xi)\mathcal{F}(g \times \delta_\Sigma) \equiv \mathbf{P}^1_{\Sigma,1} g + \mathbf{P}^1_{\Sigma,2} g,\end{aligned} \tag{2.25}$$

where $\mathcal{F}$ ($\mathcal{F}^{-1}$) is the direct (inverse) Fourier transform.

From Lemma 3 it follows that $g \times \delta_\Sigma \in \mathbb{H}^{s-1/p'}_p(\mathbb{R}^3)$ ($g \times \delta_\Sigma \in \mathbb{B}^{s-1/p'}_{p,q}(\mathbb{R}^3)$). Applying the theorem on the boundedness of pseudodifferential operators (see [14]), we obtain

$$\mathbf{P}^1_{\Sigma,1} g \in \mathbb{H}^{s+1+1/p}_p(\mathbb{R}^3) \quad \big(\mathbf{P}^1_{\Sigma,1} g \in \mathbb{B}^{s+1+1/p}_{p,q}(\mathbb{R}^3)\big). \tag{2.26}$$

For the second summand in (2.25) we have

$$\begin{aligned}\mathbf{P}^1_{\Sigma,2} g &= -\mathcal{F}^{-1}\mathcal{A}^{-1}(\xi)\mathcal{F}\big[\mathcal{F}^{-1}(\varphi(\xi)\mathcal{F}(g \times \delta_\Sigma))\big] \equiv \\ &\equiv -\mathcal{F}^{-1}\mathcal{A}^{-1}\mathcal{F} f = \Phi * f \quad \text{with} \quad f = \mathcal{F}^{-1}\varphi\mathcal{F}(g \times \delta_\Sigma).\end{aligned} \tag{2.27}$$

Since the pseudodifferential operator $\mathcal{F}^{-1}\varphi\mathcal{F}$ is of order $-\infty$, we have $f \in C^\infty(\mathbb{R}^3)$ and therefore (see (2.27))

$$\mathbf{A}(D_x)\mathbf{P}^1_{\Sigma,2} g = \mathbf{A}(D_x)(\Phi * f) = (\mathbf{A}(D_x)\Phi) * f = \delta * f = f. \tag{2.28}$$

Thus $\mathbf{P}^1_{\Sigma,2} g$ is the solution of an elliptic system with an infinitely smooth right-hand side. Therefore $\mathbf{P}^1_{\Sigma,2} g \in C^\infty(\mathbb{R}^3)$ (see, for example, [15], Chapter I, Corollary 4.1 or [16], Chapter III, Theorem 1.4).

For any $g \in \mathbb{B}^s_{p,p}(\Sigma)$ ($g \in \mathbb{B}^s_{p,q}(\Sigma)$ and any compact domain $\Omega \subset \mathbb{R}^3$ we obtain $\mathbf{P}^1_\Sigma g \in \mathbb{H}^{s+1+1/p}_p(\Omega)$ ($\mathbf{P}^1_\Sigma g \in \mathbb{B}^{s+1+1/p}_{p,q}(\Omega)$).

It should be noted that the convergence $g_n \xrightarrow{\mathbb{B}^s_{p,q}(\Sigma)} g$ as $n \to \infty$ implies the convergence $\mathbf{P}^1_\Sigma g_n \xrightarrow{\mathcal{D}'(\Omega)} \mathbf{P}^1_\Sigma g$ (see [17, §7]). Therefore the graph of the operator $\mathbf{P}^1_\Sigma : \mathbb{B}^s_{p,p}(\Sigma) \to \mathbb{H}^{s+1+1/p}_p(\Omega)$ ($\mathbf{P}^1_\Sigma : \mathbb{B}^s_{p,p}(\Sigma) \to \mathbb{B}^{s+1+1/p}_{p,q}(\Omega)$) is closed. It remains for us to apply the closed graph theorem (see, for example, [18, Theorem 2.15]).

Let us proceed to the case $s \geq 0$. Assume that $s = m+1/p'$, $m = 0, 1, \dots$. For a function $g \in \mathbb{B}^s_{p,p}(\Sigma)$ we choose a sequence $\{g_n\}_{n\in\mathbb{N}} \subset \mathbb{B}^s_{p,p}(\Sigma) \cap C^{1+\varepsilon}(\Sigma)$, $\varepsilon > 0$, such that $\lim_{n\to\infty} \|(g_n - g)|\mathbb{B}^s_{p,p}(\Sigma)\| = 0$.

