GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 2, 1997, 129-138

OSCILLATION AND NONOSCILLATION CRITERIA FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions for oscillation and nonoscillation of second-order linear equations are established.

1. Statement of the Problem and Formulation of Basic Results

Consider the differential equation

$$u'' + p(t)u = 0, \tag{1}$$

where $p : [0, +\infty[\to [0, +\infty[$ is an integrable function. By a solution of equation (1) is understood a function $u : [0, +\infty[\to] -\infty, +\infty[$ which is locally absolutely continuous together with its first derivative and satisfies this equation almost everywhere.

Equation (1) is said to be oscillatory if it has a nontrivial solution with an infinite number of zeros, and nonoscillatory otherwise.

It is known (see [1]) that if for some $\lambda < 1$ the integral $\int^{+\infty} s^\lambda p(s)ds$ diverges, then equation (1) is oscillatory. Therefore, we shall always assume below that

$$\int^{+\infty} s^\lambda p(s)ds < +\infty \quad \text{for} \;\; \lambda < 1.$$

1991 *Mathematics Subject Classification.* 34C10.

Key words and phrases. Second-order linear equation, oscillatory and nonoscillatory solutions.

1072-947X/97/0300-0129$12.50/0

Introduce the notation

$$\begin{aligned} h_\lambda(t) &= t^{1-\lambda}\int\limits_t^\infty s^\lambda p(s)ds \quad \text{for} \quad t>0 \quad \text{and} \quad \lambda<1, \\ h_\lambda(t) &= t^{1-\lambda}\int\limits_1^t s^\lambda p(s)ds \quad \text{for} \quad t>0 \quad \text{and} \quad \lambda>1, \end{aligned} \tag{2}$$
$$p_*(\lambda)=\liminf_{t\to+\infty} h_\lambda(t), \quad p^*(\lambda)=\limsup_{t\to+\infty} h_\lambda(t).$$

In [1] it is proved that equation (1) is oscillatory if $p^*(0)>1$ or $p_*(0)>\frac{1}{4}$, and nonoscillatory if $p^*(0)<\frac{1}{4}$. The oscillation criteria for equation (1) written in terms of the numbers $p_*(\lambda)$ and $p^*(\lambda)$ have been established in [2]. Below we shall give the sufficient conditions for oscillation and nonoscillation of equation (1) which make the above-mentioned results of papers [1, 2] more precise and even extend them in some cases.

First of all, for the completenes of the picture we give a proposition, which slightly generalizes one of E. Hille's theorems [1].

Proposition. *Let either $p_*(0)>\frac{1}{4}$ or $p_*(2)>\frac{1}{4}$. Then equation (1) is oscillatory.*

Theorem 1. *Let $p_*(0)\le\frac{1}{4}$ and $p_*(2)\le\frac{1}{4}$. Moreover, let either*

$$p^*(\lambda)>\frac{\lambda^2}{4(1-\lambda)}+\frac{1}{2}\Big(1+\sqrt{1-4p_*(2)}\Big) \tag{3}$$

for some $\lambda<1$ or

$$p^*(\lambda)>\frac{\lambda^2}{4(\lambda-1)}-\frac{1}{2}\Big(1-\sqrt{1-4p_*(0)}\Big) \tag{4}$$

for some $\lambda>1$. Then equation (1) is oscillatory.

Corollary 1. *Let either*

$$\lim_{\lambda\to1-}(1-\lambda)p^*(\lambda)>\frac{1}{4} \tag{5}$$

or

$$\lim_{\lambda\to1+}(\lambda-1)p^*(\lambda)>\frac{1}{4} \tag{6}$$

Then equation (1) is oscillatory.

Corollary 2 ([2]). *For some $\lambda \neq 1$ let*

$$|1-\lambda| p_*(\lambda) > \frac{1}{4} \tag{7}$$

Then equation (1) *is oscillatory.*

Remark 1. Inequalities (5)–(7) are exact and cannot be weakened. Indeed, let $p(t) = \frac{1}{4t^2}$ for $t \geq 1$. Then $|1-\lambda| p_*(\lambda) = \frac{1}{4}$, and equation (1) has oscillatory solution $u(t) = \sqrt{t}$ for $t > 1$.

