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HYPERPLANE SINGULARITIES OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. A new class of non-isolated singularities called hyperplane singularities is introduced. Special deformations with simplest critical points are constructed and an algebraic expression for the number of Morse points is given. The topology of the Milnor fibre is completely studied.

0. Introduction

This paper continues the investigation of special classes of non-isolated singularities.

In [1] and [2] germs of analytic functions having a smooth one-dimensional submanifold as a singular set were investigated, the simplest ones of such germs being obtained as limits of simple isolated singularities of series A_k and D_k. In our work germs of analytic functions $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ are considered, having singularities on the hyperplane

$$H = \{(x, y_1, y_2, \dots, y_n) \in \mathbb{C} \times \mathbb{C}^n \mid x = 0\}.$$

Such singularities are called *hyperplane singularities*.

The paper is divided into 6 sections.

In Section 1, coordinate transformations preserving the singular hyperplane H are introduced and the equivalence of germs under such transformations is defined. Moreover, simplest germs of A_∞ (local expression x^2) and D_∞ (local expression $x^2 y_1$) types are determined.

In Section 2 the notion of an isolated hyperplane singularity is introduced and investigated.

In Section 3, for isolated hyperplane singularities, a special deformation is constructed, having only A_∞ and D_∞ type singular points on the hyperplane H and only Morse points outside H.

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In Section 4 the number of Morse points is calculated for special deformation of f.

In Section 5 the topology of the Milnor fibre is studied using the special deformation. It is shown that the Milnor fibre is homotopy equivalent to the wedge of a circle S^1 with $2\mu + \sigma$ copies of the sphere S^n, where $\mu = \mu(g)$ is the Milnor number of the isolated singularity $g(0, y_1, \dots, y_n)$, while σ is the number of Morse points of the deformation of f. To this end we investigate the problem of determining a homotopy type of the complement of a nonsingular (smooth) submanifold.

In Section 6 consideration is given to germs of analytic functions representable as $f = x^k g(x, y_1, \dots, y_n)$, called hyperplane singularities of transversal type A_k. For such singularities all the results obtained in Sections 1–5 are generalized.

1. Hyperplane Singularities

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic complex function of $n+1$ complex variables $(x, y_1, \dots, y_n)$ and let the hyperplane $H = \{(x, y_1, \dots, y_n) \mid x = 0\}$ consist of points $z \in \mathbb{C}^{n+1}$ such that $\operatorname{grad} f(z) = 0$.

In the ring $\mathcal{O}_{n+1}$ of all germs at zero of holomorphic functions, single out the ideals

$$\mathbf{m} = \{f \in \mathcal{O}_{n+1} \mid f(0) = 0\},$$
$$(x) = \{f \in \mathcal{O}_{n+1} \mid f(0, y_1, \dots, y_n) = 0, \forall (y_1, \dots, y_n) \in \mathbb{C}^{n+1}\}.$$

We are going to investigate elements from the ideal (x^2). The following characterization of these elements is valid:

Lemma 1.1. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function, having H as its singular set. Then such a germ can be represented in the form $f = x^2 g(x, y_1, \dots, y_n)$, where g is a smooth germ from the ring $\mathcal{O}_{n+1}$.*

In the group D_{n+1} of germs of local diffeomorphisms of $\mathbb{C}^{n+1}$ at the origin, consider a subgroup consisting of diffeomorphisms $\varphi \in D_{n+1}$ satisfying $\varphi(H) = H$. In the following all the coordinate transformations considered will preserve the hyperplane H, i.e., belong to D_H.

Let us introduce some definitions.

Definition 1.2. A singular point $z \in H$ is called a point of type A_∞, if $\operatorname{Hess}_x f = \frac{\partial^2 f}{\partial x^2}(x, y_1, \dots, y_n)$ is nonzero at this point.

Definition 1.3. A singular point is called a point D_∞ if the gradient of the function $\operatorname{Hess}_x f$ with respect to variables y_i, $i = 1, \dots, n$, written as

$$\left(\frac{\partial}{\partial y_1} \left(\frac{\partial^2 f}{\partial x^2} \right), \frac{\partial}{\partial y_2} \left(\frac{\partial^2 f}{\partial x^2} \right), \dots, \frac{\partial}{\partial y_n} \left(\frac{\partial^2 f}{\partial x^2} \right) \right),$$

is not equal to zero at this point.

The following simple assertions are easy to prove.

Proposition 1.4. *A singular point $z \in H$ is of the type A_∞ if and only if in some neighborhood of z there exists a coordinate transform from the group D_H which reduces f to x^2.*

Proposition 1.5. *A singular point $z \in H$ has type D_∞ if and only if in some neighborhood of z there is a coordinate transform with respect to H which changes f to $x^2 y_1$.*

Let $\operatorname{Orb}(f)$ denote an orbit of the germ f under the action of D_H. As always the simplest orbits are of interest.

Having in mind to characterize isolated singularities, and in accord with the finite dimensional case, let us introduce a measure for germ complexity.

Definition 1.6. The number

$$\operatorname{codim}(f) = \dim_{\mathbb{C}} \left[(x^2)/\tau(f) \right],$$

where $\tau(f)$ is the tangent space to $\operatorname{Orb}(f)$ at f, is called the codimension of a hyperplane singularity f.

2. Isolated Hyperplane Singularities

Now we can give a simple criterion for finite determinacy:

Theorem 2.1. *Let $f \in (x^2)$ be a hyperplane singularity not of type A_∞ or D_∞ such that in the presentation $f = x^2 g(x, y_1, \dots, y_n)$ the germ $g(0, y_1, \dots, y_n)$ is, as a germ of a function of $y_1, \dots, y_n$, an isolated singularity. Then the following assertions are equivalent:*

(a) codim f *is finite;*

(b) *the function $g(x, y_1, \dots, y_n)$ has an isolated singularity;*

(c) *f has a singularity of type A_∞ outside points with $g(0, y_1, \dots, y_n) = 0$ and a singularity of typeD_∞ at points with $g(0, y_1, \dots, y_n) = 0$, except the origin.*

Proof. Let us show that (a) implies (b). Indeed, if $\operatorname{codim} f < +\infty$, then $\dim_{\mathbb{C}} \left[(x^2)/\tau(f)\right] < +\infty$, where $\tau(f)$ has type $(\xi x g + \xi x^2 g_x, \eta_1 x^2 g_{y_1}, \ldots, \eta_n x^2 g_{y_n})$, $\xi \in (x)$, $\eta_i \in m$, $i = 1, \ldots, n$. To the function g associate the ideal $(g_x, g_{y_1}, \ldots, g_{y_n})$; according to Briançon–Skoda's theorem [3], $g^{n+1} \in (g_x, g_{y_1}, \ldots, g_{y_n})$. Clearly, this implies $\tau^{n+1}(f) \in (x^2 g_x, x^2 g_{y_1}, \ldots, x^2 g_{y_n})$ and since

$$\dim_{\mathbb{C}} \left[(x^2)/\tau^{n+1}(f)\right] = \dim_{\mathbb{C}} \left[(x^2)/\tau(f)\right] + \dim_{\mathbb{C}} \left[\tau(f)/\tau^{n+1}(f)\right] < +\infty,$$

one obtains

$$\dim_{\mathbb{C}} \left[(x^2)/(x^2 g_x, x^2 g_{y_1}, \ldots, x^2 g_{y_n})\right] = \dim_{\mathbb{C}}^{\mathcal{O}_{n+1}} \left[(g_x, g_{y_1}, \ldots, g_{y_n})\right] < +\infty;$$

hence $g(x, y_1, \ldots, y_n)$ has an isolated singularity.

