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THE UNIFORM NORMING OF RETRACTIONS ON SHORT INTERVALS FOR CERTAIN FUNCTION SPACES

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Abstract. For Lizorkin–Triebel spaces the family of extension operators is constructed which yield a minimal (in order) value of the norm among all possible extensions of a given function defined initially on the interval of an arbitrary small length.

The techniques used restrict us to the one-dimensional case and spaces defined via differences of first order.

§ 1. Definitions and Formulation of the Main Result

Let $1 < p, q < \infty$, $E_{N,p}$ stand for the set of all entire analytic functions with the Fourier transform supported in $[-N, N]$ belonging to $L_p(R^1)$ (see [1], 1.4); $\{\beta_k\}$, $\{N_k\}$, $k \in \{1, 2, \dots\}$ be two sequences of positive numbers such that

$$N_{k+1} \geq \lambda N_k, \quad \lambda_1\beta_k \leq \beta_{k+1} \leq \lambda_2\beta_k, \quad \lambda > 1, \quad \lambda_2 \geq \lambda_1 > 1. \tag{1}$$

The space $L^{(\beta,N)}_{p,q}$ of Lizorkin–Triebel type consists, by the definition [2], of all functions $f(x) \in L_p(R^1)$ which can be represented as the sum of the series

$$f(x) = \sum f_k(x); \quad f_k \in E_{N_k,p}, \quad \|\{\beta_k f_k(x)\}\|_{L_p(l_q)} < \infty, \tag{2}$$

and the norm in $L^{(\beta,N)}_{p,q}$ is defined as the infimum of the last expression in (2). If $\beta_k = 2^{kr}$, $N^k = 2^k$ then one has usual (power-scaled) spaces $L^r_{p,q}$ (see [1], 2.3, 2.5).

The function $g(x)$ given on the interval $(0, b)$, $b > 0$, belongs to the retraction space $L^{(\beta,N)}_{p,q}(0, b)$ if there exists a function $f(x) \in L^{(\beta,N)}_{p,q}$ which

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coincides with $g(x)$ on $(0,b)$ and the corresponding norm is defined in the usual way as

$$A_1(g) = \inf \ \Big\{ \ \|f\|_{L^{(\beta,N)}_{p,q}} : \ f(x) = g(x), \ 0 < x < b \Big\}. \tag{3}$$

Our aim is to obtain explicit (constructive) quantities equivalent to (3) in terms of internal properties of the original function $g(x)$, $x \in (0,b)$. Let us, for $t \in R$, denote by $\Delta_t g(x)$ the difference $g(x+h) - g(x)$ provided that both points $x, x+h$ belong to $(0,b)$ (otherwise, put $\Delta_h g(x) = 0$). Introduce the averaged local oscillation (of first order) defined for $h > 0$ by the formula

$$\Omega_h(g,x) := \int\limits_{-1}^{1} |\Delta_{ht} g(x)| \ dt. \tag{4}$$

It is clear from the definition that

$$\begin{gathered} \Omega_h(g,x) \geq 0; \ \Omega_h(g,x) = 0, \quad x \notin (0,b), \\ \Omega_h(g,x) = (b/h) \ \Omega_b(g,x), \quad h \geq b. \end{gathered} \tag{5}$$

The behavior of the determining sequences $\{\beta_k\}$, $\{N_k\}$ will be reflected by the specific function first studied in [3]

$$\gamma(b) = \Big(\sum_k (\beta_k (N_k^{-1} + b)^{1/p})^{-p'} \Big)^{-p/p'} \quad (1/p + 1/p' = 1). \tag{6}$$

The series in (6) converges for all $b > 0$ and the function $\gamma(b)$ increases whereas $\gamma(b)/b$ decreases. It is easy to calculate that for the spaces $L^r_{p,q}$ and $0 < b < 1$ the function $\gamma(b)$ is equivalent to b^{1-pr} if $0 < r < 1/p$; $(\log \ 2/b)^{1-p}$ if $r = 1/p$, 1 if $r > 1/p$.

Theorem. *Let in* (1) $\lambda > \lambda_2$ (*this implies* $r < 1$ *for power-scaled spaces*). *The quantity* $A_1(g)$ *is equivalent to*

$$A_2(g) = \frac{\gamma(b)^{1/p}}{b} \ \Big| \int\limits_0^b g(x) dx \ \Big| + \|\{\beta_k \Omega_{N_k^{-1}}(g,x)\}\|_{L_p(l_q,(0,b))} \tag{7}$$

and the ratio of these two quantities is bilaterally bounded for $b > 0$. *The same remains valid if one changes the order of integration and takes the modulus of* $g(x)$ *in the first summand in* (7).