Theorem 2. *Let $p_*(0) \leq \frac{1}{4}$ and $p_*(2) \leq \frac{1}{4}$. Moreover, let either*

$$\begin{aligned} &p_*(0) > \frac{\lambda(2-\lambda)}{4} \quad \textit{and} \\ &p^*(\lambda) > \frac{p_*(0)}{1-\lambda} + \frac{1}{2}\Big(\sqrt{1-4p_*(0)} + \sqrt{1-4p_*(2)}\Big) \end{aligned} \tag{8}$$

for some $\lambda < 1$ or

$$\begin{aligned} &p_*(2) > \frac{\lambda(2-\lambda)}{4} \quad \textit{and} \\ &p^*(\lambda) > \frac{p_*(2)}{\lambda-1} + \frac{1}{2}\Big(\sqrt{1-4p_*(0)} + \sqrt{1-4p_*(2)}\Big) \end{aligned} \tag{9}$$

for some $\lambda > 1$. Then equation (1) *is oscillatory.*

Theorem 3. *Let $p_*(0) \neq 0$ and $p_*(2) \leq \frac{1}{4}$. Moreover, for some $0 < \lambda < 1$ let $p_*(\lambda) < \frac{1-\lambda^2}{4}$ and either*

$$p_*(\lambda) > \frac{p_*(0)}{1-\lambda} + \frac{\lambda}{2(1-\lambda)}\Big(\sqrt{1-4p_*(0)} + \sqrt{1-4p_*(2)}\Big)$$

and

$$\begin{aligned} p^*(\lambda) > p_*(\lambda) &+ \frac{1}{2}\Big(\lambda + \sqrt{1-4p_*(2)}\Big) + \\ &+ \sqrt{\lambda^2 + 1 - 4(1-\lambda)p_*(\lambda) + 2\lambda\sqrt{1-4p_*(2)}} \end{aligned}$$

or

$$p_*(\lambda) < \frac{p_*(0)}{1-\lambda} + \frac{\lambda}{2(1-\lambda)}\Big(\sqrt{1-4p_*(0)} + \sqrt{1-4p_*(2)}\Big)$$

and

$$p^*(\lambda) > \frac{p_*(0)}{1-\lambda} + \frac{1}{2(1-\lambda)}\Big(\sqrt{1-4p_*(0)} + \sqrt{1-4p_*(2)}\Big). \tag{10}$$

Then equation (1) *is oscillatory.*

Theorem 3′. *Let* $p_*(0) \le \frac{1}{4}$ *and* $p_*(2) \le \frac{1}{4}$. *Moreover, for some* $0 < \lambda < 1$ *let condition* (10) *be fulfilled, and let* $p_*(0) > \frac{1-\lambda^2}{4}$. *Then equation* (1) *is oscillatory.*

Corollary 3. *Let* $p_*(0) \le \frac{1}{4}$, $p_*(2) \le \frac{1}{4}$ *and*

$$p^*(0) > p_*(0) + \frac{1}{2}\Big(\sqrt{1-4p_*(0)} + \sqrt{1-4p_*(2)}\Big).$$

Then equation (1) *is oscillatory.*

Corollary 4. *For some* $\lambda \in [0, \frac{1}{4}[$ *let*

$$\frac{\lambda}{1-\lambda} < p_*(\lambda) < \frac{1}{4(1-\lambda)}$$

and

$$p^*(\lambda) > 1 + p_*(\lambda) - \frac{1}{2}\Big(1 - \lambda - \sqrt{(1+\lambda)^2 - 4(1-\lambda)p_*(\lambda)}\Big).$$

Then equation (1) *is oscillatory.*

Theorem 4. *For some* $\lambda \ne 1$ *let*

$$p_*(\lambda) > \frac{(2\lambda-1)(3-2\lambda)}{4|1-\lambda|} \quad \textit{and} \quad p^*(\lambda) < \frac{1}{4|1-\lambda|}. \tag{11}$$

Then equation (1) *is nonoscillatory.*

Remark 2. As will be seen from the proof, Theorem 4 remains also valid when the function p, generally speaking, does not have a constant sign. For such a case this result for $\lambda = 0$ is described in [3].