(b)⇒(c). Let g have an isolated singularity and $g(0, y_1, \ldots, y_n)$ have an isolated singularity at zero, i.e., $\operatorname{grad} g(0, y_1, \ldots, y_n) = 0$ only at the origin. Then for an arbitrary point z of the space $\{g(0, y_1, \ldots, y_n) = 0\}$ the gradient of this function will be nonzero; assume, for definiteness, that $\frac{\partial g}{\partial y_1}(0, y_1, \ldots, y_n)$ is not zero at z and consider a transformation from the group D_H

$$\widetilde{x} = x, \widetilde{y}_1 = g(x, y_1, \ldots, y_n), \widetilde{y}_i = y_i, i = 2, \ldots, n,$$

whose Jacobian is $\frac{\partial g}{\partial y_1}(0, y_1, \ldots, y_n) \neq 0$ and reduces f to the form $\widetilde{x}^2 \widetilde{y}_1$, i.e., f has type D_∞ at the point z.

At the points outside the set $g(0, y_1, \ldots, y_n) = 0$ on the singular hyperplane $x = 0$, consider the element of the group D_H determined by

$$\widetilde{x} = x\sqrt{g(x, y_1, \ldots, y_n)}, \widetilde{y} = y_i, \quad i = 1, \ldots, n,$$

whose Jacobian equals $\sqrt{g(0, y_1, \ldots, y_n)} \neq 0$ and which transforms f to the function $\widetilde{x}^2$, i.e., f has type A_∞ at these points.

(c)⇒(a). Let f be some representative of the germ of a given hyperplane singularity. Define on its domain a sheaf of $\mathcal{O}_{n+1}$-modules as follows:

$$\mathcal{F}(u) = (x^3)/\left(\tau(f) \cap (x^3)\right),$$

where (x^3) and $\tau(f)$ are considered as modules over the ring of holomorphic functions on $u \subset \mathbb{C}^{n+1}$, while $\mathcal{O}_{n+1}$ is the sheaf of holomorphic functions on $\mathbb{C}^{n+1}$. The sheaf $\mathcal{F}$ is coherent, hence we may use the fact that $\mathcal{F}$ has support consisting of a single point if and only if $\dim \Gamma(\mathcal{F}) < \infty$, where as usual $\Gamma(\mathcal{F})$ denotes the space of sections of $\mathcal{F}$ over u.

For $x \neq 0$ the function f is regular at $p = (x, y_1, \ldots, y_n)$, and one has $\dim \mathcal{F}_p = 0$; hence $(x^3) \cong (\mathcal{O}_{n+1})_p$ and $\tau(f) \cong (\mathcal{O}_{n+1})_p$. If $x = 0$, but $(y_1, \ldots, y_n)$ does not belong to the space $\{g(0, y_1 \ldots, y_n) = 0\}$, then the germ of f is right equivalent to x^2 under the action of D_H, and $(x^2) \cong \tau(f)$; hence at this points $\dim \mathcal{F}_p = 0$. Suppose now that $x = 0$ and $g(0, y_1, \ldots, y_n) = 0$. Then f is right equivalent to the germ of $x^2 y_1$, if

$(x, y_1, \dots, y_n) \neq (0, \dots, 0)$; hence outside the origin one obtains $\tau(f) \cong (x^3)$ and consequently $\dim_{\mathbb{C}} \mathcal{F}_p = 0$. Hence the sheaf $\mathcal{F}$ has support 0, whence, by the above remark about one-point supported sheaves, one concludes that $\dim_{\mathbb{C}} (x^3)/\big((x^3) \cap \tau(f)\big) < \infty$. This implies the finiteness of multiplicity of the hyperplane singularity f. □

Definition 2.2. A hyperplane singularity $f = x^2 g(x, y_1, \dots, y_n)$ not of type A_∞ or D_∞ is called isolated if both $g(0, y_1, \dots, y_n)$ and $g(x, y_1, \dots, y_n)$ have isolated singularities.

3. Deformations of Hyperplane Singularities

For isolated hyperplane singularities one can construct special deformations having singular points of type A_1, A_∞, D_∞.

Theorem 3.1. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated hyperplane singularity $f = x^2 g(x, y_1, \dots, y_n)$. Then there exists a deformation $f_\lambda, \lambda \in \mathbb{C}^{n+1}$, within the class of isolated hyperplane singularities, which has singular points of types A_∞ and D_∞ on the hyperplane H and only Morse points outside H, and such a deformation can be given in the form*

$$f_\lambda = x^2(g(x, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1}),$$

where $\lambda_1, \dots, \lambda_{n+1}$ are sufficiently small complex numbers.

Proof. Suppose that on the hyperplane H one has

$$g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1} \neq 0;$$

then consider the transformation from the group D_H given by

$$\widetilde{x} = x\sqrt{g(x, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1}},$$
$$\widetilde{y}_i = y_i, i = 1, \dots, n.$$

Since $\frac{\partial \widetilde{x}}{\partial x}\Big|_{x=0} = \sqrt{g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1}} \neq 0$, the Jacobian of this transformation is nonzero, and in these coordinates the singularity $\widetilde{f}$ has type A_∞.

Now suppose $g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1} = 0$ and choose λ_{n+1} from $\operatorname{Reg}(g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n)$ and $\lambda_i, i = 1, \dots, n$, from $\operatorname{Reg}\operatorname{grad}(g(0, y_1, \dots, y_n))$. This is possible, since f has an isolated hyperplane singularity and hence $g(x, y_1, \dots, y_n)$ and $g(0, y_1, \dots, y_n)$ have isolated singularities. Supposing, for definiteness, that $\frac{\partial g}{\partial y_1}(0, y_1, \dots, y_n) + \lambda_1 = 0$, and consider a transformation of the form

$$\widetilde{x} = x,$$
$$\widetilde{y}_i = g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1},$$
$$\widetilde{y}_i = y_i, i = 2, \dots, n.$$

This is an element of D_H, with the Jacobian $\frac{\partial g}{\partial y_1} + \lambda_1$, which is nonzero.

In the new coordinates, f_λ has only D_∞ type singular points on the smooth submanifold

$$\{(y_1, \dots, y_n) \in \mathbb{C} \mid g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1} = 0\},$$

while outside the submanifold the function f_λ has only A_∞ type singularities on H.

Consider the whole critical set of f_λ. It consists of a singular hyperplane H and a set determined by the system of equations

$$\begin{cases} x \neq 0 \\ 2g + 2\lambda_1 y_1 + \dots + 2\lambda_{n+1} + x g_x = 0 \\ g_{y_1} = -\lambda_1 \\ \vdots \\ g_{y_n} = -\lambda_n \end{cases} .$$

At these points $\mathrm{Hess} f_\lambda$ has the form

$$\mathrm{Hess} f_\lambda = \begin{vmatrix} 3x g_x + x^2 g_{xx} & x^2 g_{xy_1} & \dots & x^2 g_{xy_n} \\ x^2 g_{y_1 x} & x^2 g_{y_1 y_1} & \dots & x^2 g_{y_1 y_n} \\ \vdots & \vdots & \ddots & \vdots \\ x^2 g_{y_n x} & x^2 g_{y_n y_1} & \dots & x^2 g_{y_n y_n} \end{vmatrix} .$$

Consequently the set $\{\mathrm{Hess} f_\lambda = 0\}$ does not depend on $\lambda_1, \dots, \lambda_{n+1}$ and so for almost all $\lambda_1, \dots, \lambda_{n+1}$ the points given by the above system are the Morse ones. □

Following Damon [4], one can introduce the notion of a versal deformation of hyperplane singularities. The theorem on deformation implies

Corollary 3.2. *A hyperplane singularity possesses a versal deformation if and only if it is isolated, in which case the deformation can be given as* $F(x, y_1, \dots, y_n, \lambda) = f(x, y_1, \dots, y_n) + \sum_{i=1}^{\sigma} \lambda_i e_i(x, y_1, \dots, y_n)$, *where* $\sigma = \operatorname{codim} f$, *and* $e_1, \dots, e_\sigma$ *are the representatives for the* $\mathbb{C}$*-base of the space* $(x^2)/\tau(f)$.

