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ON THE BOUNDARY VALUE PROBLEM IN A DIHEDRAL ANGLE FOR NORMALLY HYPERBOLIC SYSTEMS OF FIRST ORDER

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Abstract. Some structural properties as well as a general three-dimensional boundary value problem for normally hyperbolic systems of partial differential equations of first order are studied. A condition is given which enables one to reduce the system under consideration to a first-order system with the spliced principal part. It is shown that the initial problem is correct in a certain class of functions if some conditions are fulfilled.

§ 1. Some Structural Properties of Normally Hyperbolic Systems of First Order

In the Euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$, of independent variables (x, t), $x = (x_1, \dots, x_n)$, we consider the system of partial differential equations of first order

$$A_0 u_t + \sum_{i=1}^{n} A_i u_{x_i} + Bu = F, \tag{1.1}$$

where A_i, $i = 0, 1, \dots, n$, B are the given real $m \times m$ matrix-functions, $m \geq 2$, F is the given and u is the unknown m-dimensional real vector-function. It is assumed that $\det A_0 \neq 0$.

Denote by $p(x, t; \lambda, \xi)$ the characteristic determinant of system (1.1), i.e., $p(x, t; \lambda, \xi) \equiv \det Q(x, t; \lambda, \xi)$, where

$$Q(x, t; \lambda, \xi) \equiv A_0 \lambda + \sum_{i=1}^{n} A_i \xi_i, \quad \lambda \in \mathbb{R}, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

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Since $\det A_0 \neq 0$, we have the representation

$$p(x,t;\lambda,\xi) = \det A_0 \prod_{i=1}^{l} \big(\lambda - \lambda_i(x,t;\xi)\big)^{k_i}, \quad \sum_{i=1}^{l} k_i = m,$$
$$l = l(x,t;\xi), \quad k_i = k_i(x,t;\xi), \quad i = 1,\dots,l.$$

System (1.1) is said to be hyperbolic at the point (x,t) if all roots $\lambda_1(x,t;\xi),\dots,\lambda_l(x,t;\xi)$ of the polynomial $p(x,t;\lambda,\xi)$ are real numbers.

One can easily verify that

$$k_i(x,t;\xi) \geq m - \operatorname{rank} Q\big(x,t;\lambda_i(x,t;\xi),\xi\big), \quad i = 1,\dots,l.$$

The hyperbolic system (1.1) is said be normally hyperbolic at the point (x,t), if the equalities

$$k_i(x,t;\xi) = m - \operatorname{rank} Q\big(x,t;\lambda_i(x,t;\xi),\xi\big), \quad i = 1,\dots,l,$$

are fulfilled (see, e.g., [1], [2]).

Note that strictly hyperbolic systems, i.e., when $l = m$, $k_i = 1$, $i = 1,\dots,m$, form a subclass of normally hyperbolic systems.

Since $\det A_0 \neq 0$, it can be assumed without loss of generality that $A_0 = E$, where E is the $m\times m$ unit matrix. For simplicity, we shall always assume that (i) $n = 2$, $x_1 = x$, $x_2 = y$; (ii) the matrices A_1 and A_2 are constant; (iii) system (1.1) is normally hyperbolic.

In our assumptions, in the space of independent variables x, y and t, system (1.1) is rewritten as

$$u_t + A_1 u_x + A_2 u_y + Bu = F. \tag{1.2}$$

It is easy to show that since system (1.2) is normally hyperbolic, each of the matrices A_i, $i = 1,2$, has only real characteristic roots so that the corresponding eigenvectors of the operator A_i, $1 \leq i \leq 2$, form a complete system, i.e., a basis in the space $\mathbb{R}^m$. Therefore the matrices A_i, $i = 1,2$, are diagonalizable, i.e., there exist real nondegenerate matrices C_i, $i = 1,2$, such that the matrices $C_i^{-1} A_i C_i$, $i = 1,2$, are diagonal.

The normally hyperbolic system (1.2) will be said to be diagonalizable if there exists a real nondegenerate matrix C such that the matrices $C^{-1}A_iC$, $i = 1,2$, are diagonal. We have

Lemma 1.1. *The normally hyperbolic system* (1.2) *is diagonalizable if and only if the matrices* A_1 *and* A_2 *are commutative, i.e.,* $A_1A_2 = A_2A_1$.

Proof. The necessity readily follows from the fact that the diagonal matrices $C^{-1}A_1C$ and $C^{-1}A_2C$ are commutative, $C^{-1}A_1CC^{-1}A_2C = C^{-1}A_2CC^{-1}A_1C$, i.e., $C^{-1}A_1A_2C = C^{-1}A_2A_1C$. This immediately implies $A_1A_2 = A_2A_1$.

To prove sufficiency note that since system (1.2) is normally hyperbolic, we have $\dim \mathbb{R}_{\lambda_i} = k_i$, where $\mathbb{R}_{\lambda_i} \equiv \text{Ker } (A_1 - \lambda_i E)$, $1 \le i \le l$. Clearly, $\hat{A}_1(\mathbb{R}_{\lambda_i}) \subset \mathbb{R}_{\lambda_i}$, $1 \le i \le l$, where $\hat{A}_1$ stands for the linear transform corresponding to the matrix A_1.

Let $\{\nu_{ij}\}_{j=1}^{k_i}$ be an arbitrary basis of the space $\mathbb{R}_{\lambda_i}, 1 \le i \le l$. By the definition of the space $\mathbb{R}_{\lambda_i}$, the vectors $\nu_{i1}, \dots, \nu_{ik_i}$ are the eigenvectors for the transform $\hat{A}_1$ and correspond to the eigenvalue λ_i, $1 \le i \le l$. Therefore, the matrix of the transform $\hat{A}_1$ in the basis $\{\nu_{ij}\}_{j=1}^{k_i}$ of the space $\mathbb{R}_{\lambda_i}$ will be diagonal of order $(k_i \times k_i)$ and written as diag $[\underbrace{\lambda_i, \dots, \lambda_i}_{k_i-\text{times}}]$, $1 \le i \le l$. Hence, recalling that the decomposition of the space $\mathbb{R}^m$ as the direct sum of subspaces $\mathbb{R}_{\lambda_i}$, $i = 1, \dots, l$, i.e., $\mathbb{R}^m = \mathbb{R}_{\lambda_1} \oplus \dots \oplus \mathbb{R}_{\lambda_l}$ is unique, we can write the matrix D_1 of the transform $\hat{A}_1$ in the basis $\{\nu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ as $D_1 = \text{diag } [\underbrace{\lambda_1, \dots, \lambda_1}_{k_1-\text{times}}, \dots, \underbrace{\lambda_l, \dots, \lambda_l}_{k_l-\text{times}}]$.