Corollary 5. *Let* $p_*(0) < \frac{1}{4}$ *and* $p_*(2) < \frac{1}{4}$, *and let the inequality*

$$p^*(\lambda) < \frac{1}{4|1-\lambda|}$$

hold for some $\lambda \in]-\infty, 1-\sqrt{\frac{1}{4}-p_*(0)}[\cup]1+\sqrt{\frac{1}{4}-p_*(2)}, +\infty[$. *Then equation* (1) *is nonoscillatory.*

2. Some Auxiliary Propositions

Lemma 1. *For equation* (1) *to be nonoscillatory, it is necessary and sufficient that for some* $\lambda \neq 1$ *the equation*

$$v'' = \frac{1}{t^2}\big(-h_\lambda^2(t) + \lambda \operatorname{sgn}(1-\lambda) h_\lambda(t)\big)v - \frac{2\operatorname{sgn}(1-\lambda)}{t} h_\lambda(t) v', \quad (12)$$

where h_λ *is the function defined by* (2), *be nonoscillatory.*

Proof. The equality $\rho(t) = t^\lambda \frac{u'(t)}{u(t)} - t^{\lambda-1} h_\lambda(t) \operatorname{sgn}(1-\lambda)$ determines the relation between the nonoscillatory solution u of equation (1) and the solution ρ, defined in some neighborhood of $+\infty$, of the equation

$$\begin{aligned} \rho' = -t^{-\lambda}\rho^2 + \lambda t^{-1}\rho - 2\operatorname{sgn}(1-\lambda) t^{-1} h_\lambda^2(t)\rho - \\ - t^{\lambda-2} h_\lambda^2(t) + \lambda \operatorname{sgn}(1-\lambda) t^{\lambda-1} h_\lambda(t). \end{aligned} \quad (13)$$

On the other hand, the equality $\rho(t) = t^\lambda \frac{v'(t)}{v(t)}$ determines the relation between the nonoscillatory solution v of equation (11) and the solution ρ defined in some neighborhood $+\infty$ of equation (13). Thus nonoscillation of either of equation (1) or (12) results in nonoscillation of the other. □

Lemma 2. *Let equation* (1) *be nonoscillatory. Then there exists* $t_0 > 0$ *such that the equation*

$$\rho' + p(t)\rho + \rho^2 = 0 \quad (14)$$

has a solution $\rho :]t_0, +\infty[\to [0, +\infty[$*; moreover,*

$$\rho(t_0+) = +\infty, (t - t_0)\rho(t) < 1 \quad \textit{for} \quad t_0 < t < +\infty, \quad (15)$$

$$\lim_{t\to+\infty} t^\lambda \rho(t) = 0 \quad \textit{for} \quad \lambda < 1 \quad (16)$$

and

$$\liminf_{t\to+\infty} t\rho(t) \geq A, \quad \limsup_{t\to+\infty} t\rho(t) \leq B, \quad (17)$$

where

$$A = \frac{1}{2}\Big(1 - \sqrt{1 - 4p_*(0)}\Big), \quad B = \frac{1}{2}\Big(1 + \sqrt{1 - 4p_*(2)}\Big)^{1}. \quad (18)$$

[1] Since equation (1) is nonoscillatory, we have $p_*(0) \leq \frac{1}{4}$ and $p_*(2) \leq \frac{1}{4}$.

Proof. Since equation (1) is nonoscillatory, there exists $t_0 > 0$ such that the solution u of equation (1) under the initial conditions $u(t_0) = 0$, $u'(t_0) = 1$ satisfies the inequalities

$$u(t) > 0, \quad u'(t) \ge 0 \quad \text{for} \quad t_0 < t < +\infty.$$

Clearly, the function $\rho(t) = \frac{u'(t)}{u(t)}$ for $t_0 < t < +\infty$ is the solution of equation (14), and $\lim_{t \to t_o+} = +\infty$. From (14) we have

$$\frac{-\rho'(t)}{\rho^2(t)} > 1 \quad \text{for} \quad t_0 < t < +\infty.$$

Integrating the above inequality from t_0 to t, we obtain $(t - t_0)\rho(t) < 1$ for $t_0 < t < +\infty$. In particular, equality (16) holds for any $\lambda < 1$.