4. The Number of Morse Points

Let $f \in (x^2)$ have an isolated hyperplane singularity; then according to Theorem 3.1, there exists a deformation, having, on H, A_∞ and D_∞ type singular points and a certain number s of Morse points outside H. It turns out that this number does not depend on the deformation choice and can be calculated in a purely algebraic way.

Theorem 4.1. *The number of Morse points of a deformation of f is calculated by the formula*

$$s = \dim_{\mathbb{C}} \left[(x^2)/(xf_x, f_{y_1}, \dots, f_{y_n})\right].$$

Proof. Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^\sigma, 0) \to (\mathbb{C}, 0)$ be a versal deformation of the singularity f, where $f = x^2 g(x, y_1, \dots, y_n)$. Then $F = x^2 G(x, y_1, \dots, y_n, \lambda)$, $\lambda \in \mathbb{C}^\sigma$, where $G \in \mathcal{O}_{x,y_1,\dots,y_n,\lambda}$ satisfies $G|_{\lambda=0} = g$.

Clearly, the number of Morse points of s is obtained as the number of solutions of the following system of equations lying outside the singular hyperplane $\{x = 0\}$, for some sufficiently small value of the parameter λ,

$$F_x = 0, \;\; F_{y_1} = 0, \; \dots, \; F_{y_n} = 0,$$

where F_x and F_{y_i} are the partial derivatives of the function F with respect to x and y_i, respectively.

One can trace a part consisting of values of the parameter λ_0 which obey the transversality of the intersection of the plane $\lambda = \lambda_0$ with the singular set of F_{λ_0} outside the singular plane $\{x = 0\}$; hence by the definition of the intersection index, the number of Morse points coincides with the intersection index of the plane $\{\lambda = 0\}$ with the germ of the surface $S \subset \mathbb{C}^{n+1} \times \mathbb{C}^\sigma$ determined as closure of the germ of the set

$$\{F_x = 0, F_{y_1} = 0, \dots, F_{y_n} = 0, x \neq 0\}.$$

Since $x \neq 0$, one can cancel it, which exactly corresponds to considering only the singularities outside $\{x = 0\}$; hence

$$S = \{2G + xG_x = 0, G_{y_1} = 0, \dots, G_{y_n} = 0\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^\sigma.$$

Since the set S is defined only by functions with isolated singularities, one gets by [5]

$$\begin{aligned} S &= \dim_{\mathbb{C}} [\mathcal{O}_{x,y_1,\dots,y_n,\lambda}/(2G + xG_x, G_{y_1}, \dots, G_{y_n})] = \\ &= \dim_{\mathbb{C}} [\mathcal{O}_{x,y_1,\dots,y_n}/ (\mathcal{O}_{x,y_1,\dots,y_n}(2g + xg_x, g_{y_1}, \dots, g_{y_n}))] = \\ &= \dim_{\mathbb{C}} \left[(x^2)/(2x^2 g + x^3 g_x, x^2 g_{y_1}, \dots, x^2 g_{y_n})\right] = \\ &= \dim_{\mathbb{C}} \left[(x^2)/(xf_x, f_{y_1}, \dots, f_{y_n})\right]. \qquad \square \end{aligned}$$

5. Topology of Isolated Hyperplane Singularities

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated hyperplane singularity with the singular set $H = \{x = 0\}$, and let f_λ be a deformation of the singularity f obtained by Theorem 3.1. Choose $\varepsilon_0 > 0$ such that for any ε with $0 \leq \varepsilon \leq \varepsilon_0$ one has $f^{-1}(0) \pitchfork \partial B_\varepsilon$, i.e., the fibre $f^{-1}(0)$ is transversal to the boundary of a ball of radius ε in $\mathbb{C}^{n+1}$. (This is possible since $f^{-1}(0)$ is an algebraic stratified set.) For such $\varepsilon > 0$ there is $\eta(\varepsilon)$ such that $f^{-1}(t) \pitchfork \partial B_\varepsilon$ for any $0 < |t| < \eta(\varepsilon)$. Fix $\varepsilon \le \varepsilon_0$ and consider $0 < \eta \le \eta(\varepsilon)$ and the restriction

$$f_\lambda : X_D = f^{-1}(D_\eta) \cap B_\varepsilon \to D_\eta,$$

where D_η is a disc of radius η in $\mathbb{C}$.

Lemma 5.1. *Let f_λ be a deformation of the isolated hyperplane singularity f. Consider the restriction*

$$f_\lambda : X_{D,\lambda} = f_\lambda^{-1}(D_\eta) \cap B_\varepsilon \to D_\eta,$$

for any $0 \le \|\lambda\| < \delta$ and $0 < |t| < \eta$, where δ and η are sufficiently small numbers. Then the following assertions are valid:
1. *$f_\lambda^{-1}(t) \pitchfork \partial B_\varepsilon$.*
2. *Fibrations induced over ∂D_η by f and f_λ are equivalent.*
3. *X_D and $X_{D,\lambda}$ are homeomorphic.*

Proof. At the points of $H \cap \partial B_\varepsilon$ one has A_∞ and D_∞ type singularities. If $z \in H \cap \partial B_\varepsilon$ is an A_∞ type singular point, there exist coordinates $(x, y_1, \dots, y_n)$ with $f_{\lambda(s)}(x, y_1, \dots, y_n) \sim x^2$, where $f_{\lambda(s)}$ is a one-parameter deformation of the singularity f and the coordinate x depends on λ smoothly. For $t \neq 0$ the tangent space $f_{\lambda(s)}^{-1}(t)$ is obtained from the equation

$$x_0(x - x_0) = 0,$$

i.e., $x = x_0$, which is a hyperplane parallel to H and hence transversal to ∂B_ε, as H is preserved under the coordinate transform involved.

Now assume that $z \in H \cap \partial B_\varepsilon$ is a D_∞ type singularity; then one has

$$f_{\lambda(s)}(x, y_1, \dots, y_n) \sim x^2 y_1.$$

Hence the tangent space to $f_{\lambda(s)}^{-1}(t)$ at $(x_0, y_1^0, \dots, y_n^0)$ has the form

$$(x - x_0)x_0 y_1^0 + (y_1 - y_1^0)x_0^2 = 0,$$

i.e., $x = x_0, y_1 = y_1^0$, which is transversal to ∂B_ε, since the set $y_1 = y_1^0$ coincides with $\{g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1} = y_1^0\}$, and this set is compact and intersects $\partial B_\varepsilon \cap H$ transversally.

We have thus established that at the points $z \in H \cap \partial B_\varepsilon$ the transversality condition holds, while at the points from $\partial B_\varepsilon \setminus H$ the map is a submersion. Since $f^{-1}(0) \cap \partial B_\varepsilon$ is compact and transversality is an open property, this implies $f_{\lambda(s)}^{-1}(t) \pitchfork \partial B_\varepsilon$, $0 \le \|\lambda\| < \delta$ and $0 < |t| < \eta$, which concludes the proof of assertion (1).

Let us prove (2). Consider the mapping

$$F(x, y_1, \dots, y_n, s) = (f_{\lambda(s)}(x, y_1, \dots, y_n), s).$$

Define

$$Y_{D,s_0} = F^{-1}(D_\eta \times [0,s_0]) \cap (B_\varepsilon \times [0,s_0])$$

and the mapping

$$F_{D,s} : Y_{D,s_0} \to D_\eta \times [0,s_0] \to [0,s_0]$$

which is well defined for any $s \in [0,s_0]$ $f_{\lambda(s)} : X_{D,s} \to D_\eta$; the map $F_{D,s}$ is submersive at the internal points of

$$F^{-1}(\partial D_\eta \times [0,s_0]) \cap (\mathrm{int} B_\varepsilon \times [0,s_0]),$$

since $df_{\lambda(s)}$ has a maximal rank over the boundary of D_η. The restriction of $F_{D,s}$ to the boundary of

$$F^{-1}(\partial D_\eta \times [0,s_0]) \cap (\partial B_\varepsilon \times [0,s_0])$$

is also a submersion, since $f^{-1}_{\lambda(s)}(t) \pitchfork \partial B_\varepsilon$ for any $t \in D_\eta, \lambda(s), s \in [0,s_0]$. Now one can apply the theorem of Ehresmann [6] to find that $F_{D,s}$ is a trivial fibration over the contractible set $[0,s_0]$ and, consequently, for any s the maps $f_{\lambda(s)}$ determine equivalent fibrations over the boundary of D_η.