Let $\widetilde{A_{2i}}$ be the matrix corresponding to the linear transform $\hat{A}_2$ of the subspace $\mathbb{R}_{\lambda_i}$, $1 \le i \le l$ in the basis $\{\nu_{ij}\}_{j=1}^{k_i}$. Since the matrices A_1 and A_2 are commutative, the subspace $\mathbb{R}_{\lambda_i}$ is invariant with respect to the linear transform $\hat{A}_2$, i.e., $\hat{A}_2(\mathbb{R}_{\lambda_i}) \subset \mathbb{R}_{\lambda_i}$, $1 \le i \le l$ (see, e.g., [3]). Therefore, in the basis $\{\nu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$, the matrix $\widetilde{A_2}$ corresponding to $\hat{A}_2$ will be block-diagonal and have, on its principal diagonal, matrices $\widetilde{A_{2i}}$, $i = 1, \dots, l$. It is well known that matrices giving the same linear transform in different bases are similar. At the same time, similar matrices have the same characteristic equation. Therefore we have

$$\det(A_2 - \lambda E) = \det(\widetilde{A_2} - \lambda E) = \det(\widetilde{A_{21}} - \lambda E_{k1}) \times \dots \times \det(\widetilde{A_{2l}} - \lambda E_{kl}).$$

Since system (1.2) is normally hyperbolic, the matrix A_2 has only real characteristic roots. Thus for the linear transform $\hat{A}_2 : \mathbb{R}_{\lambda_i} \to \mathbb{R}_{\lambda_i}$ there exists a basis $\{\mu_{ij}\}_{j=1}^{k_i}$ which consists of the real vectors of the subspace $\mathbb{R}_{\lambda_i}$, and where the matrix A_{2i}^* of the above-mentioned transform is of Jordan form, $1 \le i \le l$. Therefore, in the basis $\{\mu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$, the matrix A_2^* of the transform $\hat{A}_2$ will also be of Jordan form. But, since system (1.2) is normally hyperbolic, in the space $\mathbb{R}^m$ there exists a basis $\{\sigma_i\}_{i=1}^m$ in which the matrix of the transform $\hat{A}_2$ is diagonal. Further, as is well known, a Jordan matrix similar to the diagonal one is diagonal too. Therefore in the basis $\{\mu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$ the matrix A_2^* is diagonal, but the matrix of the transform $\hat{A}_1$ is diagonal in any basis of the transform $\mathbb{R}_{\lambda_i}$, in particular, in the basis $\{\mu_{ij}\}_{j=1}^{k_i}$, $1 \le i \le l$. Therefore the matrices of the transforms $\hat{A}_1$ and $\hat{A}_2$ will be diagonal in the basis $\{\mu_{ij};\ i = 1, \dots, l;\ j = 1, \dots, k_i\}$ of the space $\mathbb{R}^m$. □

§ 2. Statement of the Problem and Some Notations

In the discussion below the matrices A_1 and A_2 will always be assumed to be commutative, i.e., the equality

$$A_1A_2 = A_2A_1 \tag{2.1}$$

is valid.

After introducing a new unknown function v by the formula $u = Cv$ with the nondegenerate matrix C whose existence was proved by condition (2.1) in §1, system (1.2) takes the form

$$v_t + D_1v_x + D_2v_y + B_0v = F_0, \tag{2.2}$$

where by virtue of Lemma 1.1 the matrices $D_i = C^{-1}A_iC$, $i = 1, 2$, are diagonal, i.e., $D_1 = \operatorname{diag}[\nu_1, \dots, \nu_m]$, $D_2 = \operatorname{diag}[\mu_1, \dots, \mu_m]$, $B_0 = C^{-1}BC$, $F_0 = C^{-1}F$.

It is obvious that the directions defined by the vectors $l_i = (\nu_i, \mu_i, 1)$, $i = 1, \dots, m$, are bicharacteristic.

Let $\hat{A}_j$ be the linear transform corresponding to he matrix A_j, $1 \le j \le 2$. Denote by Λ_i an m-dimensional vector which is the eigenvalue of the transform $\hat{A}_1$, corresponding to the eigenvalue ν_i, $1 \le i \le m$. By virtue of (2.1) the vector Λ_i is also the eigenvector of the transform $\hat{A}_2$ corresponding to the eigenvalue μ_i, $1 \le i \le m$. By Lemma 1.1, the vectors Λ_i, $i = 1, \dots, m$, can be chosen such that the $(m \times m)$ matrix $C = [\Lambda_1, \dots, \Lambda_m]$, whose columns consist of these vectors, will reduce the matrices A_1 and A_2 to the diagonal form, namely, to $D_i = C^{-1}A_iC$, $i = 1, 2$.

Obviously, the vectors $l_i = (\nu_i, \mu_i, 1)$ and $l_j = (\nu_j, \mu_j, 1)$ define the same bicharacteristic direction if the equalities $\nu_i = \nu_j$, $\mu_i = \mu_j$, $1 \le i \ne j \le m$, are fulfilled. In this context, we divide the set of vectors $\{l_1, \dots, l_m\}$ into nonintersecting classes $\{l_{11}, \dots, l_{1s_1}\}, \dots, \{l_{m_01}, \dots, l_{m_0s_{m_0}}\}$ whose representatives with respective "multiplicities" $s_1, \dots, s_{m_0}$, will be denoted by $\widetilde{l}_1, \dots, \widetilde{l}_{m_0}$, $m_0 \le m$. Now the matrix $C = [\Lambda_1, \dots, \Lambda_m]$ can be represented as

$$C = [\Lambda_{11}, \dots, \Lambda_{1s_1}; \dots; \Lambda_{m_01}, \dots, \Lambda_{m_0s_{m_0}}],$$

or as $C = (\widetilde{C_1}, \widetilde{C_2})$, where

$$\widetilde{C_1} = [\Lambda_{11}, \dots, \Lambda_{1s_1}; \dots; \Lambda_{q1}, \dots, \Lambda_{qs_q}],$$
$$\widetilde{C_2} = [\Lambda_{q+11}, \dots, \Lambda_{q+1s_{q+1}}; \dots; \Lambda_{m_01}, \dots, \Lambda_{m_0s_{m_0}}]$$

and q will be defined below.

Denote by D^* the dihedral angle

$$D^* \equiv \{(x, y, t) \in \mathbb{R}^3,\ t - y > 0,\ t + y > 0\}.$$

For bicharacteristic directions of system (1.2) we make the following assumption: bicharacteristics passing through any point of the edge $\Gamma^* \equiv \{(x,y,t) \in \mathbb{R}^3 : y = t = 0, x \in \mathbb{R}\}$ of the angle D^* have no common points with the set $\overline{D^*}\backslash\Gamma^*$. This is equivalent to the fulfillment of the inequalities

$$|\mu_i| > 1, \quad i = 1, \dots, m. \tag{2.3}$$

Let $P_0 = P_0(x_0, y_0, t_0)$ be an arbitrary fixed point of the set $\overline{D^*}\backslash\Gamma^*$, and let $S_1 \supset \Gamma^*$ and $S_2 \supset \Gamma^*$ be the two-dimensional edges of D^*, i.e., $\partial D^* = S_1 \cup S_2$, $S_1 \equiv \{(x,y,t) \in \mathbb{R}^3 : x \in \mathbb{R},\ y = t,\ t \in \overline{\mathbb{R}}_+\}$, $S_2 \equiv \{(x,y,t) \in \mathbb{R}^3 : x \in \mathbb{R},\ y = -t,\ t \in \overline{\mathbb{R}}_+\}$, $\mathbb{R}_+ \equiv (0,\infty)$. From the point P_0 we draw the bicharacteristic beam $\widetilde{L_i}(P_0)$ of system (2.2) which corresponds to the vector $\widetilde{l}_i$, is directed towards the decreasing values of the t-coordinate of a moving point $\widetilde{L_i}(P_0)$, and intersects one of the edges S_1 or S_2 at a point $\widetilde{P_0^i}$, $1 \le i \le m_0$. It can be assumed without loss of generality that bicharacteristic beams defined by the vectors $\widetilde{l}_1, \dots, \widetilde{l}_q$ and passing through the point P_0 intersect the edge S_1, while those defined by $\widetilde{l}_{q+1}, \dots, \widetilde{l}_{m_0}$ intersect the edge S_2.

Below, it will be assumed for simplicity that $q = 2$ and $m_0 = 3$, $\widetilde{l}_i = (\widetilde{\nu}_i, \widetilde{\mu}_i, 1)$, $i = 1,2,3$, and also rank $(\widetilde{l}_1, \widetilde{l}_2, \widetilde{l}_3) = 3$.