By Lemma 1 we have $\mathbf{P}^1_\Sigma g_n \in C^{2+\varepsilon}(\overline{\Omega}) \subset \mathbb{W}^2_p(\Omega)$ representing the solution of the boundary value problem

$$\mathbf{A}(D_x)U(x) = 0, \quad x \in \Omega, \tag{2.29}$$

$$U\big|_\Sigma = \mathbf{P}^1_\Sigma g_n\big|_\Sigma, \tag{2.30}$$

where

$$\mathbf{P}^1_\Sigma g_n\big|_\Sigma \in C^{2+\varepsilon}(\Sigma).$$

By Theorem 2 (see (2.19))

$$\|\mathbf{P}^1_\Sigma g_n|\mathbb{B}^{s+1}_{p,p}(\Sigma)\| \leq C\|g_n|\mathbb{B}^s_{p,p}(\Sigma)\|, \tag{2.31}$$

where $C = const$ is independent of g_n. Using the a priori estimates (see [19], [20] or [21], Chapter V), we obtain $\mathbf{P}^1_\Sigma g_n \in \mathbb{W}^{m+2}_p(\Omega) = \mathbb{H}^{s+1+1/p}_p(\Omega)$ and

$$\|\mathbf{P}^1_\Sigma g_n|\mathbb{H}^{s+1+1/p}_p(\Omega)\| \leq C_1\big(\|\mathbf{P}^1_\Sigma g_n\big|_\Sigma|\mathbb{B}^{s+1}_{p,p}(\Sigma)\| + \|\mathbf{P}^1_\Sigma g_n|\mathbb{L}_p(\Omega)\|\big).$$

The results proved above for $s < 0$ and the embedding theorem yield

$$\|\mathbf{P}^1_\Sigma g_n|\mathbb{L}_p(\Omega)\| \leq C_2\|g_n|\mathbb{B}^s_{p,q}(\Sigma)\|,$$

since $g_n \in \mathbb{B}^{-\varepsilon}_{p,q}(\Sigma)$ implies $\mathbf{P}^1_\Sigma g_n \in \mathbb{B}^{-\varepsilon+1+1/p}_{p,q}(\Omega) \subset \mathbb{L}_p(\Omega)$.

Taking into account (2.31), we arrive at

$$\|\mathbf{P}^1_\Sigma g_n|\mathbb{H}^{s+1+1/p}_p(\Omega)\| \leq C_2\|g_n|\mathbb{B}^s_{p,p}(\Sigma)\|. \tag{2.32}$$

A similar inequality holds for $g_n - g_m$. Therefore $\{\mathbf{P}^1_\Sigma g_n\}_{n\in\mathbb{N}}$ represents a fundamental sequence in $\mathbb{H}^{s+1+1/p}_p(\Omega)$. From the proven part of the theorem and the above embedding theorem we obtain

$$\lim_{n\to\infty} \|(\mathbf{P}^1_\Sigma g - \mathbf{P}^1_\Sigma g_n)|\mathbb{L}_p(\Omega)\| = 0.$$

Therefore $\mathbf{P}^1_\Sigma g \in \mathbb{H}^{s+1+1/p}_p(\Omega)$ and

$$\|\mathbf{P}^1_\Sigma g|\mathbb{H}^{s+1+1/p}_p(\Omega)\| \leq C_2\|g|\mathbb{B}^s_{p,p}(\Sigma)\|. \tag{2.33}$$

For $\mathbf{P}^1_\Sigma$ the proof is completed using interpolation (see [9]). For the operator $\mathbf{P}^2_\Sigma$ the proof is similar, the only difference being that for $s \geq 0$ we should begin with $s = \frac{1}{p'} + 1$ instead of $s = \frac{1}{p'}$ and apply (2.20). □

§ 3. Main Theorems

The next five theorems can be regarded as the main results of both parts of this work. Two of them (Theorems 7 and 8) will be proved in §5.