§ 2. Auxilary Assertions

First we shall show that the theorem is a consequence of the following lemmas.

Lemma 1. *Let a positive integer l be chosen so that $\lambda^l > \lambda_2$ and let C^l stand for the class of all functions $f(x)$, $-\infty < x < \infty$, having the derivative $f^{(l-1)}(x)$ which is absolutely continuous on any finite interval. The norm in the space $L^{(\beta,N)}_{p,q}$ (see (1)) is equivalent to the following two quantities:*

$$\|f\|^{(2)}_{L^{(\beta,N)}_{p,q}} := \inf\left\{ \left\| \left\{ \beta_k \sum_{s=0}^{l} N_k^{-s} |f^{(s)}_k(x)| \right\} \right\|_{L_p(l_q,R)} : \quad f_k(x) \in C^l, \quad f(x) = \sum f_k(x) \right\}, \tag{8}$$

$$\|f\|^{(3)}_{L^{(\beta,N)}_{p,q}} := \|f\|_p + \|\{\beta_k \Omega_{N_k^{-1}}(f,x)\}.\|_{L_p(l_q;R)} \tag{9}$$

These equivalences have been established in [3], [4] (see also [2]).

Lemma 2. *For any $b > 0$ and any function $f \in L^{(\beta,N)}_{p,q}$ the inequality*

$$\frac{\gamma(b)^{1/p}}{b} \int_0^b |f(x)|dx \le c_1 \ \|f\|_{L^{(\beta,N)}_{p,q}} \tag{10}$$

holds, and there exists a function $f_b(x) \in L^{(\beta,N)}_{p,q}$ such that

$$f_b(x) = 1, \quad \forall x \in (0,b); \qquad \|f_b\|_{L^{(\beta,N)}_{p,q}} \le c_2 \gamma(b)^{1/p}, \tag{11}$$

where $\gamma(b)$ is defined in (3) and $c_1 > 0, c_2 > 0$ do not depend on b.

These estimates have been established in [5] (see also [2], Theorem 5.4).

Lemma 3. *If $\lambda > \lambda_2$ and $\int_0^b g(x)dx = 0$ then there exists a function $f(x), x \in R$ such that $f(x) = g(x)$ for all $x \in (0,b)$ and the estimate*

$$\|f\|_{L^{(\beta,N)}_{p,q}} \le c_0 \|\{\beta_k \Omega_{N_k^{-1}}(g,x)\}\|_{L_p(l_q;(0,b))} \tag{12}$$

holds, where the constant c_0 depends neither on g nor on b.

This lemma is the central part of our discussion; its proof is given in the next section.

Now let us suppose that this assertion has already been proved. Consider an arbitrary function $g(x)$ defined on $(0,b)$, which we shall represent as

$$g(x) = B + g_1(x); \quad B := \frac{1}{b}\int_0^b g(x)dx; \quad g_1(x) := g(x) - B. \tag{13}$$

By construction, $\int_0^b g_1(x)dx = 0$ and from Lemma 3 implies (see (9)) that there exists $f_1(x)$ such that $f_1(x) = g_1(x), x \in (0, b)$ and

$$\begin{aligned} \|f_1\|_{L_{p,q}^{(\beta,N)}} &\le c_0 \|\{\beta_k \Omega_{N_k^{-1}}(g_1, x)\}\|_{L_p(l_q;(0,b))} = \\ &= c_0 \|\{\beta_k \Omega_{N_k^{-1}}(g, x)\}\|_{L_p(l_q;(0,b))} \end{aligned} \tag{14}$$

because one has identically $\Delta_h g_1(x) = \Delta_h g(x)$.

On the other hand, according to (8) there exists $f_2(x)$ which equals B on $(0, b)$ such that

$$\|f_2\|_{L_{p,q}^{(\beta,N)}} \le c|B|\gamma^{1/p}(b). \tag{15}$$

Then for $f(x) = f_1(x) + f_2(x)$ one has the estimate

$$\|f\|_{L_{p,q}^{(\beta,N)}} \le c\big(|B|\gamma^{1/p}(b) + \|\{\beta_k \Omega_{N_k^{-1}}(g, x)\}\|_{L_p(l_q;(0,b))}\big) \tag{16}$$

and because $f(x) = g(x)$, $\forall x \in (0, b)$ we conclude that $A_1(g) \le cA_2(g)$.