Through the point P_0 draw a plane $P_{\widetilde{l}_i,\widetilde{l}_j}$, parallel to the vectors $\widetilde{l}_i$ and $\widetilde{l}_j$, $1 \le i < j \le 3$. We introduce the following notation:

$P_{\widetilde{l}_2,\widetilde{l}_3}$ and $P_{\widetilde{l}_1,\widetilde{l}_3}$ are respectively the intersection points of the planes Q_1 and Q_2 with the edge Γ^*;

D is a the domain forming a pentahedron with the vertices at the points $P_0, \widetilde{P_0^2}, \widetilde{P_0^1}, Q_2, \widetilde{P_0^3}, Q_1$;

Δ_1 and Δ_2 are respectively a triangle and rectangle with the vertices at the points $Q_2, \widetilde{P_0^3}, Q_1$ and $Q_1, \widetilde{P_0^2}, \widetilde{P_0^1}, Q_2$, respectively.

For system (1.2) we consider the boundary value problem formulated as follows: Find, in the domain D, a regular solution $u(x,y,t)$ of system (1.2) satisfying the boundary conditions

$$B_i u\big|_{\overline{\Delta_i}} = f_i, \tag{2.4}$$

where B_i are the given $(m_i \times m)$ matrix-functions and f_i are the given m_i-dimensional vector-functions, $i = 1,2, m_1 = s_1 + \cdots + s_q, m_2 = s_{q+1} + \cdots + s_{m_0}$. It is obvious that $m_1 + m_2 = m$ though we do not exclude the cases with $m_1 = 0$ or $m_2 = 0$, which correspond to the Cauchy problem. Below it will always be assumed that $0 < m_i < m$, $i = 1,2$.

A function $u(x,y,t)$ which, together with its partial derivatives u_x, u_y, u_t, is continuous in D and satisfies system (1.2) is called a regular solution of system (1.2).

Some analogs of the Goursat problem for hyperbolic systems of first order with two independent variables have been studied in [4]–[8]. A lot of papers are devoted to general boundary value problems of the Darboux type for normally hyperbolic systems of second order on a plane (see, e.g., [1], [2]). Some multidimensional problems of the Goursat and Darboux type are considered in several papers (see, e.g., [9]–[11]) both for a hyperbolic equation and for a system of equations in a dihedral angle. For hyperbolic equations of third order, a boundary value problem in a dihedral angle is investigated in [12].

Denote by Δ_i^* the orthogonal projection of the polygons Δ_i^*, $i=1,2$, onto the plane x,t. The restrictions of B_i and f_i on the sets $\overline{\Delta_i^*}$, $i=1,2$, will be denoted as before.

In the domains D and Δ_i^*, we introduce the following functional spaces

$$\overset{0}{C}_\alpha(\overline{D}) \equiv \Big\{ w \in C(\overline{D}) : w|_\Gamma = 0, \sup_{(x,y,t)\in\overline{D}\setminus\Gamma^*} \rho^{-\alpha} ||w(x,y,t)||_{\mathbb{R}^m} < \infty \Big\},$$

$$\overset{0}{C}_\alpha(\overline{\Delta_i^*}) \equiv \Big\{ \psi \in C(\overline{\Delta_i^*}) : \psi|_{\Gamma_1} = 0, \sup_{(x,t)\in\overline{\Delta_i^*}\setminus\Gamma_1^*} t^{-\alpha} ||\psi(x,t)||_{\mathbb{R}^m} < \infty \Big\},$$

where $\Gamma \equiv \overline{D}\cap\Gamma^*, \Gamma_1 \equiv \overline{\Delta_i^*}\cap\Gamma_1^*, i=1,2, \Gamma_1^* \equiv \{(x,t)\in\mathbb{R}^2 : x\in\mathbb{R}, t=0\}$, ρ is the distance from the point $(x,y,t)\in\overline{D}\setminus\Gamma^*$ to the edge Γ^* of the domain D^*, the real parameter $\alpha = const \geq 0$. For $a=(a_1,\dots,a_m)\in\mathbb{R}^m$, $m\geq 2$, denote $\|a\|_{\mathbb{R}_m} = |a_1|+\cdots+|a_m|$.

Obviously, the spaces $\overset{0}{C}_\alpha(\overline{D})$ and $\overset{0}{C}_\alpha(\overline{\Delta_i^*})$, $i=1,2$, are Banach ones with the norms

$$\|w\|_{\overset{0}{C}_\alpha(\overline{D})} = \sup_{(x,y,t)\in\overline{D}\setminus\Gamma^*} \rho^{-\alpha}||w(x,y,t)||_{\mathbb{R}^m},$$

$$\|\psi\|_{\overset{0}{C}_\alpha(\overline{\Delta_i^*})} = \sup_{(x,t)\in\overline{\Delta_i^*}\setminus\Gamma_1^*} t^{-\alpha}||\psi(x,t)||_{\mathbb{R}^m}.$$

Remark 2.1. Since the estimate $1 \leq \rho/t \leq \sqrt{2}$, $(x,y,t)\in D^*$, is uniform, the value ρ in the definition of the space $\overset{0}{C}_\alpha(\overline{D})$ below will be replaced by the variable t.

It is easy to verify that the fact that $w \in \overset{0}{C}(\overline{D})$ and $\psi \in \overset{0}{C}(\overline{\Delta_i^*})$ belong to the spaces $\overset{0}{C}_\alpha(\overline{D})$ and $\overset{0}{C}_\alpha(\overline{\Delta_i^*})$, respectively, is equivalent to the fulfillment of the inequalities

$$\begin{aligned} &||w(x,y,t)||_{\mathbb{R}^m} \leq ct^\alpha, \ \ (x,y,t)\in\overline{D}, \\ &||\psi(x,t)||_{\mathbb{R}^m} \leq ct^\alpha, (x,t)\in\overline{\Delta_i^*}, \ \ i=1,2. \end{aligned} \tag{2.5}$$

We shall investigate the boundary value problem (1.2), (2.4) in the Banach space

$$\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D}) \equiv \Big\{ u : \frac{\partial^{|i|}u}{\partial x^{i_1}\partial y^{i_2}\partial t^{i_3}} \in \overset{0}{C}_{\alpha}(\overline{D}),\ |i| \le 1,\ |i| = \sum_{j=1}^{3} i_j \Big\},$$

with respect to the norm

$$\|u\|_{\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})} = \sum_{|i|\le 1} \Big\| \frac{\partial^{|i|}u}{\partial x^{i_1}\partial y^{i_2}\partial t^{i_3}} \Big\|_{\overset{0}{C}_{\alpha}(\overline{D})}$$

assuming that the matrix-functions $B \in C(\overline{D})$, $B_i \in C(\overline{\Delta_i^*})$ and the vector-functions $F \in \overset{0}{C}_{\alpha}(\overline{D})$, $f_i \in \overset{0}{C}_{\alpha}(\overline{\Delta_i^*})$, $i = 1, 2$.

§ 3. Equivalent Reduction of Problem (1.2), (2.4) to a System of Integro-Differential Equations

From an arbitrary point $P(x,y,t) \in \overline{D}\backslash\Gamma$ we draw the bicharacteristic beam $\widetilde{L_i}(P)$ of system (2.2) which corresponds to the vector $\widetilde{l_i}$ and is directed towards the decreasing values of the t-coordinate of a moving point of $\widetilde{L_i}(P)$, $1 \le i \le 3$. The points of intersection of beams $\widetilde{L_i}(P)$, $i = 1,2,3$, with the faces S_1 and S_2 are $\widetilde{P^i} \in S_1$, $i = 1,2$, and $\widetilde{P^3} \in S_2$. Denote by $\big(\omega_i^1(x,y,t), \omega_i^2(x,y,t)\big)$ the coordinates of orthogonal projection of the point $\widetilde{P^i}$ onto the plane (x,t), $1 \le i \le 3$. A simple calculation yields

$$\left.\begin{aligned} \omega_i^1(x,y,t) &= x + \widetilde{\nu}_i(1-\widetilde{\mu}_i)^{-1}(y-t),\\ \omega_i^2(x,y,t) &= t + (1-\widetilde{\mu}_i)^{-1}(y-t), \end{aligned}\right. \quad i = 1,2,$$

$$\omega_3^1(x,y,t) = x - \widetilde{\nu}_3(1+\widetilde{\mu}_3)^{-1}(y+t), \quad \omega_3^2(x,y,t) = t - (1+\widetilde{\mu}_3)^{-1}(y+t).$$

Let $\xi = x_i(x,y,t;\tau)$, $\eta = y_i(x,y,t;\tau)$, $\zeta = \tau$ be the parametrization of a segment $\widetilde{L_i}(P) \cap \overline{D}$, where $\omega_i^2(x,y,t) \le \tau \le t$, $1 \le i \le 3$.