Finally, (3) follows from Thom's lemma on isotopy [7] needed for describing a homotopy type of the Milnor fibre. □

Now let us turn to the main construction.

Let $b_1,\dots,b_\sigma$ be Morse points for the deformation $f_\lambda := \widetilde{f}$ with critical values $\widetilde{f}(b_1),\dots,\widetilde{f}(b_\sigma)$. Define $B_1,\dots,B_\sigma$ to be disjoint $2n+2$-dimensional balls in $\mathbb{C}^{n+1}$ centered at $b_1,\dots,b_\sigma$ respectively, and let $D_1,\dots,D_\sigma$ be disjoint 2-dimensional discs centered at $\widetilde{f}(b_1),\dots,\widetilde{f}(b_\sigma)$. Let

$$\widetilde{f} : B_i \cap \widetilde{f}^{-1}(D_i) \to D_i, \quad i = 1,\dots,\sigma,$$

be locally trivial Milnor fibrations satisfying the transversality condition

$$\widetilde{f}^{-1}(t) \pitchfork \partial B_i, t \in D_i, \quad i = 1,2,\dots,\sigma.$$

Choose furthermore a small cylinder B_0 around H and a 2-dimensional disc $D_0 \subset \mathrm{int}\widetilde{f}(B_0)$, satisfying

$$\partial B_0 \pitchfork \widetilde{f}^{-1}(t) \quad \text{for} \quad t \in D_0.$$

First of all, let us investigate the fibration $\widetilde{f} : B_0 \cap \widetilde{f}^{-1}(D_0) \to D_0$. Its fibre $\widetilde{f}^{-1}(t) \cap B_0$ can be in turn fibred over $B_\varepsilon \cap (H \backslash U)$ using the projection π, where U is a tubular neighborhood of the smooth nonsingular subvariety

$$g(0,y_1,\dots,y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1} = 0,$$

with $\pi(x,y_1,\dots,y_n) = (0,y_1,\dots,y_n)$. This projection may have singularities. To describe them, consider the mapping

$$\varphi_{\widetilde{f}} : \widetilde{f}^{-1}(D_0) \cap B_0 \to \mathbb{C} \times \mathbb{C}^n,$$

defined by

$$\varphi_{\widetilde{f}}(x, y_1, \dots, y_n) = (\widetilde{f}(x, y_1, \dots, y_n), y_1, \dots, y_n).$$

The Jacobi matrix of this mapping has the form

$$\begin{pmatrix} \frac{\partial \widetilde{f}}{\partial x} & \frac{\partial \widetilde{f}}{\partial y_1} & \cdots & \frac{\partial \widetilde{f}}{\partial y_n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Hence the critical set of $\varphi_{\widetilde{f}}$ is given by the equation

$$\frac{\partial \widetilde{f}}{\partial x} = 0. \qquad (\Gamma)$$

The hypersurface Γ contains the hyperplane H, i.e., $\Gamma = H \cup \Gamma_{\widetilde{f}}$; and the projection

$$\pi := \widetilde{f}^{-1}(t) \cap B_0 \to B_\varepsilon \cap (H \backslash U)$$

is smooth outside $\Gamma_{\widetilde{f}}$.

Lemma 5.2. *The hypersurface $\Gamma_{\widetilde{f}}$ meets H at D_∞ type points.*

Proof. We shall prove that if $\Gamma_{\widetilde{f}}$ meets H at D_∞ type points, then $\Gamma_{\widetilde{f}}$ coincides with H.

Let $\widetilde{f} = x^2 \widetilde{g}$, where $\widetilde{g} = g(x, y_1, \dots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1}$ and $\widetilde{g}(0, \dots, 0) \neq 0$. Then

$$\frac{\partial \widetilde{f}}{\partial x} = 2x\widetilde{g} + x^2 \widetilde{g}$$

and since $\widetilde{g}(0, \dots, 0) \neq 0$, x can be expressed by the module x^2. Hence $(x) \subset (\frac{\partial \widetilde{f}}{\partial x}) + (x^2)$.

By Nakayama's lemma this implies $(x) = (\frac{\partial \widetilde{f}}{\partial x})$. Consequently, the set defined by the equality $\frac{\partial \widetilde{f}}{\partial x} = 0$ coincides with the set $x = 0$, i.e., $\Gamma_{\widetilde{f}} = H$. This concludes the proof. □

The lemma implies that the projection π is a locally trivial fibration outside D_∞ type singular points, and its fibre is given by the equation $\widetilde{f} = t$. And since the set $\pi^{-1}(B_\varepsilon \cap (H \backslash U))$ is compact and consists of A_∞ type singular points, ($\widetilde{f} \sim x^2$), the fibre locally consists of two points. By compactness of the aforementioned set one can choose a radius for B_0 in such a way that π will define a double covering over $B_\varepsilon \cap (H \backslash U)$.

Let us introduce the space $B_\varepsilon \cap (H \backslash U) = \widetilde{B}_\varepsilon \backslash U$, where $\widetilde{B}_\varepsilon$ is a $2n$-dimensional ball in the space $\mathbb{C}^n$ and U is a small tubular neighborhood of the smooth nonsingular variety

$$\widetilde{V} = \{g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1} = 0\};$$

obviously, $\widetilde{B}_\varepsilon \backslash \widetilde{V}$ and $\widetilde{B}_\varepsilon \backslash U$ are of the same homotopy type.

Our aim is to investigate a homotopy type of the complement to $\widetilde{V}$.

The homology of that space is easily computed from Leray's exact homological sequence

$$\xrightarrow{\delta_*} H_q(\widetilde{B}_\varepsilon \backslash \widetilde{V}) \xrightarrow{j_*} H_q(\widetilde{B}_\varepsilon) \xrightarrow{i_*} -H_{q-2}(\widetilde{V}) \xrightarrow{\delta_*} H_{q-1}(\widetilde{B}_\varepsilon \backslash \widetilde{V}) \xrightarrow{j_*}$$

obtained from Leray's exact cohomological sequence [8] by the Poincaré duality

$$\xrightarrow{\delta_*} H^p(\widetilde{B}_\varepsilon \backslash \widetilde{V}) \xrightarrow{j_*} H^p(\widetilde{B}_\varepsilon) \xrightarrow{i_*} H^p(\widetilde{V}) \xrightarrow{\delta_*} H^{p+1}(\widetilde{B}_\varepsilon \backslash \widetilde{V}) \xrightarrow{j_*},$$

where j_* are induced by the embedding $j : \widetilde{B}_\varepsilon \backslash \widetilde{V} \subset \widetilde{B}_\varepsilon$, i_* is the intersection of cycles from $H^*(\widetilde{B})$ and $H^*(\widetilde{V})$, and δ_* is the Leray coboundary.