Conversely, let us take arbitrary $\varphi(x) \in L_{p,q}^{(\beta,N)}$ which coincides with $g(x)$, $0 < x < b$. According to Lemma 2 one has

$$\begin{aligned} &\|\{\beta_k \Omega_{N_k^{-1}}(g, x)\}\|_{L_p(l_q;(0,b))} \le \\ &\le \|\{\beta_k \Omega_{N_k^{-1}}(\varphi, x)\}\|_{L_p(l_q;R)} \le c\|\varphi(x)\|_{L_{p,q}^{(\beta,N)}}. \end{aligned} \tag{17}$$

From Lemma 3 it follows that

$$\frac{\gamma(b)^{1/p}}{b} \int_0^b |g(x)|dx \le \ c_2\|\varphi\|_{L_{p,q}^{(\beta,N)}} \tag{18}$$

and by combining these two inequalities we come finally to the estimates

$$A_1(g) \ge \|\varphi(x)\|_{L_{p,q}^{(\beta,N)}} \ge c_0 A_2(g) \tag{19}$$

which in connection with the inverse estimate yield $A_1(g) \asymp A_2(g)$.

§ 3. Proof of Lemma 3

Step 1. Let us introduce a function

$$\rho: \ R^1 \to [0, b]; \quad x \to \rho(x) := \min \{ \ |x - 2mb| : \ m \in \mathbb{Z} \ \}, \tag{20}$$

i.e., $\rho(x)$ denotes the minimal distance between the point $x \in R^1$ and the points of the mesh $\{2mb\}$, $m \in \mathbb{Z}$. Now consider a function $G(x) := g(\rho(x))$

which extends $g(x)$ onto the whole axis. By construction, it immediately follows that

$$G(-x) \equiv G(x);\ G(x+2b) \equiv G(x),$$
$$\int\limits_{x}^{x+2b} G(y)\ dy = 2\int\limits_{0}^{b} g(y)\ dy = 0\ ,\ \forall x \in R^1. \tag{21}$$

Here we have used the assumption that the total integral of $g(x)$ over the interval $(0, b)$ equals zero. Moreover (which is the most important), by (21) and (4) we have

$$\Omega_h(G, x) \le 2\Omega_h(g, \rho(x)) \tag{22}$$

for any $x \in R^1$ and $0 < h \le b$.

Step 2. Choose a kernel function $\Phi(x)$ such that

$$\Phi(x) \in C_0^\infty,\ \ \operatorname{supp}\Phi \subset (-1, 1),\ \ \int \Phi(x)\, dx = 1. \tag{23}$$

Here and in the sequel the integration without indication of the lower and upper limits extend onto the whole axis.

Introduce the family of averaged functions

$$G(x, h) := \int \Phi(t) G(x+ht)\ dt = h^{-1}\int \Phi((y-x)/h)\ G(y)\ dy. \tag{24}$$

Using (23) and (4) we obtain the estimate

$$|G(x, h) - G(x)| = |\int \Phi(t)\ (G(x+ht) - G(x))\ dt| \le c_0\Omega_h(G, x) \tag{25}$$

and thus $G(x, h) \to G(x)$, $h \to +0$ for almost all x.

Similarily for the derivatives of these function we have

$$|G'_x(x, h)| = h^{-2}|\int \Phi_1((y-x)/h)\ G(y)\ dy\ | =$$
$$= h^{-1}|\int \Phi_1(t)\ (G(x+ht) - G(x))\ dt\ | \le c_0 h^{-1}\Omega_h(G, x). \tag{26}$$

Here we have used the notation $\Phi_1(t) = -(d\Phi(t)/\ dt)$, taking into account that the integral over the whole axis of the function $\Phi_1(t)$ equals 0.

Step 3. Denote by $m = m_b$ the greatest k for which $N_k^{-1} \ge b$ so that $N_m^{-1} \ge b$, $N_{m+1}^{-1} < b$. In case $N_1^{-1} < b$ (this would only mean that the interval $(0, b)$ is not "small") we put simply $m = 0$. Denote by $\tilde N_k$ the numbers N_k, $k > m$, $\tilde N_m := b^{-1}$.

Introduce the sequence of functions

$$G_k(x) \equiv 0, \quad k < m, \quad G_m(x) := G(x, b),$$
$$G_k(x) := G(x, \tilde N_k^{-1}) - G(x, \tilde N_{k-1}^{-1}), \quad k > m. \tag{27}$$

(Please note the difference between the cases $k = m$ and $k > m$!) These functions belong to C^∞, are $2b-$ periodic, and their integrals over any interval of the length $2b$ equal zero. Therefore (recall that we deal with the real-valued functions) *on the interval* $(-2b, 0)$ *there exist points* x_k *such that* $G_k(x_k) = 0,\ k \ge m$.