After integrating the (q_i+j)-th equation of system (2.2), where $q_1 = 0$, $q_i = s_1 + \cdots + s_{i-1}$, $i \ge 2$, $j = 1,\dots,s_i$, along the i-th bicharacteristic $\widetilde{L_i}(P)$ drawn from an arbitrary point $P(x,y,t) \in \overline{D}\backslash\Gamma$ and lying between the point $P(x,y,t)$ and the point of intersection of $\widetilde{L_i}(P)$ with the face S_1 or S_2 (depending on the index i of $\widetilde{L_i}(P)$), we obtain

$$\begin{aligned} v_{q_i+j}(x,y,t) &= v_{q_i+j}\big(\omega_i^1(x,y,t), \omega_i^2(x,y,t)\big) + \\ &+ \int_{\omega_i^2(x,y,t)}^{t} \Big(\sum_{p'=1}^{m} b_{ijp'}v_{p'}\Big)\big(x_i(x,y,t;\tau), y_i(x,y,t;\tau), \tau\big)d\tau + \\ &+ F_{ij}(x,y,t), \quad 1 \le i \le 3, \quad 1 \le j \le s_i, \end{aligned} \tag{3.1}$$

where v_{q_i+j} are the components of the vector v, $b_{ijp'}$ and F_{ij} are the well-defined functions depending only on the coefficients and the right-hand side of system (2.2).

We set

$$\begin{aligned}\varphi^1_{q_i+j}(x,t)&\equiv v_{q_i+j}\big|_{\overline{\Delta_1}}\equiv v_{q_i+j}(x,t,t),\ (x,t)\in\overline{\Delta_1^*}, i=1,2; j=1,\dots,s_i,\\ \varphi^2_{q_i+j}(x,t)&\equiv v_{q_i+j}\big|_{\overline{\Delta_2}}\equiv v_{q_i+j}(x,-t,t),\ (x,t)\in\overline{\Delta_2^*}, i=3; j=1,\dots,s_i.\end{aligned}\tag{3.2}$$

It is obvious that the number of components of the vectors

$$\begin{aligned}\varphi^1(x,t)&\equiv\big(\varphi^1_{q_i+j}(x,t)\big),\ \ (x,t)\in\overline{\Delta_1^*},\ i=1,2;\ j=1,\dots,s_i,\\ \varphi^2(x,t)&\equiv\big(\varphi^2_{q_i+j}(x,t)\big),\ \ (x,t)\in\overline{\Delta_2^*},\ i=3;\ j=1,\dots,s_i,\end{aligned}$$

is equal to the numbers m_1 and m_2, respectively.

By substituting the expressions of v from equality (3.1) into the boundary conditions (2.4) and taking into account (3.2) we have

$$\begin{aligned}&Q^1_0(x,t)\varphi^1(x,t)+Q^1_3(x,t)\varphi^2\big(\sigma_3(x,t)\big)+(T_1v)(x,t)=f^1(x,t),\ (x,t)\in\overline{\Delta_1^*},\\ &Q^2_0(x,t)\varphi^2(x,t)+\sum_{i=1}^{2}Q^2_i(x,t)\varphi^1\big(\sigma_i(x,t)\big)+(T_2v)(x,t)=\\ &\qquad=f^2(x,t),\ \ (x,t)\in\overline{\Delta_2^*},\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}\big(T_1v\big)(x,t)&\equiv\int\limits_{\omega_3^2(x,t,t)}^{t}\big(\widetilde{A_3}v\big)\big(x_3(x,t,t;\tau),y_3(x,t,t;\tau),\tau\big)d\tau,\\ \big(T_2v\big)(x,t)&\equiv\sum_{i=1}^{2}\int\limits_{\omega_i^2(x,-t,t)}^{t}\big(\widetilde{A_i}v\big)\big(x_i(x,-t,t;\tau),y_i(x,-t,t;\tau),\tau\big)d\tau,\end{aligned}\tag{3.4}$$

and Q^1_3, $\widetilde{A_3}$, Q^2_i, $\widetilde{A_i}$, f^i, $i=1,2$, are respectively the well-defined matrices and vectors.

It is obvious that Q^i_0 from (3.3) are matrices of order $(m_i\times m_i)$ which can be represented as the product

$$Q^i_0=B_i\times\widetilde{C_i},\quad i=1,2,\tag{3.5}$$

and the functions σ_i are defined by the equalities

$$\begin{aligned}\sigma_i&:(x,t)\to\big(\omega^1_i(x,-t,t),\omega^2_i(x,-t,t)\big),\ i=1,2,\\ \sigma_i&:(x,t)\to\big(\omega^1_i(x,t,t),\omega^2_i(x,t,t)\big),\ i=3.\end{aligned}$$

Assuming that

$$\det Q_0^i\big|_{\overline{\Delta_i^*}} \neq 0, \quad i=1,2, \tag{3.6}$$

where the matrices Q_0^i are given by (3.5), we rewrite system (3.3) as

$$\begin{aligned}
&\varphi^1(x,t)-\sum_{i=1}^{2} G_i^1(x,t)\varphi^1\big(J_i^1(x,t)\big)+\big(T_3 v\big)(x,t)=f^3(x,t),\ (x,t)\in\overline{\Delta_1^*},\\
&\varphi^2(x,t)-\sum_{i=1}^{2} G_i^2(x,t)\varphi^2\big(J_i^2(x,t)\big)+\big(T_4 v\big)(x,t)=f^4(x,t),\ (x,t)\in\overline{\Delta_2^*},
\end{aligned} \tag{3.7}$$

where G_i^p are the known matrix-functions of order $(m_p\times m_p)$, $p=1,2$, f^{i+2} are the known vector-functions,

$$\begin{aligned}
&J_i^1(x,t)\equiv \sigma_i\big(\sigma_3(x,t)\big), \quad (x,t)\in\overline{\Delta_1^*},\\
&J_i^2(x,t)\equiv \sigma_3\big(\sigma_i(x,t)\big), \quad (x,t)\in\overline{\Delta_2^*}, \ \ i=1,2.
\end{aligned}$$