Let us now show that inequalities (17) are valid. Assume $p_*(0) \ne 0$ and $p_*(2) \ne 0$ (inequalities (17) are trivial, otherwise). Let us introduce the notation

$$r = \liminf_{t \to +\infty} t\rho(t), \quad R = \limsup_{t \to +\infty} t\rho(t).$$

From (14) we easily find that for any $t_1 > t_0$

$$\begin{aligned} t\rho(t) &= t \int_t^{+\infty} p(s)ds + t \int_t^{+\infty} \rho^2(s)ds, \\ t\rho(t) &= \frac{t_1^2\rho(t_1)}{t} - t^{-1} \int_{t_1}^{t} s^2 p(s)ds + t^{-1} \int_{t_1}^{t} s\rho(s)(2 - s\rho(s))ds \\ &\qquad \text{for} \quad t_1 < t < +\infty. \end{aligned} \tag{19}$$

This implies that $r \ge p_*(0)$ and $R \le 1 - p_*(2)$.

It is easily seen that for any $0 < \varepsilon < \min\{r, 1 - R\}$ there exists $t_\varepsilon > t_1$ such that

$$r - \varepsilon < t\rho(t) < R + \varepsilon, \quad t \int_t^{+\infty} p(s)ds > P_*(0) - \varepsilon,$$

$$\frac{1}{t} \int_{t_1}^{t} s^2 p(s)ds > p_*(2) - \varepsilon \quad \text{for} \quad t_\varepsilon < t < +\infty.$$

Taking into account the above argument, from (19) we have

$$t\rho(t) > p_*(0) - \varepsilon + (r - \varepsilon)^2 \quad \text{for} \quad t_\varepsilon < t < +\infty,$$

$$t\rho(t) < \frac{t_\varepsilon^2\rho(t_\varepsilon)}{t} - p_*(2) + \varepsilon + (R + \varepsilon)(2 - R - \varepsilon) \quad \text{for} \quad t_\varepsilon < t < +\infty,$$

whence

$$r \geq p_*(0) + r^2, \quad R \leq -p_*(2) + R(2-R),$$

that is, $r \geq A$ and $R \leq B$, where A and B are the numbers defined by equalities (18). Hence (17) holds. $\square$

Lemma 3. *Let the functions $g, q : [a, +\infty[\to R$ be integrable in every finite interval, and let $v : [a, +\infty[\to]0, +\infty[$ be absolutely continuous together with its first derivative on every compactum contained in $[a, +\infty[$. Moreover, let the inequality*

$$v''(t) \leq g(t)v(t) + q(t)v'(t) \tag{20}$$

hold almost everywhere in $[a, +\infty[$. Then the equation $u'' = g(t)u + q(t)u'$ is nonoscillatory.

3. Proof of the Basic Results

Proof of Theorem 1. Assume the contrary. Let equation (1) be nonoscillatory. Then, according to Lemma 2, equation (14) has the solution $\rho :]t_0, +\infty[\to [0, +\infty[$ satisfying conditions (15)–(17). Suppose $\lambda < 1$ $(\lambda > 1)$. Because of (17) we have that for any $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that

$$A - \varepsilon < t\rho(t) < B + \varepsilon \quad \text{for} \quad t_\varepsilon < t < +\infty. \tag{21}$$

Multiplying equality (14) by t^λ, integrating it from t to $+\infty$ (from t_ε to t), and taking into account (15)–(17), we get

$$\int_t^{+\infty} s^\lambda p(s)ds = t^\lambda \rho(t) + \frac{\lambda^2 t^{\lambda-1}}{4(1-\lambda)} - \int_t^{+\infty} (s^{\frac{\lambda}{2}}\rho(s) - \frac{1}{2}s^{\frac{\lambda}{2}-1})^2 ds <$$

$$< t^{\lambda-1}\big(B + \varepsilon + \frac{\lambda^2}{4(1-\lambda)}\big) \quad \text{for} \quad t_\varepsilon < t < +\infty$$