For the functions $G_k(x)$ it follows from (25),(26) that

$$\tilde N_k^{-1}\ |G_k'(x)| + |G_k(x)| \le c_0(\Omega_{\tilde N_k^{-1}}(G, x) + \Omega_{\tilde N_{k-1}^{-1}}(G, x)),\ k > m;$$
$$b|G_m'(x)| \le c_0 \Omega_b(G, x). \tag{28}$$

As for the function $G_m(x)$ we shall use the fact that $G_m(x_m) = 0$ at some point $x_m \in (-2b, 0)$. This implies that for any $x \in (-2b, 4b)$

$$|G_m(x)| = \left| \int_{x_k}^{x} G_m'(y)\ dy \right| \le c_0 b^{-1} \int_{-2b}^{4b} \Omega_b(G, y)\ dy \tag{29}$$

and consequently

$$\|G_m(x)\|_{L_p(-2b,4b)} \le c_0 \|\Omega_b(G, x)\|_{L_p(-2b,4b)}. \tag{30}$$

Note that only now we need the condition that the integral of $g(x)$ is zero.

Step 4. Let us consider the sequence of functions defined on the whole axis

$$f_k(x) := G_k(x), \quad x \in (x_k, x_k + 4b);$$
$$f_k(x) := 0, \quad x \le x_k \quad or \quad x \ge x_k + 4b. \tag{31}$$

These functions are absolutely continuous because $G_k(x_k) = 0$, they coincide with $G_k(x)$ for $0 \le x \le b$ because $[0, b] \subset (x_k, x_k + 4b)$, and for all x except two points x_k and $x_k + 4b$ the estimates

$$|f_k(x)| \le |G_k(x)|, \quad |f_k'(x)| \le |G_k'(x)| \tag{32}$$

hold (except two points x_k and $x_k + 4b$ where the derivatives$f_k'(x)$ may not exist). Therefore from (27) it follows that

$$N_k^{-1}\ |f_k'(x)| + |f_k(x)| \le c_0(\Omega_{N_k^{-1}}(G, x) + \Omega_{\tilde N_{k-1}^{-1}}(G, x)) \tag{33}$$

for $k < m$ and $x \in (-2b, 4b)$. By construction, the left-hand side equals 0 outside $(-2b, 4b)$. Thus, taking also into account estimate (22), we obtain

$$\begin{aligned}
&\Big\| \Big(\sum_{k>m} (\beta_k (N_k^{-1}\, |f_k'(x)| + |f_k(x)|))^q \Big)^{1/q} \Big\|_{L_p(R^1)} \leq \\
&\leq c_0 \Big\| \Big(\sum_{k>m} (\beta_k (\Omega_{N_k^{-1}}(G,x) + \Omega_{\tilde{N}_{k-1}^{-1}}(G,x)))^q \Big)^{1/q} \Big\|_{L_p(-2b,4b)} \leq \\
&\leq c_0 \bigg(\Big\| \Big(\sum_{k>m} (\beta_k (\Omega_{N_k^{-1}}(g,x))^q \Big)^{1/q} \Big\|_{L_p(0,b)} + \beta_m \|\Omega_b(g,x)\|_{L_p(0,b)} \bigg). \qquad (34)
\end{aligned}$$

As for the case $k = m$, we have from estimates (28) (the second part), (30), and (5)

$$\begin{aligned}
&N_m^{-1} \|f_m'(x)\|_{L_p(R^1)} + \\
&+ \|f_m(x)\|_{L_p(R^1)} \leq c_0 (N_m b)^{-1} \|\Omega_b(G,x)\|_{L_p(-2b,4b)} \leq \\
&\leq c_0 (N_m b)^{-1} \|\Omega_b(g,x)\|_{L_p(0,b)} = c_0 \|\Omega_{N_m^{-1}}(g,x)\|_{L_p(0,b)}. \qquad (35)
\end{aligned}$$

By combining (34), (35), and (27), we come to the conclusion that

$$\begin{aligned}
&\|\{\beta_k (|f_k(x)| + N_k^{-1} |f_k'(x)|)\}\|_{L_p(l_q, R^1)} \leq \\
&\leq c_0 \|\{\beta_k \Omega_{N_k^{-1}}(g,x)\}\|_{L_p(l_q,(0,b))}. \qquad (36)
\end{aligned}$$

According to Lemma 1 this implies that the function

$$f(x) = \sum_{k=1}^{\infty} f_k(x) \qquad \text{(convergence in } L_p) \qquad (37)$$

which, by construction (see (25), (27), (31)), coincides with $g(x)$ on $(0, b)$, belongs to the space $L_{p,q}^{(\beta,N)}$ and estimate (12) holds.

This completes the proof of Lemma 3 and thus of the theorem. □

Remark 1. The extension operator constructed in the proof of Lemma 3 uses the zeros of functions $G_k(x)$ and is thus nonlinear. The author's conjecture is that the *linear* operator must exist and the result of the theorem remains valid also for differences of higher order (the number l having been chosen as in Lemma 1).

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