It is easy to verify that by virtue of equalities (3.4) the linear integral operators T_3 and T_4 can be represented as

$$\begin{aligned}
\big(T_3 v\big)(x,t)=\sum_{i=1}^{2}\int\limits_{\widetilde{\omega}_i^2(x,t)}^{\omega_3^2(x,t,t)} &\big(\widetilde{B_i}v\big)\big(x_i\big(\omega_3^1(x,t,t),-\omega_3^2(x,t,t),\omega_3^2(x,t,t);\tau\big),\\
&y_i\big(\omega_3^1(x,t,t),-\omega_3^2(x,t,t),\omega_3^2(x,t,t);\tau\big),\tau\big)d\tau+\\
&+\int\limits_{\omega_3^2(x,t,t)}^{t}\big(\widetilde{B_3}v\big)\big(x_3(x,t,t;\tau),y_3(x,t,t;\tau),\tau\big)d\tau,
\end{aligned}$$

$$\begin{aligned}
\big(T_4 v\big)(x,t)=\sum_{i=1}^{2}\int\limits_{\widetilde{\omega}_3^2(x,t)}^{\omega_i^2(x,-t,t)} &\big(E_i v\big)\big(x_3\big(\omega_i^1(x,-t,t),\omega_i^2(x,-t,t),\omega_i^2(x,-t,t);\tau\big),\\
&y_3\big(\omega_i^1(x,-t,t),\omega_i^2(x,-t,t),\omega_i^2(x,-t,t);\tau\big),\tau\big)d\tau+\\
&+\sum_{i=1}^{2}\int\limits_{\omega_i^2(x,-t,t)}^{t}\big(H_i v\big)\big(x_i(x,-t,t;\tau),y_i(x,-t,t;\tau),\tau\big)d\tau,
\end{aligned}$$

where $\widetilde{\omega}_i^2(x,t) \equiv \omega_i^2\big(\omega_3^1(x,t,t),\ -\omega_3^2(x,t,t),\ \omega_3^2(x,t,t)\big)$, $\widetilde{\omega}_3^2(x,t) \equiv \omega_3^2\big(\omega_i^1(x,-t,t),\ \omega_i^2(x,-t,t),\ \omega_i^2(x,-t,t)\big)$, and E_i, H_i, $i=1,2$, $\widetilde{B_j}$, $j=1,2,3$, are the well-defined matrices.

For the functions $J_i^k : \overline{\Delta_k^*} \to \overline{\Delta_k^*}$ we have the formulas

$$J_i^k : (x,t) \to (x + \delta_i^k t, \tau_i t), \quad (x,t) \in \overline{\Delta_k^*},$$

where $\delta_i^k, \tau_i,\ i,k = 1,2$, are the well-defined constants written in terms of $\widetilde{\nu}_i,\ \widetilde{\mu}_i,\ i = 1,2,3$.

Remark 3.1. Note that by virtue of condition (2.3) it is easy to establish that $0 < \tau_i < 1$, $i = 1,2$.

Remark 3.2. It is obvious that when conditions (3.6) are fulfilled, problem (1.2), (2.4) in the class $\overset{0}{C}{}_\alpha^{1,1,1}(\overline{D})$ is equivalently reduced to system (3.7) for the unknown vector-function φ^i of the class $\overset{0}{C}{}_\alpha(\overline{\Delta_i^*})$, $i = 1,2$. Furthermore, if $u \in \overset{0}{C}{}_\alpha^{1,1,1}(\overline{D})$, then $\varphi^i \in \overset{0}{C}{}_\alpha(\overline{\Delta_i^*})$, $i = 1,2$. Vice versa, if $\varphi^i \in \overset{0}{C}{}_\alpha(\overline{\Delta_i^*})$, $i = 1,2$, then with regard to inequality (2.5) equalities (3.1), (3.2) and $u = Cv$ readily imply that $u \in \overset{0}{C}{}_\alpha^{1,1,1}(\overline{D})$.

§ 4. Investigation of the System of Integro-Functional Equations (3.1), (3.7) and the Proof of the Main Result

Let us consider the system of functional equations

$$(K_p\varphi^p)(x,t) \equiv \varphi^p(x,t) - \sum_{i=1}^{2} G_i^p(x,t)\varphi^p\big(J_i^p(x,t)\big) = g_p(x,t),\ (x,t) \in \overline{\Delta_p^*}, \quad (4.1)$$

and introduce the notation

$$\begin{gathered} h_p(\rho) \equiv \sum_{i=1}^{2} \eta_{ip}\tau_i^\rho, \quad \eta_p \equiv \max_{1\le i\le 2} \sup_{(x,t)\in\overline{\Delta_p^*}} \|G_i^p(x,t)\|, \\ \eta_{ip} \equiv \sup_{x\in[Q_1,Q_2]} \|G_i^p(x,0)\|, \quad i,p = 1,2, \quad \rho \in \mathbb{R}. \end{gathered} \quad (4.2)$$

Here and in what follows by $\|.\|$ we understand the norm of a matrix operator acting from one Euclidean space into another.

If all values $\eta_{ip} = 0$, then it is assumed that $\rho_p = -\infty$, $i = 1,2$, $1 \le p \le 2$. Let now for some value of the index i the value η_{ip}, $1 \le p \le 2$, be different from zero. In that case, by Remark 3.1, the function $h_p : \mathbb{R} \to \overline{\mathbb{R}}_+$ is continuous and strictly decreasing on $\mathbb{R}$; also, $\lim_{\rho\to-\infty} h_p(\rho) = +\infty$ and $\lim_{\rho\to+\infty} h_p(\rho) = 0$, $1 \le p \le 2$. Therefore there exists a unique real number ρ_p such that $h_p(\rho_p) = 1$, $1 \le p \le 2$. It is assumed that $\rho_0 \equiv \max(\rho_1,\rho_2)$.

Lemma 4.1. *If $\alpha > \rho_0$, then equation* (4.1) *is uniquely solvable in the space* $\overset{0}{C}\alpha(\overline{\Delta_p^*})$ *and for the solution* $\varphi^p = K_p^{-1}g^p$ *the estimate*

$$\|\varphi^p(x,t)\|_{\mathbb{R}^{m_p}} = \|(K_p^{-1}g_p)(x,t)\|_{\mathbb{R}^{m_p}} \leq$$
$$\leq C_{2+p}t^\alpha\|g_p\|_{\overset{0}{C}\alpha(\overline{\Delta_p^*}\cap\{t_1\leq t\})}, \quad (x,t)\in\overline{\Delta_p^*}, \tag{4.3}$$

holds, where C_{2+p} is a positive constant not depending on the function g_p, $1\leq p\leq 2$.

Proof. We shall consider the case $p=1$, since the case $p=2$ is considered analogously. The condition $\alpha > \rho_0$ and the definition of the function h_1 from (4.2) imply

$$h_1(\alpha) = \sum_{i=1}^{2} \eta_{i1}\tau_i^\alpha < 1. \tag{4.4}$$

By inequality (4.4) and the continuity of the functions G_i^1, $i=1,2$, there exist positive numbers ε_1 ($\varepsilon_1 < t^0$) and δ_1 such that the inequalities

$$\|G_i^1(x,t)\| \leq \eta_{i1}+\delta_1, \quad i=1,2, \tag{4.5}$$
$$\sum_{i=1}^{2}(\eta_{i1}+\delta_1)\tau_i^\alpha \equiv \gamma_1 < 1 \tag{4.6}$$

hold for $(x,t)\in\overline{\Delta_1^*}\cap\{0\leq t\leq\varepsilon_1\}$.

By Remark 3.1 there is a natural number r_0 such that for $r\geq r_0$

$$\tau_{i_r}\tau_{i_{r-1}}\cdots\tau_{i_1}t\leq\varepsilon_1, \quad 0\leq t\leq t^0, \tag{4.7}$$

where $1\leq i_s\leq 2$, $s=1,\dots,r$.

We introduce the operators Λ_1 and K_1^{-1} acting by the formulas

$$(\Lambda_1\varphi^1)(x,t) = \sum_{i=1}^{2} G_i^1(x,t)\varphi^1\Big(J_i^1(x,t)\Big), \ \ (x,t)\in\overline{\Delta_1^*}, \ \ K_1^{-1} = I + \sum_{r=1}^{\infty}\Lambda_1^r,$$

where I is the identical operator. Obviously, the operator K_1^{-1} is the formally inverse operator to the operator K_1 defined by equality (4.1). Hence it is sufficient for us to show that K_1^{-1} is continuous in the space $\overset{0}{C}_\alpha(\overline{\Delta_1^*})$.