Since $\widetilde{V}$ is a smooth nonsingular submanifold of real codimension two which is homotopy equivalent to the wedge of $\mu(g)$ copies of the $n-1$-spheres, where $\mu(g)$ is the Milnor number of the isolated singularity g, one obtains $H_0(\widetilde{V}) = \mathbb{Z}, H_{n-1}(\widetilde{V}) = \mathbb{Z}^{\mu(g)}$, and $H_i(\widetilde{V}) = 0$ for $i \neq 0, n-1$. Taking this in account, one obtains from Leray's exact homological sequence that

$$H_1(\widetilde{B}_\varepsilon \backslash \widetilde{V}) = \mathbb{Z}, H_n(\widetilde{B}_\varepsilon \backslash \widetilde{V}) = \mathbb{Z}^{\mu(g)} \text{ and } H_i(\widetilde{B}_\varepsilon \backslash \widetilde{V}) = 0, \text{ if } i \neq 0, 1, n.$$

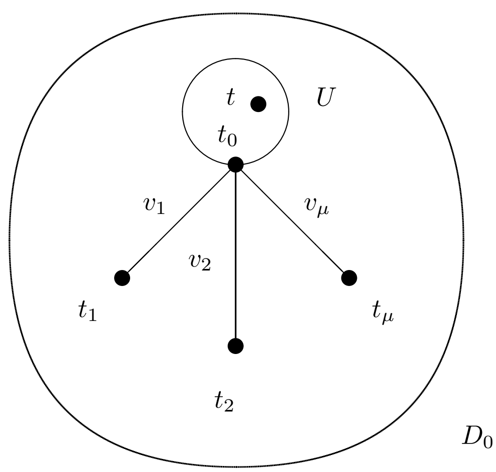

Figure 1

To find the homotopy type of the Milnor fibre, let us prove

Lemma 5.3. *For sufficiently small t the complement $\widetilde{B}_\varepsilon \backslash \widetilde{V}$ of the non-singular hypersurface $\widetilde{V}$ inside the ball $\widetilde{B}_\varepsilon$ is homotopy equivalent to the space obtained from the direct product $S^1 \times \widetilde{V}$ by filling all the vanishing spheres $S_i^{n-1}, i = 1, 2, \dots, \mu$, with n-dimensional balls, in one of the fibres $\{t_0\} \times V$ for some $t_0 \in S^1$, where μ is the Milnor number of the isolated singularity $g(0, y_1, \dots, y_n)$.*

Proof. Take a small neighborhood u of the point $t_0 \in D_0$ and let $\bar{u}$ be its closure. Let $t_0 \in \partial u$. Connect the critical values t_i of the mapping $g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1}$ with t by disjoint paths $v_i(\tau)$, where $v_i(0) = t_i$ and $v_i(1) = t_0$ (see Figure 1).

The disc $D_0 \backslash t$ is a deformation retract of the set $\bigcup_{i=1}^{\mu} v_i(\tau) \cup (\bar{u} \backslash t)$. Since $\widetilde{g}(0, y_1, \dots, y_n) = g(0, y_1, \dots, y_n) + \lambda_1 y_1 + \dots + \lambda_n y_n + \lambda_{n+1}$ is a locally trivial Milnor fibration in the ball $\widetilde{B}_\varepsilon$, by the homotopy lifting property one obtains that $\widetilde{g}^{-1}(D_0 \backslash t)$ is homotopy equivalent to $\widetilde{g}^{-1} \bigcup_{i=1}^{\mu} v_i(\tau) \cup (\bar{u} \backslash t)$. Restrictions of the locally trivial fibration on the contractible set are trivial; consequently $\widetilde{g}^{-1}(\bar{u} \backslash t)$ is a total space of the trivial fibration over $\bar{u} \backslash t$, i.e., over a circle, with fibre $\{\widetilde{g}(0, y_1, \dots, y_n) = t\} \cap \widetilde{B}_\varepsilon$, diffeomorphic to $\widetilde{V}$, hence $\widetilde{g}^{-1}(\bar{u} \backslash t)$ is homotopy equivalent to the direct product $S^1 \times \widetilde{V}$.

Following [9], we shall show that the space $Y = \widetilde{g}^{-1}(\bigcup_{i=1}^{\mu} v(\tau))$ is obtained, up to homotopy type, from the fibre V by filling all the spheres Δ_i^{n-1}, $i = 1, 2, \dots, \mu$ with n-dimensional balls T_i. Let

$$S_i(t) : S_i^{n-1} \longrightarrow S_i(t) \subset F_{v_i(t)} \quad (0 \le t \le 1)$$

be the family of maps of the standard $n-1$-dimensional sphere S_i^{n-1} (the index i counts copies of the sphere), determining the vanishing cycle $\Delta_i = S_i(1)(S_i(0) : S_i^{n-1} \to P_i)$. Let T_i be the n-dimensional ball constructed as the cone over the sphere S_i^{n-1},

$$T = [0, 1] \times S_i^{n-1} / \{0\} \times S_i^{n-1}.$$

The space $\widetilde{V} \cup_{\Delta_i} \{T_i\}$ obtained from the fibre $\widetilde{V}$ by filling in the vanishing cycles Δ_i with n-balls T_i is the quotient of $\widetilde{V} \cup \bigcup_{i=1}^{\mu} T_i$ under the equivalence relation

$$S_i(1)(a) \sim (1, a), a \in S_i^{n-1}, \ \ (1, a) \in T_i, \ \ i = 1, \dots, \mu,$$

and its mapping to the space Y can be written as

$$\varphi(x) = x \text{ for } x \in \widetilde{V} \subset Y; \varphi(t, a) = S_i(t)(a) \ \text{ for } \ (t, a) \in T_i, \ \ 0 \le t \le 1,$$

$a \in S_i^{n-1}$. Let us construct the inverse mapping $\psi : Y \to \widetilde{V} \cup_{\Delta_i} \{T_i\}$ by putting $\psi(y) = y$ for $y \in \widetilde{V}$, $\psi(y) = (t, a)$ for $y \in \widetilde{V}_{v_i(t)}$, if under the homotopy equivalence between $\widetilde{V}_{v_i(t)}$ and the wedge $\bigvee_{i=1}^{\mu} \Delta_i$ the point y passes $S_i(t)(a)$ for $a \in S_i^{n-1}$. Consider the composition

$$\psi \circ \varphi : \widetilde{V} \cup_{\Delta_i} \{T_i\} \to \widetilde{V} \cup_{\Delta_i} \{T_i\}.$$

Then $\psi(\varphi(x)) = x$ for $x \in \widetilde{V}$ and $\psi \circ \varphi$ is homotopic to the identity mapping of $\widetilde{V} \cup_{\Delta_i} \{T_i\}$, since $\widetilde{V}_{v_i(t)}$ for $0 < t \leq 1$ is homotopy equivalent to the wedge of the spheres Δ_i, while $\widetilde{V}_{v_i(0)}$ – to that without one of them (the one vanishing along v_i). Similarly, $\varphi \circ \psi : Y \to Y$ is homotopic to Id_y, which proves the homotopy equivalence.

The space $(\bar{u} \backslash t) \cup \bigcup_{i=1}^{\mu} v_i(\tau)$ is the amalgam [10] of the diagram

$$\bigcup_{i=1}^{\mu} v_i(\tau) \longleftarrow \{t_0\} \longrightarrow \bar{u} \backslash t,$$

whence the inverse image of this space under the mapping $\widetilde{g}$ will be the amalgam of the inverse image of the diagram [10], i.e., of the diagram

$$\widetilde{g}^{-1}\left(\bigcup_{i=1}^{\mu} v(\tau)\right) \longleftarrow \widetilde{V}_{t_0} \longrightarrow \widetilde{g}^{-1}(\bar{u} \backslash t),$$

where $\widetilde{V}_{t_0}$ is diffeomorphic to $\widetilde{V}$. We arrived at the amalgam of

$$\widetilde{V} \cup_{\Delta_i} \bigcup_{i=1}^{\mu} \{T_i\} \longleftarrow \widetilde{V} \longrightarrow \widetilde{V} \times S^1,$$

which is the space $\widetilde{V} \times S^1$ with all the vanishing spheres in the fibre over t_0 filled with n-balls. $\square$

An even more general fact can be proved.