Theorem 5. *Let φ^+ and φ^- be given vector-functions satisfying* (1.19). *Then $U \in \mathbb{W}^1_{p,loc}(\mathbb{R}^3_S)$ is a solution of Problem $\mathcal{D}$ if and only if*

$$U(x) = (\mathbf{P}^2_S\varphi^0)(x) - (\mathbf{P}^1_S\varphi)(x), \quad x \in \mathbb{R}^3_S, \tag{3.1}$$

where $\varphi^0 = \varphi^+ - \varphi^- \in \widetilde{\mathbb{B}}^{1/p'}_{p,p}(S)$, while $\varphi \in \widetilde{\mathbb{B}}^{-1/p}_{p,p}(S)$ solves the system of pseudodifferential equation

$$\mathbf{P}^1_S\varphi = f \quad on \quad S, \tag{3.2}$$

with

$$f = \mathbf{P}^2_S\varphi^0 - \frac{1}{2}(\varphi^+ + \varphi^-).$$

Proof. By Theorem 4 and formula (2.7) we obtain the following representation of an arbitrary solution U of the homogeneous equation (1.4), $F = 0$,

$$\pm\left\{(\mathbf{P}^2_\Sigma U^\pm)(x) - (\mathbf{P}^1_\Sigma(\mathbf{B}U)^\pm)(x)\right\} = \begin{cases} U(x), & x \in \Omega^\pm, \\ 0, & x \in \Omega^\mp, \end{cases} \tag{3.3}$$

which, upon taking the differences, yields (3.1) with $\varphi^0 = U^+ - U^-$ and $\varphi = (\mathbf{B}U)^+ - (\mathbf{B}U)^-$, if we take into account that

$$(U)^+ - (U)^- = 0, \quad (\mathbf{B}U)^+ - (\mathbf{B}U)^- = 0 \quad \text{on} \quad \Sigma\backslash\overline{S}.$$

Applying Theorem 4, from (3.1) it follows that

$$U^\pm = \pm\frac{1}{2}\varphi^0 + \mathbf{P}^2_S\varphi^0 - \mathbf{P}^1_S\varphi = \varphi^\pm$$

and, upon taking the sum $U^+ + U^-$, we obtain equation (3.2). □

Theorem 6. *Let $1 < p < \infty$ and $\psi^\pm \in \mathbb{B}^{-1/p}_{p,p}(S)$, $\psi^0 = \psi^+ - \psi^- \in \widetilde{\mathbb{B}}^{-1/p}_{p,p}(S)$ be given functions. Then $U \in \mathbb{W}^1_{p,loc}(\mathbb{R}^3_S)$ is a solution of Problem $\mathcal{N}$ if and only if*

$$U(x) = (\mathbf{P}^2_S\psi)(x) - (\mathbf{P}^1_S\psi^0)(x), \quad x \in \mathbb{R}^3_S, \tag{3.4}$$

where $\psi \in \widetilde{\mathbb{B}}^{1/p}_{p,p}(S)$ solves the pseudodifferential equation

$$\mathbf{P}^4_S\psi = g, \tag{3.5}$$

where

$$g = \frac{1}{2}(\psi^+ + \psi^-) + \mathbf{P}_S^3 \psi^0.$$

Proof. The theorem is proved similarly to Theorem 5. We would like only to note that if equation (3.5) has a solution, then ψ and the boundary values $U^\pm$ are related by the equality

$$\psi = U^+ - U^- \in \widetilde{\mathbb{B}}^{1/p}_{p,p}(S). \qquad \square$$

Theorem 7. (i) *The operators*

$$\mathbf{P}_S^1 : \widetilde{\mathbb{B}}^\nu_{p,q}(S) \to \mathbb{B}^{\nu+1}_{p,q}(S), \tag{3.6}$$

$$\mathbf{P}_S^1 : \widetilde{\mathbb{H}}^\nu_p(S) \to \mathbb{H}^{\nu+1}_p(S) \tag{3.7}$$

are bounded for any $1 < p < \infty$, $1 \le q \le \infty$, $\nu \in \mathbb{R}$*;*

(ii) (3.6) *is a Fredholm operator if the condition*

$$1/p - 3/2 < \nu < 1/p - 1/2 \tag{3.8}$$

is fulfilled;

(iii) (3.7) *is a Fredholm operator if and only if condition* (3.8) *is fulfilled;*

(iv) *operators* (3.6) *and* (3.7) *are invertible for all* ν *satisfying* (3.8).