As is easily seen, the expression $\Lambda_1^r g_1$ is the sum consisting of terms of the form

$$I_{i_1\cdots i_r}(x,t) = G_{i_1}^1(x,t)G_{i_2}^1(J_{i_1}^1(x,t))G_{i_3}^1(J_{i_2}^1(J_{i_1}^1(x,t)))\cdots$$
$$\cdots G_{i_r}^1(J_{i_{r-1}}^1(J_{i_{r-2}}^1(\cdots(J_{i_1}^1(x,t))\cdots)))g_1(J_{i_r}^1(J_{i_{r-1}}^1(\cdots(J_{i_1}^1(x,t))\cdots))),$$

where $1\leq i_s\leq 2$, $s=1,\dots,r$.

Hence, using (4.2), (4.5), (3.7) and Remark 3.1, we obtain: for $r > r_0$, $g_1 \in \overset{0}{C}_\alpha(\overline{\Delta_1^*})$

$$\|I_{i_1\cdots i_r}(x,t)\|_{\mathbb{R}^{m_1}} \le$$
$$\le \|G^1_{i_1}(x,t)\|\cdots\|G^1_{i_{r_0}}(J^1_{i_{r_0-1}}(J^1_{i_{r_0-2}}(\cdots(J^1_{i_1}(x,t))\cdots)))\|\times$$
$$\times\|G^1_{i_{r_0+1}}(J^1_{i_{r_0}}(J^1_{i_{r_0-1}}(\cdots(J^1_{i_1}(x,t))\cdots)))\|\cdots$$
$$\cdots\|G^1_{i_r}(J^1_{i_{r-1}}(J^1_{i_{r-2}}(\cdots(J^1_{i_1}(x,t))\cdots)))\|\times$$
$$\times\|g_1(J^1_{i_r}(J^1_{i_{r-1}}(\cdots(J^1_{i_1}(x,t))\cdots)))\|_{\mathbb{R}^{m_1}} \le$$
$$\le \eta_1^{r_0}(\eta_{i_{r_0+1}1}+\delta_1)\cdots(\eta_{i_r1}+\delta_1)(\tau_{i_r}\tau_{i_{r-1}}\cdots\tau_{i_1}t)^\alpha\|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} \le$$
$$\le \eta_1^{r_0}\Big(\prod_{s=r_0+1}^{r}(\eta_{i_s1}+\delta_1)\Big)\Big(\prod_{s=r_0+1}^{r}\tau_{i_s}^\alpha\Big)t^\alpha\|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} =$$
$$= \eta_1^{r_0}\Big(\prod_{s=r_0+1}^{r}(\eta_{i_s1}+\delta_1)\tau_{i_s}^\alpha\Big)t^\alpha\|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})}, \tag{4.8}$$

and for $1 \le r \le r_0$

$$\|I_{i_1\cdots i_r}(x,t)\|_{\mathbb{R}^{m_1}} \le \eta_1^r(\tau_{i_r}\tau_{i_{r-1}}\cdots\tau_{i_1}t)^\alpha\|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} \le$$
$$\le \eta_1^r t^\alpha\|g_1\|_{\overset{0}{C}\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})}. \tag{4.9}$$

By (4.8), (4.9), and (4.6) we have: for $r > r_0$

$$\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} = \Big\|\sum_{i_1,\dots,i_r} I_{i_1\cdots i_r}(x,t)\Big\|_{\mathbb{R}^{m_1}} \le$$
$$\le \Big(\sum_{i_1,\dots,i_{r_0}}1\Big)^{r_0}\eta_1^{r_0}\Big[\sum_{i=1}^{2}(\eta_{i1}+\delta_1)\tau_i^\alpha\Big]^{r-r_0}t^\alpha\|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})} \le$$
$$\le C_5\gamma_1^r t^\alpha\|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})}, \tag{4.10}$$

and for $1 \le r \le r_0$

$$\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} \le C_6 t^\alpha\|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\le t\})}, \tag{4.11}$$

where $C_5 \equiv \eta_1^{r_0}\gamma_1^{-r_0}\Big(\sum_{i_1,\dots,i_{r_0}}1\Big)^{r_0}$, $C_6 \equiv \eta_1^r\Big(\sum_{i_1,\dots,i_r}1\Big)$.

Inequalities (4.10) and (4.11) finally imply

$$\|\varphi^1(x,t)\|_{\mathbb{R}^{m_1}} = \|(K_1^{-1}g_1)(x,t)\|_{\mathbb{R}^{m_1}} \le$$
$$\le \|g_1(x,t)\|_{\mathbb{R}^{m_1}} + \sum_{r=1}^{r_0}\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} + \sum_{r=r_0+1}^{\infty}\|(\Lambda_1^r g_1)(x,t)\|_{\mathbb{R}^{m_1}} \le$$

$$\leq \big(1 + C_6 r_0 + C_5 \gamma_1^{r_0+1}(1-\gamma_1)^{-1}\big) t^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\leq t\})} =$$
$$= C_3 t^\alpha \|g_1\|_{\overset{0}{C}_\alpha(\overline{\Delta_1^*}\cap\{t_1\leq t\})},$$

where $C_3 \equiv 1 + C_6 r_0 + C_5 \gamma_1^{r_0+1}(1-\gamma_1)^{-1}$. Hence we conclude that the operator K_1^{-1} is continuous in the space $\overset{0}{C}_\alpha(\overline{\Delta_1^*})$ and therefore Lemma 4.1 is true. □

On the basis of this lemma we have

Theorem 4.1. *Let conditions* (3.6) *be fulfilled. If* $\alpha > \rho_0$, *then problem* (1.2), (2.4) *is uniquely solvable in the space* $\overset{0}{C}{}^{1,1,1}_\alpha(\overline{D})$.

Proof. First we solve the system of equations (3.1), (3.7) with respect to the unknown functions $v \in \overset{0}{C}{}^{1,1,1}_\alpha(\overline{D})$ and $\varphi^p \in \overset{0}{C}_\alpha(\overline{\Delta_p^*})$, $p = 1,2$, using the method of successive approximations.

Let

$$v_0(x,y,t) \equiv 0, \ \ (x,y,t) \in \overline{D}; \ \ \varphi_0^p(x,t) \equiv 0, \ \ (x,t) \in \overline{\Delta_p^*}, \ \ p = 1,2;$$
$$v_{q_i+j,k}(x,y,t) = \varphi_{q_i+j,k}\big(\omega_i^1(x,y,t), \omega_i^2(x,y,t)\big) +$$
$$+ \int\limits_{\omega_i^2(x,y,t)}^{t} \Big(\sum_{p'=1}^{m} b_{ijp'} v_{p',k-1}\Big)\big(x_i(x,y,t;\tau), y_i(x,y,t;\tau), \tau\big) d\tau + \tag{4.12}$$
$$+ \widetilde{F_{ij}}(x,y,t), \quad 1 \leq i \leq 3, \quad 1 \leq j \leq s_i,$$

where

$$\varphi_{q_i+j,k}\big(\omega_i^1(x,y,t), \omega_i^2(x,y,t)\big) =$$
$$= \begin{cases} \varphi^1_{q_i+j,k}\big(\omega_i^1(x,y,t), \omega_i^2(x,y,t)\big), & 1 \leq i \leq 2, \ \ 1 \leq j \leq s_i, \\ \varphi^2_{q_i+j,k}\big(\omega_i^1(x,y,t), \omega_i^2(x,y,t)\big), & i = 3, \ \ 1 \leq j \leq s_i, \end{cases} \quad (x,y,t) \in \overline{D}.$$

The values $\varphi_k^p(x,t)$ are defined by the equations

$$(K_p \varphi_k^p)(x,t) + (T_{2+p} v_{k-1})(x,t) = f^{2+p}(x,t), \tag{4.13}$$
$$(x,t) \in \overline{\Delta_p^*}, \ \ p = 1,2, \ \ k \geq 1,$$

where the operators K_p, $p = 1,2$, act by (4.1).