Proposition 5.4. *Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of an isolated singularity; then $V = \{g = t\} \cap B_\varepsilon$, where $\widetilde{B}_\varepsilon$ is a small ball in $\mathbb{C}^n$, is, for small t, homotopy equivalent inside $\widetilde{B}_\varepsilon$ to $S^1 \times V$ with n-dimensional balls filling in all the vanishing spheres of one of its fibres V.*

Proof. Let $\widetilde{g}$ be the morsification of the isolated singularity g in the ball $\widetilde{B}_\varepsilon$, having nondegenerate critical points p_i with different critical values $t_i = \widetilde{g}(p_i)$; then $V = \{g = t\} \cap \widetilde{B}_\varepsilon$ is diffeomorphic to V.

We shall show that $\widetilde{B}_\varepsilon \backslash V$ is homotopy equivalent to $\widetilde{B}_\varepsilon \backslash \widetilde{V}$, which by Lemma 5.3 will imply our proposition.

Let $\varepsilon > 0$ and $\delta > 0$ be chosen in such a way that $\widetilde{g}^{-1}(t) = \widetilde{g}_\lambda^{-1}(t)$ is transversal to $\partial \widetilde{B}_\varepsilon$ for any $0 \leq \|\lambda\| \leq \delta$. Consider the mapping given by $F(x, \lambda) = (g_\lambda(x), \lambda)$ and its restriction

$$F_{t,\delta} = F^{-1}(\{t\} \times [0, \delta]) \cap (\widetilde{B}_\varepsilon \times [0, \delta]) \to \{t\} \times [0, \delta] \to [0, \delta].$$

The mapping $F_{t,\delta}$ is submersive at the interior points of

$$F^{-1}(\{t\} \times [0,\delta]) \cap (\mathrm{int}\widetilde{B}_\varepsilon \times [0,\delta]),$$

since dg_λ has a maximal rank over $\{t\}$. Moreover, the restriction of $F_{t,\delta}$ on the boundary of $F^{-1}(\{t\}\times[0,\delta])\cap(\delta\widetilde{B}_\varepsilon\times[0,\delta])$ is submersive as $\widetilde{g}^{-1}(t) \pitchfork \partial\widetilde{B}_\varepsilon$ for any $\lambda \in [0,\delta]$. By Ehresmann's theorem [6] one obtains a locally trivial fibration over the contractible space $[0,\delta]$, which is trivial. Consequently, V is diffeomorphic to the fibre $\widetilde{V}$.

Let T be a tubular neighborhood of the submanifold V. Choose $0 < \delta_1 < \delta$ sufficiently small for the fibre $F^{-1}(\delta_1)$ to lie inside T. Making δ_1 still smaller, one can make T into a tubular neighborhood for $F^{-1}(\delta_1)$ too. This will imply that $\widetilde{B}_\varepsilon\backslash T$ is homotopy equivalent to $\widetilde{B}_\varepsilon\backslash V$ and $\widetilde{B}_\varepsilon\backslash F^{-1}(\delta_1)$ simultaneously; hence $\widetilde{B}_\varepsilon\backslash V$ is homotopy equivalent to $\widetilde{B}_\varepsilon\backslash F^{-1}(\delta_1)$. By the compactness of $[0,\delta]$ in a finite number of steps one obtains the homotopy equivalence of $\widetilde{B}_\varepsilon\backslash V$ to $\widetilde{B}_\varepsilon\backslash\widetilde{V}$. □

Corollary 5.5. *The complement $\widetilde{B}_\varepsilon\backslash V$ is homotopy equivalent to a wedge of S^1 and μ copies of the n-sphere S^n, where μ is the Milnor number of the isolated singularity $g(y_1,\dots,y_n)$.*

Proof. By Lemma 5.4, $\widetilde{B}_\varepsilon\backslash V$ is homotopy equivalent to the direct product $S^1 \times V$, where over a point $t_0 \in S^1$ the fibre V is contracted to a point, hence such a space is homotopy equivalent to the suspension of V with the identified vertices, i.e., suspension of the wedge of $(n-1)$-spheres S_i^{n-1}, $i = 1,\dots,\mu(g)$, with identified vertices, which is obviously a wedge of $\mu(g(y_1,\dots,y_n))$ copies of the n-sphere and a circle (see Figure 2).

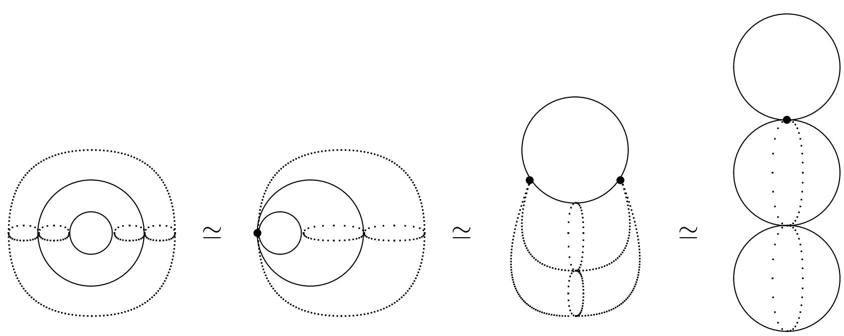

Figure 2

We obtain that $B_\varepsilon \cap (H\backslash U)$, where U is a tubular neighborhood of the smooth nonsingular subvariety $g(0,y_1,\dots,y_n)+\lambda_1 y_1+\dots+\lambda_n y_n+\lambda_{n+1}$, is homotopy equivalent to the wedge of a circle S^1 and $\mu = \mu(g(0,y_1,\dots,y_n))$

copies of the n-sphere and one has a double covering

$$\pi : \widetilde{f}^{-1}(t) \cap B_0 \to B_\varepsilon \cap (H \backslash U).$$

Represent $B_\varepsilon \cap (H\backslash U)$ as a union of V_1 and V_2, where V_1 has homotopy type of a circle, while V_2 has homotopy type of a wedge of n-spheres, and where $V_1 \cap V_2$ is contractible. Since π is a double cover, $\pi^{-1}(V_1)$ is homotopy equivalent to the circle S^1, while over the simply connected space V_2 the covering π is trivial, hence $\pi^{-1}(V_2)$ consists of a disjoint union of two wedges of $\mu = \mu(g(0, y_1, \dots, y_n))$ copies of the n-sphere S^n, and, since $\pi^{-1}(V_1)$ is a doubly winded circle, one obtains that $\widetilde{f}^{-1}(t) \cap B_0$ is homotopy equivalent to the wedge of S^1 and $2\mu(g(0, y_1, \dots, y_n))$ copies of the n-spheres S^n. Hence we arrive at

Lemma 5.6. *Let an isolated hyperplane singularity be not of type A_∞; then the fibre of the Milnor fibration in a small cylinder B_0*

$$\widetilde{X}_t = \widetilde{f}^{-1}(t) \cap B_0$$

is homotopy equivalent to the wedge of a circle S^1 and 2μ copies of the n-sphere, where μ is the Milnor number of the isolated singularity $g(0, y_1,...,y_n)$.

We have already established that the critical set $\widetilde{f}$ consists of
(a) the hyperplane H;
(b) Morse points $b_1, \dots, b_\sigma$.