Theorem 8. (i) *The operators*

$$\mathbf{P}_S^4 : \widetilde{\mathbb{B}}^{\nu+1}_{p,q}(S) \to \mathbb{B}^{\nu}_{p,q}(S), \tag{3.9}$$

$$\mathbf{P}_S^4 : \widetilde{\mathbb{H}}^{\nu+1}_p(S) \to \mathbb{H}^{\nu}_p(S) \tag{3.10}$$

are bounded for any $1 < p < \infty$, $1 \le q \le \infty$, $\nu \in \mathbb{R}$*;*

(ii) (3.9) *is a Fredholm operator if* (3.8) *is fulfiled;*

(iii) (3.10) *is a Fredholm operator if and only if* (3.8) *is fulfilled;*

(iv) *operators* (3.9) *and* (3.10) *are invertible for all* ν *satisfying* (3.8).

Let us assume that M is a smooth manifold with the boundary $\partial M \neq \varnothing$ and introduce the notation $\mathcal{H}^s_\infty(M) := \bigcap_{2<p<\infty} \mathbb{H}^s_p(M) = \bigcap_{2<p<\infty} \mathbb{B}^s_{p,q}(M)$, $\widetilde{\mathcal{H}}^s_\infty(M) := \bigcap_{2<p<\infty} \widetilde{\mathbb{H}}^s_p(M) = \bigcap_{2<p<\infty} \widetilde{\mathbb{B}}^s_{p,q}(M)$. Obviously, $\mathcal{H}^s_\infty(M) = \widetilde{\mathcal{H}}^s_\infty(M)$ if $-1/2 \le s \le 0$.

Theorem 9. *Let* $\varphi^\pm \in \mathcal{H}^{1/2}_\infty(S)$, $\varphi^+ - \varphi^- \in \widetilde{\mathcal{H}}^{1/2}_\infty(S)$ *and* $\psi^\pm \in \mathcal{H}^{-1/2}_\infty(S)$. *Then the solutions of Problems* $\mathcal{D}$ *and* $\mathcal{N}$ *are real analytic vectors in* $\mathbb{R}^3_S$, *vanishing at infinity. For their restrictions to* $\Omega^\pm$ *we have the inclusions* $U\big|_{\Omega^\pm} \in \mathcal{H}^{1/2+1/p}_\infty(\Omega^\pm)$.

Therefore

$$U \in \bigcap_{\alpha<1/2} C^\alpha(\mathbb{R}^3_S),$$

where

$$C^\alpha(\mathbb{R}^3_S) := \big\{\varphi \,:\, \varphi \in C^\alpha(\overline{\Omega}^\pm),\ \varphi^+(x) = \varphi^-(x) \ \ if \ x \in \Sigma\backslash S\big\}.$$

Proof. Assume that Theorems 7 and 8 are proved (see §5). Due to representation (3.1) and (3.4) $U(x)$ is a real analytic vector satisfying condition (1.11) at infinity. For its traces we have $U^\pm \in \bigcap\limits_{1<p<\infty} \mathbb{H}^{1/2}_p(S)$. If we now take a sufficiently large p, the proof will follow from Theorems 2 and 4 and the well-known embedding

$$\mathbb{H}^\nu_p(\mathbb{R}^3_S) \subset C^{\nu-\mu}(\mathbb{R}^3_S), \quad \mu > 3/p \tag{3.11}$$

(see [9]). □

Theorem 9 implies that the traces $U^\pm$ on both faces of the crack surface S belong to the Hölder space $\bigcap\limits_{\alpha<1/2} C^\alpha(S)$ and $U^+(x) = U^-(x)$ for $x \in \ell = \partial S$.

Remark 10. Operators (2.16) and (2.18) are invertible for any $k = 0, 1, ...$ and $0 < \gamma < 1$ if $\mathrm{Re}\,\tau > 0$ (see [2]).

Remark 11. Theorem 9 shows the advantage of considering equations (3.2) and (3.5) in the spaces $\mathbb{B}^\nu_{p,p}(S)$ (or $\mathbb{H}^\nu_p(S)$) with $p \neq 2$, since if we stick to the case $p = 2$, we shall not be able to obtain the above results on smoothness for $U|_{\Omega^\pm}$ and $U^\pm$.

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