For convenience, system (4.12) is rewritten as

$$v_k(x,y,t) = \varphi_k(x,y,t) +$$

$$+\sum_{i=1}^{3}\int\limits_{\omega_i^2(x,y,t)}^{t}\Big(\Omega_i v_{k-1}\Big)\Big(x_i(x,y,t;\tau),y_i(x,y,t;\tau),\tau\Big)d\tau+$$
$$+\widetilde{F}(x,y,t),\quad (x,y,t)\in\overline{D},\tag{4.14}$$

where the (q_i+j)-th component of the vector $\varphi_k(x,y,t)$ is equal to $\varphi_{q_i+j,k}\big(\omega_i^1(x,y,t),\omega_i^2(x,y,t)\big)$, $1\le i\le 3$, $1\le j\le s_i$, $k\ge 1$; Ω_i, $i=1,2,3$, and $\widetilde{F}$ are respectively the well-defined matrices and vector-functions.

We shall now show that the following estimates are true:

$$\|v_{k+1}(x,y,t)-v_k(x,y,t)\|_{\mathbb{R}^m}\le M^*\frac{M_*^k}{k!}t^{k+\alpha},\ \ (x,y,t)\in\overline{D},\tag{4.15}$$
$$\|\varphi_{k+1}^p(x,t)-\varphi_k^p(x,t)\|_{\mathbb{R}^{m_p}}\le M^*\frac{M_*^k}{k!}t^{k+\alpha},\ \ (x,t)\in\overline{\Delta_p^*},\tag{4.16}$$

where M_* and M^* are well-defined sufficiently large numbers not depending on k, $k\ge 1$, $p=1,2$.

Indeed, by the assumptions for f_p and F we have $f^{2+p}\in\overset{0}{C}_{\alpha}(\overline{\Delta_p^*})$, $\widetilde{F}\in\overset{0}{C}_{\alpha}(\overline{D})$, $p=1,2$. Hence, on account of inequalities (2.5) from §2, we conclude that the estimates

$$\|\widetilde{F}(x,y,t)\|_{\mathbb{R}^m}\le\Theta_1 t^\alpha,\quad (x,y,t)\in\overline{D},\tag{4.17}$$
$$\|f^{2+p}(x,t)\|_{\mathbb{R}^{m_p}}\le\Theta_{1+p}t^\alpha,\quad (x,t)\in\overline{\Delta_p^*},\tag{4.18}$$
$$p=1,2,\quad \Theta_i=\text{const}\ge 0,\quad i=1,2,3,$$

are fulfilled.

By $v_0\equiv 0$, $\varphi_0^p\equiv 0$, $p=1,2$ and the conditions of Theorem 4.1 estimate (4.3) is true so that (4.13), (4.18) imply

$$\begin{gathered}\|\varphi_1^p(x,t)-\varphi_0^p(x,t)\|_{\mathbb{R}^{m_p}}=\|\varphi_1^p(x,t)\|_{\mathbb{R}^{m_p}}\le C_5\Theta_4 t^\alpha,\ \ p=1,2,\\ C_7=\max\,(C_3,C_4),\quad \Theta_4=\max\,(\Theta_2,\Theta_3)\end{gathered}\tag{4.19}$$

which in turn gives rise to

$$\begin{aligned}&\|\varphi_1(x,y,t)-\varphi_0(x,y,t)\|_{\mathbb{R}^m}=\|\varphi_1(x,y,t)\|_{\mathbb{R}^m}=\\ &=\sum_{1\le i\le 3}\sum_{1\le j\le s_i}\big|\varphi_{q_i+j,1}\big(\omega_i^1(x,y,t),\omega_i^2(x,y,t)\big)\big|\le\\ &\le\sum_{1\le i\le 3}\sum_{1\le j\le s_i}C_7\Theta_4\big(\omega_i^2(x,y,t)\big)^\alpha\le mC_7\Theta_4 t^\alpha,\end{aligned}\tag{4.20}$$

since $\sum\limits_{1\le i\le 3}\sum\limits_{1\le j\le s_i}1=m$ and, as shown in §3, $0\le\omega_i^2(x,y,t)\le t$, $i=1,2,3$.

By virtue of (4.17) and (4.20), from (4.14) we have

$$\|v_1(x,y,t)-v_0(x,y,t\|_{\mathbb{R}^m}=\|v_1(x,y,t)\|_{\mathbb{R}^m}\le\|\varphi_1(x,y,t)\|_{\mathbb{R}^m}+ \\ +\|\widetilde{F}(x,y,t)\|_{\mathbb{R}^m}\le mC_7\Theta_4t^\alpha+\Theta_1t^\alpha=(mC_7\Theta_4+\Theta_1)t^\alpha. \quad (4.21)$$

Now, assuming that estimates (4.15), (4.16) are fulfilled for k, $k>0$, we shall show that they hold for $k+1$ when M_* and M^* are sufficiently large.

Using (4.13), for $p=1$ we have

$$\{K_1(\varphi^1_{k+2}-\varphi^1_{k+1})\}(x,t)=-\{T_3(v_{k+1}-v_k)\}(x,t),\quad (x,t)\in\overline{\Delta^*_1}. \quad (4.22)$$

It is obvious that for the right-hand side of equation (4.22) we have the estimate

$$\big\|\{T_3(v_{k+1}-v_k)\}(x,t)\big\|_{\mathbb{R}^{m_1}}\le \\ \le\sum_{i=1}^2\int\limits_{\widetilde{\omega}_i^2(x,t)}^{\omega_3^2(x,t,t)}\|\widetilde{B_i}\|\|v_{k+1}-v_k\|_{\mathbb{R}^m}\big(x_i\big(\omega_3^1(x,t,t),-\omega_3^2(x,t,t), \\ \omega_3^2(x,t,t);\tau\big),y_i\big(\omega_3^1(x,t,t),-\omega_3^2(x,t,t),\omega_3^2(x,t,t);\tau\big),\tau\big)d\tau+ \\ +\int\limits_{\omega_3^2(x,t,t)}^{t}\|\widetilde{B_3}\|\|v_{k+1}-v_k\|_{\mathbb{R}^m}\big(x_3(x,t,t;\tau),y_3(x,t,t;\tau),\tau\big)d\tau. \quad (4.23)$$

Denote by ξ_1 the largest of the numbers $\max\limits_{x,t,\tau}\|\widetilde{B_i}(x,t,\tau)\|$, $i=1,2,3$. Since $0\le\widetilde{\omega}_i^2(x,t)\le\omega_3^2(x,t,t)\le t$, by (4.15) we find from (4.23) that

$$\big\|\{T_3(v_{k+1}-v_k)\}(x,t)\big\|_{\mathbb{R}^{m_1}}\le \\ \le\xi_1M^*\frac{M_*^k}{k!}\Big(\sum_{i=1}^2\int\limits_{\widetilde{\omega}_i^2(x,t)}^{\omega_3^2(x,t,t)}\tau^{k+\alpha}d\tau+\int\limits_{\omega_3^2(x,t,t)}^{t}\tau^{k+\alpha}d\tau\Big)\le \\ \le\xi_1M^*\frac{M_*^k}{k!}\Big(\sum_{i=1}^2 1+1\Big)\int\limits_0^t\tau^{k+\alpha}d\tau\le \\ \le 3\xi_1M^*\frac{M_*^k}{k!}\,\frac{1}{k+\alpha+1}t^{k+\alpha+1}\le 3\xi_1M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha}. \quad (4.24)$$

Now (4.22), (4.24), and (4.3) (for $p=1$) imply

$$\big\|\varphi^1_{k+2}(x,t)-\varphi^1_{k+1}(x,t)\big\|_{\mathbb{R}^{m_1}}\le 3C_3\xi_1M^*\frac{M_*^k}{(k+1)!}t^{k+1+\alpha}. \quad (4.25)$$

Similarly, (4.13) (for $p=2$), (4.15), and (4.3) (for $p=2$) give

$$\left\|\varphi_{k+2}^2(x,t)-\varphi_{k+1}^2(x,t)\right\|_{\mathbb{R}^{m_2}} \leq 4C_4\xi_2 M^* \frac{M_*^k}{(k+1)!} t^{k+1+\alpha}, \qquad (4.26)$$

where ξ_2 denotes the largest of the numbers $\max\limits_{x,t,\tau}\|E_i(x,t,\tau)\|$, $\max\limits_{x,t,\tau}\|H_i(x,t,\tau)\|$, $i=1,2$.