$$\Bigg(\int_{t_\varepsilon}^t s^\lambda p(s)ds < t^{\lambda-1}\big(\frac{\lambda^2}{4(1-\lambda)} - A + \varepsilon + t_\varepsilon^\lambda \rho(t_\varepsilon)\big)$$

$$\text{for} \quad t_\varepsilon < t < +\infty \quad \Bigg),$$

whence we have $p^*(\lambda) \leq \frac{\lambda^2}{4(1-\lambda)} + \frac{1}{2}(1 + \sqrt{1 - 4p_*(2)})$ $(p^*(\lambda) \leq \frac{\lambda^2}{4(\lambda-1)} - \frac{1}{2}(1 - \sqrt{1 - 4p_*(0)}))$, which contradicts equality (3) ((4)). $\square$

To convince ourselves that Corollary 1 is valid, let us note that (5) ((6)) imply

$$\lim_{\lambda\to 1+}\Big[(1-\lambda)p^*(\lambda)-\frac{\lambda^2}{4}-\frac{1-\lambda}{2}\big(1+\sqrt{1-4p_*(2)}\big)\Big]>0$$
$$\Big(\lim_{\lambda\to 1-}\Big[(\lambda-1)p^*(\lambda)-\frac{\lambda^2}{4}-\frac{\lambda-1}{2}\big(1+\sqrt{1-4p_*(0)}\big)\Big]>0\Big).$$

Consequently, (3) ((4)) is fulfilled for some $\lambda<1$ $(\lambda>1)$. Thus, according to Theorem 1, equation (1) is oscillatory. As for Corollary 2, taking into account that the mapping $\lambda\longmapsto(1-\lambda)p_*(\lambda)$ for $\lambda<1$ $(\lambda\longmapsto(\lambda-1)p_*(\lambda)$ for $\lambda>1$) is non-decreasing (non-increasing), we easily find from (7) that (5) ((6)) is fulfilled for some λ.

Proof of Theorem 2. Assume the contrary. Let equation (1) be nonoscillatory. Then according to Lemma 2, equation (14) has the solution $\rho:\,]t_0,+\infty[\to[0,+\infty[$ satisfying conditions (15)–(17). Suppose $\lambda<1$ $(\lambda>1)$. By the conditions of the theorem, $p_*(0)>\frac{\lambda(2-\lambda)}{4}$ $(p_*(2)>\frac{\lambda(2-\lambda)}{4})$, which implies that $A>\frac{\lambda}{2}$ $(B<\frac{\lambda}{2})$. On account of (17), for any $0<\varepsilon<A-\frac{\lambda}{2}$ $(0<\varepsilon<\frac{\lambda}{2}-B)$ there exists $t_\varepsilon>t_0$ such that (21) holds.

Multiplying equality (14) by t^λ, integrating it from t to $+\infty$ (from t_ε to t), and taking into account (15)–(17), we easily find that

$$t^{1-\lambda}\int_t^{+\infty}s^\lambda p(s)ds=t\rho(t)+t^{1-\lambda}\int_t^{+\infty}s^{\lambda-2}s\rho(s)(\lambda-s\rho(s))ds<$$
$$<B+\varepsilon+\frac{1}{1-\lambda}(A-\varepsilon)(\lambda-A+\varepsilon)\quad\text{for}\quad t_\varepsilon<t<+\infty$$
$$\Big(t^{1-\lambda}\int_{t_\varepsilon}^{t}s^\lambda p(s)ds<\varepsilon-A+\frac{1}{\lambda-1}(B+\varepsilon)(\lambda-B-\varepsilon)+$$
$$+t^{1-\lambda}t_\varepsilon\rho(t_\varepsilon)\quad\text{for}\quad t_\varepsilon<t<+\infty\Big).$$