We have defined the small discs D_i around the points $\widetilde{f}(b_i)$, the disjoint disc balls B_i over the points b_i and the cylinder B_0 over the singular set H. Now choose $t_i \in \partial D_i, t \in D_\eta$ and a system of separate paths $\gamma_0, \gamma_1, \dots, \gamma_\sigma$ from the point t to t_i (see Figure 3)

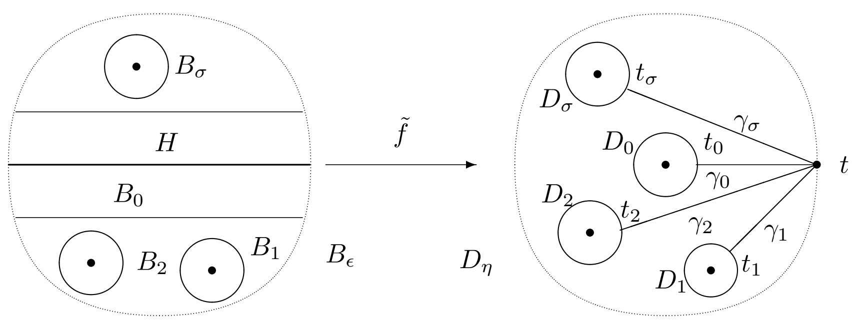

Figure 3

Let us introduce the notations

$$D=\bigcup_{i=0}^{\sigma} D_i, X_A=\tilde{f}^{-1}(A)\cap B_\varepsilon,\quad A\subset D_\eta,\quad \widetilde{X}_s=f^{-1}(s)\cap B_\varepsilon,\quad s\in D_\eta.$$

Define suitable neighborhoods of the critical sets as follows:
(1) for the hyperplane $H: r_0(z)=|x|^2$ and let

$$B_0(\widetilde{\varepsilon})=\{z\in B_\varepsilon \mid r_0(z)\leq\widetilde{\varepsilon},\widetilde{\varepsilon}\ll\varepsilon\};$$

(2) for the Morse points $b_i: r_i(z)=|z-b_i|^2$ and

$$B_i(\widetilde{\varepsilon})=\{z\in B_\varepsilon \mid r_i(z)\leq\widetilde{\varepsilon},\widetilde{\varepsilon}\ll\varepsilon\}.$$

As shown by Lemma 5.1, for $\widetilde{f}$ there exists ε_0 such that for any $0<\widetilde{\varepsilon}\leq\widetilde{\varepsilon}_0$ the set X_0 is transversal to $\partial B_0(\widetilde{\varepsilon})$ and there is $\widetilde{\varepsilon}_i$ such that for any $\widetilde{\varepsilon}$ with $0<\widetilde{\varepsilon}\leq\widetilde{\varepsilon}_i$ the set $X_{f(b_i)}$ is transversal to $\partial B_i(\widetilde{\varepsilon}), i=1,2,\dots,\sigma$, as the points b_i are the Morse ones [11].

Since the transversality condition is open, for any $0<\widetilde{\varepsilon}\leq\widetilde{\varepsilon}_i, i=0,1,\dots,\sigma$, there exists $\tau_i=\tau_i(\widetilde{\varepsilon})$, such that $X_i \pitchfork \partial B_i(\varepsilon)$ for any

$$0<|t-\widetilde{f}(b_i)|\leq\tau_i, i=0,1,\dots,\sigma,\ \text{where}\ f(b_0)=0.$$

Now fix $\widetilde{\varepsilon}>0$ and $\tau>0$ and require $B_i(\widetilde{\varepsilon})$ and $D_i(\tau)$ to be disjoint balls and discs, respectively.

Denote

$$B_i=B_i(\varepsilon),\quad D_i=D_i(\tau),\quad E^i=B_i\cap X_{D_i},$$
$$E=B_\varepsilon\cap X_{D_\eta},\quad F^i=B_i\cap X_{t_i},\quad F=B_\varepsilon\cap X_t.$$

We shall need the isomorphism (see [2])

$$H_*(E,F)\simeq\bigoplus_{i=0}^{\sigma} H_*(E^i,F^i).$$

This implies that the homology groups $H_*(E,F)$ are direct sums of homology groups over all critical sets. The situation for the Morse points $b_1,\dots,b_\sigma$ is very well known [11]

$$H_{k+1}(E^i,F^i)=H_k(F^i)=\begin{cases}\mathbb{Z}, & k=0,n,\\ 0, & k\neq n.\end{cases}$$

Hence we finally obtain

$$\begin{cases}H_{n+1}(E,F)=H_n(E^0,F^0)\oplus\mathbb{Z}^\sigma, & \\ H_k(E,F)=H_k(E^0,F^0), & k\neq n+1.\end{cases}$$

To calculate the homology groups $H_k(E^0, F^0)$, write down an exact sequence of the pair (E^0, F^0) [12]

$$\cdots \to H_k(E^0) \to H_k(E^0, F^0) \to H_{k-1}(F^0) \to H_{k-1}(E^0) \to \dots .$$

The spaces $E^0 = E \cap B_0$ and E are homotopy equivalent, while E is contractible [11]. Therefore

$$0 \to H_k(E^0, F^0) \to H_{k-1}(F^0) \to 0.$$

This implies $H_k(E^0, F^0) \cong H_{k-1}(F^0)$.

Similarly, one obtains $H_k(E, F) = H_{k-1}(F)$ so that we have

$$\begin{cases} H_n(F) = H_n(F_0) \oplus \mathbb{Z}^\sigma & , \\ H_{k-1}(F) = H_{k-1}(F_0) & , k \neq n. \end{cases}$$

By Lemma 5.6 we obtain

Proposition 5.7. *Homology groups of the Milnor fibre are calculated as follows:*

$$\begin{cases} H_0(F) = \mathbb{Z}, \\ H_1(F) = \mathbb{Z}, \\ H_n(F) = \mathbb{Z}^{2\mu+\sigma}, \\ H_i(F) = 0, & i \neq 0, 1, n, \end{cases}$$

where $\mu = \mu(g(0, y_1, \dots, y_n))$ *is the Milnor number of an isolated singularity and* σ *is the number of the Morse critical points for* $\widetilde{f}$, *which by Theorem* 4.1 *equals*

$$\sigma = \dim_{\mathbb{C}} \left[(x^2)/(xf_x, f_{y_1}, \dots, f_{y_n}) \right].$$

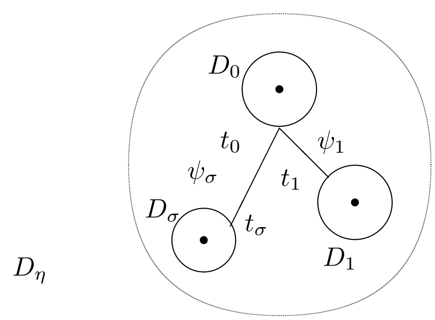

Figure 4

Now let us determine a homotopy type of the fibre F. We have

Theorem 5.8. *Let f be an isolated hyperplane singularity (not of A_∞ type). Then the Milnor fibre of f is homotopy equivalent to the wedge of a circle S^1 and $2\mu+\sigma$ copies of the n-dimensional sphere, where $\mu=\mu(g)$ is the Milnor number of the isolated singularity $g(0,y_1,\dots,y_n)$, while σ is the number of Morse points of the deformation $\widetilde{f}$.*

Proof. Let $D_\eta, D_0, \dots, D_\sigma$ and $B_\varepsilon, B_0, \dots, B_\sigma$ be as before. Let t be a point in ∂D_0, and choose a system of separate paths $\psi_1, \dots, \psi_\sigma$ from t to $D_1, \dots, D_\sigma$ (see Figure 4).

Applying the Morse lemma [13] to $|f|$ which has $b_1, \dots, b_\sigma$ as Morse points of the index $n+1$, one obtains the homotopy equivalences

$$(X_{D_\eta}, X_t) \cong (X_{D_0} \cup_{\psi_1} e_1^{n+1} \cup \dots \cup \psi_\sigma e_\sigma^{n+1}, X_t),$$
$$(X_{D_0}, X_t) \simeq (X_{D_0} \cap B_0 \cup X_t, X_t).$$

Let $\varphi_1, \varphi_2, \dots, \varphi_{2\mu} : S^n \to F^0$ and $\varphi_0 : S^1 \to F^0$ represent the generators of $\pi_n(F^0)$ and $\pi_1(F^0)$, respectively. Use $\varphi_0, \dots, \varphi_{2\mu}$ to attach a 2-cell and $n+1$-cells $e_0^2, e_1^{n+1}, \dots, e_{2\mu}^{n+1}$ to $F_0 = X_t \cap B_0$.