Using the same arguments as in deriving estimate (4.20), from (4.25) and (4.26) we obtain

$$\|\varphi_{k+2}(x,y,t)-\varphi_{k+1}(x,y,t)\|_{\mathbb{R}^m} \leq \xi_4 M^* \frac{M_*^k}{(k+1)!} t^{k+1+\alpha}, \qquad (4.27)$$

where $\xi_4 \equiv 4mC_7\xi_3$, $\xi_3 \equiv \max\,(\xi_1,\xi_2)$.

We denote by η the largest of the numbers $\max\limits_{\overline{D}}\|\Omega_i\|$, where the matrices Ω_i, $i=1,2,3$, are defined by (4.14). By (4.27) and (4.15), from system (4.14) we have

$$\begin{aligned}
&\|v_{k+2}(x,y,t)-v_{k+1}(x,y,t)\|_{\mathbb{R}^m} \leq \|\varphi_{k+2}(x,y,t)-\varphi_{k+1}(x,y,t)\|_{\mathbb{R}^m} + \\
&+\sum_{i=1}^{3}\int\limits_{\omega_i^2(x,y,t)}^{t} \|\Omega_i\|\|v_{k+1}-v_k\|_{\mathbb{R}^m}\big(x_i(x,y,t;\tau),y_i(x,y,t;\tau),\tau\big)d\tau \leq \\
&\leq \xi_4 M^* \frac{M_*^k}{(k+1)!} t^{k+1+\alpha} + 3\eta\int\limits_0^t M^* \frac{M_*^k}{k!}\tau^{k+\alpha} d\tau \leq \\
&\leq (\xi_4+3\eta) M^* \frac{M_*^k}{(k+1)!} t^{k+1+\alpha}, \quad (x,y,t)\in\overline{D}, \qquad (4.28)
\end{aligned}$$

since $0\leq\omega_i^2(x,y,t)\leq t$, $i=1,2,3$.

If we set

$$M^* = mC_7\Theta_4+\Theta_1, \quad M_* = \max\big(3C_3\xi_1,\, 4C_4\xi_2,\, \xi_4+3\eta\big),$$

then by (4.19), (4.21), (4.25), (4.26), (4.28) immediately imply that estimates (4.15), (4.16) hold for any integer $k\geq 0$.

It follows from (4.15), (4.16) that the series

$$v(x,y,t) = \lim_{k\to\infty} v_k(x,y,t) = \sum_{k=1}^{\infty}\big(v_k(x,y,t)-v_{k-1}(x,y,t)\big),\ (x,y,t)\in\overline{D},$$

$$\varphi^p(x,t) = \lim_{k\to\infty}\varphi_k^p(x,t) = \sum_{k=1}^{\infty}\big(\varphi_k^p(x,t)-\varphi_{k-1}^p(x,t)\big),\ (x,t)\in\overline{\Delta_p^*},\ p=1,2,$$

converge in the spaces $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$, $\overset{0}{C}{}_{\alpha}(\overline{\Delta^*_p})$, $p=1,2$, and by (4.13), (4.14) the limit functions v, φ^p, $p=1,2$, satisfy system (3.1), (3.7). Finally, since problem (1.2), (2.4) is equivalent to system (3.1), (3.7) and the equality $u=Cv$ holds, we conclude that the obtained function $u(x,y,t)$ is really a solution of problem (1.2), (2.4) in the class $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$, $\alpha>\rho_0$.

Now we shall show that under the conditions of Theorem 4.1 problem (1.2), (2.4) has no other solutions in the class $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$. Indeed, if $u\in\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})$ is the solution of the homogeneous problem corresponding to (1.2), (2.4), then the corresponding functions v, φ^p, $p=1,2$, satisfy the homogeneous system of equations

$$\begin{gathered} v_{q_i+j}(x,y,t)=\varphi_{q_i+j}\big(\omega^1_i(x,y,t),\omega^2_i(x,y,t)\big)+\\ +\int\limits_{\omega^2_i(x,y,t)}^{t}\Big(\sum_{p'=1}^{m}b_{ijp'}v_{p'}\Big)\big(x_i(x,y,t;\tau),y_i(x,y,t;\tau),\tau\big)d\tau,\ (x,y,t)\in\overline{D},\\ 1\le i\le 3,\quad 1\le j\le s_i,\\ \big(K_p\varphi^p\big)(x,t)+\big(T_{2+p}v\big)(x,t)=0,\ (x,t)\in\overline{\Delta^*_p},\ \ p=1,2. \end{gathered}\tag{4.29}$$

We apply the method of successive approximations to system (4.29), assuming that v, φ^1, φ^2, are zero approximations. Since these values satisfy system (4.29), each next approximation will coincide with it so that we shall have $v_k(x,y,t)\equiv v(x,y,t)$, $(x,y,t)\in\overline{D}$, $\varphi^p_k(x,t)\equiv\varphi^p(x,t)$, $(x,t)\in\overline{\Delta^*_p}$ for $k\ge 1$ and $p=1,2$. Recalling that these values satisfy estimates of form (4.17), (4.18) and arguing as in the case of deriving estimates (4.15), (4.16), we obtain

$$\|v(x,y,t)\|_{\mathbb{R}^m}=\|v_{k+1}(x,y,t)\|_{\mathbb{R}^m}\le\widetilde{M}^*\frac{\widetilde{M}^k_*}{k!}t^{k+\alpha},\ \ (x,y,t)\in\overline{D},$$

$$\|\varphi^p(x,t)\|_{\mathbb{R}^{m_p}}=\|\varphi^p_{k+1}(x,t)\|_{\mathbb{R}^{m_p}}\le\widetilde{M}^*\frac{\widetilde{M}^k_*}{k!}t^{k+\alpha},\ (x,t)\in\overline{\Delta^*_p},\ k\ge 1,\ p=1,2,$$

whence, as $k\to\infty$, we find in the limit that

$$v(x,y,t)\equiv 0,\ \ (x,y,t)\in\overline{D},\ \ \varphi^p(x,t)\equiv 0,\ \ (x,t)\in\overline{\Delta^*_p},\ \ p=1,2.\quad\square$$

Next, using inequality (4.15) and recalling that the value M^* is defined by Θ_i, $i=1,2,3$, which are given by the right-hand sides F and f_i, $i=1,2$, of problem (1.2), (2.4), we can readily show that for a regular solution of the considered problem the estimate

$$\|u\|_{\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D})}\le c\Big(\sum_{i=1}^{2}\|f_i\|_{\overset{0}{C}{}_{\alpha}(\overline{\Delta_i})}+\|F\|_{\overset{0}{C}{}_{\alpha}(\overline{D})}\Big)\tag{4.30}$$

holds, where the positive constant c does not depend on f_i, $i = 1, 2$, and F. Estimate (4.30) implies that a regular solution of problem (1.2), (2.4) is stable in the space $\overset{0}{C}{}^{1,1,1}_{\alpha}(\overline{D}), \alpha > \rho_0$.

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