This implies

$$p^*(\lambda)\le\frac{p_*(0)}{1-\lambda}+\frac{1}{2}\Big(\sqrt{1-4p_*(0)}+\sqrt{1-4p_*(2)}\Big)$$
$$\Big(p^*(\lambda)\le\frac{p_*(2)}{\lambda-1}+\frac{1}{2}\Big(\sqrt{1-4p_*(0)}+\sqrt{1-4p_*(2)}\Big)\Big),$$

which contradicts condition (8) ((9)). □

Proof of Theorems 3 *and* 3′. Assume the contrary. Let equation (1) be nonoscillatory. Then according to Lemma 2, equation (14) has the solution $\rho :]t_0, +\infty[\to [0, +\infty[$ satifying conditions (15)–(17). Multiplying equality (14) by t^λ, integrating it from t to $+\infty$, and taking into account (16), we easily obtain

$$t\rho(t) = h_\lambda(t) - \lambda t^{1-\lambda} \int_t^{+\infty} s^{\lambda-1}\rho(s)ds + \\ + t^{1-\lambda} \int_t^{+\infty} s^\lambda \rho^2(s)ds \quad \text{for} \quad t_0 < t < +\infty, \tag{22}$$

where h_λ is the function defined by equality (2).

Introduce the notation

$$r = \liminf_{t\to+\infty} t\rho(t).$$

On account of (17) we have $r > 0$. Therefore for any $0<\varepsilon<\max\{r, p_*(\lambda)\}$ there exists $t_\varepsilon > t_0$ such that

$$r - \varepsilon < t\rho(t) < B + \varepsilon, h_\lambda(t) > p_*(\lambda) - \varepsilon \quad \text{for} \quad t_\varepsilon < t < +\infty.$$

Owing to the above arguments, we find from (22) that

$$(1 - \lambda)h_\lambda < B + \varepsilon - (r - \varepsilon)^2 \quad \text{for} \quad t_\varepsilon < t < +\infty,$$
$$t\rho(t) > p_*(\lambda) - \varepsilon - \frac{\lambda}{1-\lambda}(B + \varepsilon) + \frac{1}{1-\lambda}(r - \varepsilon)^2 \quad \text{for} \quad t_\varepsilon < t < +\infty,$$

which implies

$$p^*(\lambda) \le \frac{B - r^2}{1 - \lambda}, \tag{23}$$

$$r \ge p_*(\lambda) - \frac{\lambda}{1-\lambda}B + \frac{r^2}{1-\lambda}.$$

The latter inequality results in $r \ge x_1$, where x_1 is the least root of the equation

$$\frac{1}{1-\lambda}x^2 - x + p_*(\lambda) - \frac{\lambda}{1-\lambda}B = 0.$$

Thus $r \ge \max\{A, x_1\}$. From (23) we have that if $A < x_1$, then

$$p^*(\lambda) \le B + p_*(\lambda) - x_1,$$

but if $A \ge x_1$, then

$$p^*(\lambda) \le \frac{1}{1-\lambda}B - \frac{1}{1-\lambda}A^2,$$

which contradicts the conditions of the theorem. □

Proof of Theorem 4. From (11) we have that for some $t_0 > 0$

$$\frac{(2\lambda-1)(3-2\lambda)}{4|1-\lambda|} < h_\lambda(t) < \frac{1}{4|1-\lambda|} \quad \text{for} \quad t_0 < t < +\infty,$$

whence

$$h_\lambda^2(t) + \frac{2\lambda^2-4\lambda+1}{2|1-\lambda|}h_\lambda(t) + \frac{(2\lambda-1)(3-2\lambda)}{16(1-\lambda)^2} < 0 \quad \text{for} \quad t_0 < t < +\infty.$$

Taking into consideration the latter inequality, we can easily see that (20) holds, where

$$v(t) = t^{\frac{1-2\lambda}{4(1-\lambda)}} \quad \text{for} \quad t_0 < t < +\infty,$$

$$g(t) = -\frac{1}{t^2}\big(h_\lambda^2(t) - \lambda\,\mathrm{sgn}(1-\lambda)h_\lambda(t)\big) \quad \text{for} \quad t_0 < t < +\infty,$$

and

$$q(t) = -\frac{2}{t}\,\mathrm{sgn}(1-\lambda)h_\lambda(t) \quad \text{for} \quad t_0 < t < +\infty.$$

Consequently, according to Lemmas 1 and 3, equation (1) is nonoscillatory. □

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(Received 03.05.1995)

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