The inclusion $X_t \cap B_0 \subset X_D \cap B_0$ extends to the homotopy equivalence

$$X_t \cup_{\varphi_0} e_1^{n+1} \cup \dots \cup_{\varphi_{2\mu}} e_{2\mu}^{n+1} \to X_{D_0} \cap B_0$$

as both spaces are contractible. This gives the homotopy equivalence

$$(X_{D_0}, X_t) \simeq (X_t \cup_{\varphi_0} e_1^{n+1} \cup \dots \cup_{\varphi_{2\mu}} e_{2\mu}^{n+1}, X_t).$$

Finally, we obtain the contractible space X_{D_η} from the fibre X_t by attaching $\sigma+2\mu$ copies of the $n+1$-cell and one 2-cell, and since attaching $n+1$-cells does not change homotopy groups in dimension $n-1$, it follows that $X_t \cup_{\varphi_0} e_0^2$ is $n-1$-connected.

The homology group $H_n(X_t \cup_{\varphi_0} e_1^{n+1})$ must be free abelian, since any torsion elements would give rise to nonzero elements in the $(n+1)$th cohomology group, which would contradict the fact that $X_t \cup_{\varphi_0} e_0^2$ is an n-dimensional CW-complex. According to Hurewich's theorem [12] there is an isomorphism $\pi_n(X_t \cup_{\varphi_0} e_0^2) \simeq H_n(X_t \cup_{\varphi_0} e_0^2)$. Hence $\pi_n(X_t \cup_{\varphi_0} e_0^2)$ is a free abelian group, and one can choose a finite number of maps

$$(S_i^{n-1}, \text{basepoint}) \to (X_t \cup_{\varphi_0} e_0^2, \text{basepoint})$$

representing the basis in the group $\pi_n(X_t \cup_{\varphi_0} e_0^2)$. Wedging these maps gives the map

$$S^n \bigvee \dots \bigvee S^n \to X_t \cup_{\varphi_0} e_0^2$$

inducing an isomorphism in homology, which, consequently, by Whitehead's theorem [12], is a homotopy equivalence. Therefore $X_t \cup_{\varphi_0} e_0^2$ is homotopy equivalent to the wedge $S^n \bigvee \dots \bigvee S^n$ of n-spheres.

This implies that $\pi_1(X_t)$ is generated by one element, and since $H_1(X_t) = \mathbb{Z}$, we obtain $\pi_1(X_t) = \mathbb{Z}$.

Consider the map

$$(S^1 \vee S^n \vee \cdots S^n, \text{basepoint}) \to (X_t, \text{basepoint})$$

defined as follows: the sphere S^n maps to X_t as the representative of a generator in the homology group $H_n(X_t)$, whereas S^1 maps as the representative of the generator of $\pi_1(X_t)$. The constructed map induces an isomorphism of homology groups and fundamental groups, hence, by Whitehead's theorem, the constructed map is a homotopy equivalence. This concludes the proof of the theorem. □

6. Hypersurface Singularities of Transversal Type A_k

In this section the germs of analytic functions $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of $n+1$ complex variables shall be considered, having the hyperplane $H = \{(x, y_1, \dots, y_n) \mid x = 0\}$ as their singular set and representable in the form $f = x^k g(x, y_1, \dots, y_n)$, where $k > 2$.

Let us introduce some definitions.

Definition 6.1. A singular point z in H is called an $A_{k\infty}$ type singular point if in some neighborhood U of the point z there exists a local coordinate system $(x, y_1, \dots, y_n)$ such that

$$H = \{x = 0\}, \quad x(z) = 0, \quad y_i(z) = 0, \quad i = 1, 2, \dots, n,$$

and in U the identity $f = x^k$ holds.

Definition 6.2. A singular point z in H is called a $D_{k\infty}$ type singular point if in some neighborhood U of the point z there exists a local coordinate system $(x, y_1, \dots, y_n)$ such that

$$H = \{x = 0\}, \quad x(z) = 0, \quad y_i(z) = 0, \quad i = 1, 2, \dots, n,$$

and in U the identity $f = x^k y_1$ holds.

Definition 6.3. The codimension of the singularity of a germ is called

$$\operatorname{codim} f = \dim_{\mathbb{C}} \left[(x^k)/\tau(f) \right].$$

Definition 6.4. A singularity $f \in (x^k)$ is called an isolated hyperplane singularity of transversal type A_k if $\operatorname{codim} f < +\infty$.

Similarly to the isolated hyperplane singularity case one can prove

Theorem 6.5. *Let $f \in (x^k)$ be not of type $A_{k\infty}$ or $D_{k\infty}$, and, moreover, let the germ $g(0, y_1, \ldots, y_n)$ from the representation $f = x^k g(x, y_1, \ldots, y_n)$ be not identically zero; then the following assertions are equivalent:*

(a) codim f *is finite;*

(b) *the function $g(x, y_1, \ldots, y_n)$ has an isolated singularity at zero;*

(c) *outside the points with $g(0, y_1, \ldots, y_n) = 0$ the germ f has type $A_{k\infty}$, while at the points with $g(0, y_1, \ldots, y_n) = 0$, except for the origin, it has $D_{k\infty}$ type singular points.*

Theorem 6.6. *Let $f \in (x^k)$ have an isolated hyperplane singularity of transversal type A_k on the hyperplane H; then there exists a deformation $\widetilde{f}$ of the form*

$$\widetilde{f} = x^k(g(x, y_1, \ldots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n + \lambda_{n+1}),$$

where

$$\lambda_{n+1} \in \operatorname{Reg}(g(x, y_1, \ldots, y_n) + \lambda_1 y_1 + \cdots + \lambda_n y_n)$$

and

$$\lambda_i \in \operatorname{Reg} \operatorname{grad} g(0, y_1, \ldots, y_n)),$$

satisfying the condition: $\widetilde{f}$ has only $A_{k\infty}$ and $D_{k\infty}$ type singular points on H and only Morse type singular points outside H. Moreover, the number σ of Morse points is calculated by the formula

$$\sigma = \dim_{\mathbb{C}} \left[(x^k)/(x f_x, f_{y_1}, \ldots, f_{y_n})\right]. \tag{1}$$

Lemma 6.7. *The Milnor fibre $f^{-1}(t) \cap B_0$, where B_0 is a cylinder around the singular set H, admits a k-fold covering of a wedge of the circle S^1 and μ copies of the n-sphere S^n, where $\mu = \mu(g(0, y_1, \ldots, y_n))$ is the Milnor number of the isolated singularity $g(0, y_1, \ldots, y_n)$.*

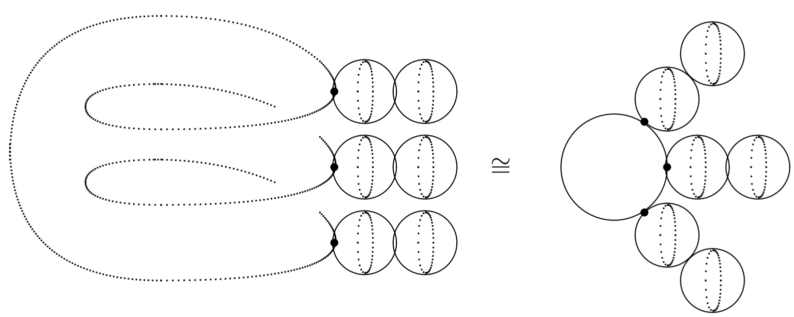

Figure 5

This enables the proof of

Lemma 6.8. *The Milnor fibre $f^{-1}(t) \cap B_0$ has the homotopy type of a wedge of the circle S^1 and $k \cdot \mu$ copies of the n-sphere S^n (see Figure 5).*

This implies

Theorem 6.9. *The Milnor fibre of the isolated hyperplane singularity of transversal type A_k in the ball $B_\varepsilon \subset \mathbb{C}^{n+1}$ is homotopy equivalent to the wedge of a circle and $\mu k + \sigma$ copies of the n-sphere, where μ is the Milnor number of the isolated singularity $g(0, y_1, \dots, y_n)$, while σ is the number of Morse points calculated by formula (1).*

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